

O(2)-equivariant Bifurcation Equations of mode (1, 2)

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Abstract

We consider mappings of one real and two complex variables with a range in two dimensional complex space. We introduce the notion of O(2)-equivariance with mode (m, n) , where m and n are distinct positive integers. We consider only O(2)-equivariant mappings with mode (1, 2). Normal forms and universal unfoldings are given in three cases; one is a generic case, the other two cases deal with some degeneracies.

§ 1. Introduction

We consider bifurcation equations which stem from interaction of two distinct modes in the presence of symmetry group O(2). Since the bifurcation equations to be considered here involve five variables, it is not immediately clear what order of the Taylor expansions of the equations suffice to the analysis of the zero-point set. Our aim here is to determine where we should truncate the Taylor expansion of the bifurcation equation without changing topological nature of the zero-point set. We think the singularity theoretic approach by Golubitsky and Schaeffer [6, 7] and Golubitsky, Stewart and Schaeffer [8] is very useful for understanding structure of the bifurcation diagrams associated with the bifurcation equations. We use their terminology and follow the method there.

What we do in this paper is loosely stated as follows. In §2 we introduce a concept of O(2)-equivariance which is satisfied by mappings of the following form:

$$G: \mathbf{R} \times \mathbf{C}^2 \longrightarrow \mathbf{C}^2.$$

Roughly speaking, O(2)-equivariance means that a certain action of O(2) is defined on \mathbf{C}^2 and that G commutes with it. A brief statement of the main result is given at the end of §2. In §§3, 4 and 5 we consider the following problems.

- i) Under what condition, G is $O(2)$ -equivalent to some normal form?
- ii) Of what form the universal unfolding is?

In these sections we consider only $O(2)$ -equivariance of mode $(1, 2)$ (see §2). Namely we consider bifurcation phenomena described in an eigenspace spanned by $(\exp(i\theta), 0)$ and $(0, \exp(2i\theta))$. We consider the problem in three different cases. §3 is devoted to the generic case, the analysis of which is begun by Fujii, Mimura and Nishiura [4]. By means of the universal unfolding in Theorem 3.3, we can prove that the bifurcation diagrams in [4] give all the possible cases and no new diagram other than those presented in [4] can be seen in the generic case. §§4 and 5 are devoted to some degenerate cases, which are partially considered in Armbruster and Dangelmayr [1, 2] and Fujii, Nishiura and Hosono [5]. Let us explain the difference of the result here and those in [1, 2, 5]. The context of pp.181-185 of [5] corresponds to §4. The bifurcation diagrams in [5] are, however, included as a part of those presented in this paper. Since we compute a universal unfolding, we can give a complete set of bifurcation diagrams. We also think it is worthwhile to re-observe their results from a singularity theoretic viewpoint. Our approach has an advantage that the case in §5, which is not considered in [5], can be treated in a way similar to §4. The approach in [1, 2] is close to ours. But it seems to us that some of the unfoldings in [1, 2] are different from ours. Since [1, 2] do not contain detailed proof, we do not know whether our proof is the same as theirs. We finally remark that we consider only bifurcations of stationary solutions. Therefore the bifurcation of travelling waves considered in Kokubu [9] is out of our scope. The study of them are left to consideration in the future.

The analysis presented in this paper has a variety of applications. In particular, we can refer to [5] of reaction-diffusion equations, [11] of the Taylor problem and [12, 14] of water waves. This paper is a revised version of the author's thesis in 1985 at the University of Tokyo. A part of it is published in [10]. Although [10] contains some errors, we correct them and add some improvements in this paper. We include all the details of the proofs except that computations of determinants of some matrices are left to the readers. We checked the computations both by hand computations and by a algebraic programming system REDUCE.

We give here a list of symbols which will be used later. Definitions will appear in §§2 and 3.

List of Symbols

- $\mathcal{E} = \{f; f \text{ is a smooth function of } (\lambda, u, v, r)\}.$
- $\mathcal{M} = \{f \in \mathcal{E}; f(0, 0, 0, 0) = 0\}.$
- $\mathcal{E}_\lambda = \{A; A \text{ is a smooth function of } \lambda\}.$
- $\mathcal{M}_\lambda = \{A \in \mathcal{E}_\lambda; A(0) = 0\}.$
- $\tilde{E} = \{X: \mathbf{R} \times \mathbf{C}^2 \longrightarrow \mathbf{C}^2; X(\lambda, \gamma(\xi, \zeta)) = \gamma X(\lambda, \xi, \zeta) (\gamma \in O(2))\}.$

$$\begin{aligned}\tilde{\Gamma}G &= \{dG(X)\} + \{TG\}. \\ \Gamma G &= \tilde{\Gamma}G + \mathcal{E}_\lambda \frac{\partial G}{\partial \lambda}.\end{aligned}$$

§ 2. O(2)-equivariant equations

We begin this section by

DEFINITION 2.1. *Given two integers m and n satisfying $0 < m < n$ and a mapping*

$$G = (G_1, G_2): \mathbf{R} \times \mathbf{C}^2 \longrightarrow \mathbf{C}^2,$$

we say that G is O(2)-equivariant with mode (m, n) if the following conditions (2.1, 2) are satisfied:

$$(2.1) \quad G(\lambda, e^{im\alpha}\xi, e^{in\alpha}\zeta) = (e^{im\alpha}G_1(\lambda, \xi, \zeta), e^{in\alpha}G_2(\lambda, \xi, \zeta))$$

for any $\lambda \in \mathbf{R}$, $(\xi, \zeta) \in \mathbf{C}^2$ and $\alpha \in [0, 2\pi)$. Hereafter $i = \sqrt{-1}$.

$$(2.2) \quad G(\lambda, \bar{\xi}, \bar{\zeta}) = \overline{(G_1(\lambda, \xi, \zeta), G_2(\lambda, \xi, \zeta))}$$

for any $\lambda \in \mathbf{R}$ and $(\xi, \zeta) \in \mathbf{C}^2$.

Let us clarify this definition by a motivation. In many problems of bifurcation, we are given an equation like

$$(2.3) \quad F(a, u) = 0 \quad (a \in \mathbf{R}^k, u \in X),$$

where $F(a, \cdot)$ is a mapping from a Banach space X into another Banach space Y ; X, Y are spaces of functions of θ of period 2π and k -dimensional parameter a represents physical parameters ($k \geq 2$). The orthogonal group O(2) naturally acts on periodic functions. For instance, we often encounter the following action O(2) on X and Y : if we parametrize O(2) by $\alpha \in [0, 2\pi)$ and the reflection, then

$$(2.4) \quad u(\theta) \longrightarrow u(\theta - \alpha) \quad \text{for } \alpha \in [0, 2\pi)$$

and

$$(2.5) \quad u(\theta) \longrightarrow u(-\theta) \quad \text{for the reflection}$$

define an action of O(2) on X and Y . We are concerned with equations which commute with this action of O(2), i.e., $\gamma F(a, u) = F(a, \gamma u)$ ($\gamma \in O(2)$). We assume that $F(a, 0) \equiv 0$.

In general, we have a variety V_n of codimension one in \mathbf{R}^k , on which the linearized operator $D_u F(a, 0)$ has a nontrivial kernel spanned by $\cos(n\theta)$ and $\sin(n\theta)$, where n is a positive integer. Since $k \geq 2$, V_n and V_m ($m \neq n$), generically have a non-empty intersection of V_n and V_m . Let V_{mn} be the intersection of

V_m and V_n . Then V_{mn} is of codimension two in \mathbf{R}^k and, at generic points, the kernel of $D_u F(a, 0)$ is spanned exactly by $\cos(m\theta)$, $\sin(m\theta)$, $\cos(n\theta)$ and $\sin(n\theta)$, where m and n are positive integers satisfying $0 < m < n$. Since the bifurcation diagram of (2.3) is strongly influenced by these singularities with 4-dimensional kernel, it is important to analyze the bifurcation at these points. Let N denote the 4-dimensional space spanned by $\cos(m\theta)$, $\sin(m\theta)$, $\cos(n\theta)$, $\sin(n\theta)$. In order to obtain nontrivial solutions to (2.3), we follow the Lyapunov-Schmidt procedure. Let P be a projection from X onto N . Then (2.3) is equivalent to (2.6, 7) below :

$$(2.6) \quad PF(a, x \cos(m\theta) + y \sin(m\theta) + z \cos(n\theta) + w \sin(n\theta) + \phi(a, x, y, z, w)) = 0,$$

$$(2.7) \quad (I - P)F(a, x \cos(m\theta) + y \sin(m\theta) + z \cos(n\theta) + w \sin(n\theta) + \phi(a, x, y, z, w)) = 0,$$

where x, y, z and w are real variable and ϕ is a function of (a, x, y, z, w) with its range in $(I - P)X$. The implicit function theorem enables (2.7) to define ϕ uniquely. Therefore (2.6), which determines the zero set of F , can be regarded as a mapping defined in some open set of $\mathbf{R}^k \times \mathbf{R}^4$ with its range in \mathbf{R}^4 . Introducing $\xi = x + iy$ and $\zeta = z + iw$, we identify \mathbf{R}^4 with \mathbf{C}^2 . N is identified with \mathbf{C}^2 , too. In this setting, the action (2.4, 5) of $O(2)$ is represented as

$$(\xi, \zeta) \longrightarrow (e^{im\alpha} \xi, e^{in\alpha} \zeta),$$

$$(\xi, \zeta) \longrightarrow (\bar{\xi}, \bar{\zeta}).$$

Let us fix $a_0 = (a_0^1, a_0^2, \dots, a_0^k) \in V_{mn}$. For simplicity, let us assume that a moves transversally to $V_m \cup V_n$ as we change a^1 near a_0^1 . Putting $\lambda = a^1 - a_0^1$ and fixing a_0^2, \dots, a_0^k , we can regard the right hand side of (2.6) as a mapping of (λ, ξ, ζ) . Since F commutes with the $O(2)$ -action, so does the equation (2.6) (see, e.g., Sattinger [13] or [7]). Thus we get to a mapping satisfying the properties in Definition 2.1. In what follows we fix G and analyze it by the Golubitsky-Schaeffer theory.

REMARK 2.1. In Definition 2.1 we write as if the defining domain of G is the whole space $\mathbf{R} \times \mathbf{C}^2$. The bifurcation equation G is actually defined only in some open set. Accordingly, we should work with mapping germs as in [6]. We, however, believe that no confusion is caused by writing as if G is defined in the whole space. Therefore we say mapping, although it is actually a mapping germ.

PROPOSITION 2.1. *Let G be an $O(2)$ -equivariant C^∞ -mapping with mode (m, n) . Then it must be of the following form*

$$(2.8) \quad G_1 = f_1(\lambda, u, v, r) \xi + f_2(\lambda, u, v, r) \bar{\xi}^{n'} - 1 \zeta^{m'}$$

$$(2.9) \quad G_2 = f_3(\lambda, u, v, r) \zeta + f_4(\lambda, u, v, r) \xi^{n'} \bar{\zeta}^{m'-1}$$

where m' and n' are positive integers with no common divisor such that $n'/m' = n/m$. $f_j(\lambda, u, v, r)$ are real-valued C^∞ -functions of $\lambda, u \equiv |\xi|^2, v \equiv |\zeta|^2$ and

$$r \equiv \mathbf{Re}(\xi^{n'} \bar{\zeta}^{m'}).$$

This proposition is proved in Fujii, Mimura and Nishiura [4] with a slightly different statement. Also we can refer to Dangelmayr and Armbruster [3], Golubitsky, Stewart and Schaeffer [8] or we can prove it with a method with which we will prove Proposition 2.3 below. Let \mathcal{E} be the set of all real valued C^∞ -functions $f: \mathbf{R} \times \mathbf{C}^2 \rightarrow \mathbf{R}$ which is of the following form

$$f = g(\lambda, u, v, r) \quad (g \in C^\infty(\mathbf{R}^4)).$$

Then \mathcal{E} is a commutative ring with a unit. By definition, we say that $\phi = \phi(\lambda, \xi, \zeta): \mathbf{R} \times \mathbf{C}^2 \rightarrow \mathbf{R}$ is invariant with respect to O(2) if

$$\phi(\lambda, e^{im\alpha} \xi, e^{in\alpha} \zeta) \equiv \phi(\lambda, \xi, \zeta) \quad (\alpha \in [0, 2\pi)),$$

and

$$\phi(\lambda, \bar{\xi}, \bar{\zeta}) \equiv \phi(\lambda, \xi, \zeta)$$

are satisfied. In a way similar to Proposition 2.1, we can prove that \mathcal{E} equals the ring of all invariant functions. Using \mathcal{E} , we can state Proposition 2.1 in an equivalent way:

PROPOSITION 2.1'. *The set of all the O(2)-equivariant C^∞ -maps with mode (m, n) is an \mathcal{E} -module generated by $(\xi, 0)$, $(0, \zeta)$, $(\bar{\xi}^{n'-1} \zeta^{m'}, 0)$ and $(0, \xi^{n'} \bar{\zeta}^{m'-1})$.*

Since we consider bifurcation equations, we assume that the Jacobi matrix of G vanishes at the origin. This means that

$$(2.10) \quad f_1(0; 0, 0, 0) = f_3(0; 0, 0, 0) = 0$$

Under this condition, we may generically assume that

$$(2.11) \quad f_2(0; 0, 0, 0) \neq 0, \quad f_4(0; 0, 0, 0) \neq 0.$$

If, however, we have more than two parameters, the condition (2.11) may be violated at some point in the parameter space. We encounter this situation in the Taylor problem of viscous incompressible fluid in two concentric cylinders ([10]) and in a problem of water waves of permanent configuration (Shōji [14]). Therefore we consider the following cases separately:

$$(I) \quad f_2(0; 0, 0, 0) \neq 0, \quad f_4(0; 0, 0, 0) \neq 0.$$

$$(II) \quad f_2(0; 0, 0, 0) = 0, \quad f_4(0; 0, 0, 0) \neq 0.$$

$$(III) \quad f_2(0; 0, 0, 0) \neq 0, \quad f_4(0; 0, 0, 0) = 0.$$

We now give a definition which characterizes changes of coordinates preserving O(2)-equivariance.

DEFINITION 2.2. We say that G is $O(2)$ -equivalent to H if there is a change of coordinates Θ , T and A which satisfy

$$H(\lambda, \theta) = T(\lambda, \theta)G(A(\lambda), \Theta(\lambda, \theta)) \quad (\lambda \in \mathbf{R}, \theta = (\xi, \zeta) \in \mathbf{C}^2)$$

and the following properties (2.12-14): $\Theta \in \mathbf{C}^2$ is a smooth function of (λ, ξ, ζ) satisfying

$$(2.12) \quad \Theta(\lambda, \gamma\theta) = \gamma\Theta(\lambda, \theta) \quad (\gamma \in O(2), \theta \in \mathbf{C}^2).$$

The Jacobi matrix of Θ with respect to θ has positive determinant at the origin $(\lambda, \theta) = (0, 0, 0)$. T is a 4×4 real matrix valued smooth function of (λ, θ) satisfying

$$(2.13) \quad T(\lambda, \gamma\theta) = \gamma T(\lambda, \theta) \gamma^{-1} \quad (\gamma \in O(2)).$$

T is nonsingular at the origin $(\lambda, \theta) = (0, 0, 0)$. A is a smooth function of λ satisfying

$$(2.14) \quad \frac{dA}{d\lambda}(0) > 0.$$

REMARK 2.2. This definition is borrowed from [6]. Θ and T may depend on both λ and θ . But A is a function of λ only. By the definition, H is $O(2)$ -equivariant if G is $O(2)$ -equivariant and if H is $O(2)$ -equivalent to G .

From Proposition 2.1' and Definition 2.2 the following proposition immediately follows:

PROPOSITION 2.2. If α and β are real-valued function of (λ, u, v, r) , then $H = (H_1, H_2)$ with

$$H_1(\lambda, \xi, \zeta) = G_1(\lambda, \xi + \alpha \bar{\xi}^{n'-1} \zeta^{m'}, \zeta + \beta \bar{\xi}^{n'} \zeta^{m'-1}),$$

$$H_2(\lambda, \xi, \zeta) = G_2(\lambda, \xi + \alpha \bar{\xi}^{n'-1} \zeta^{m'}, \zeta + \beta \bar{\xi}^{n'} \zeta^{m'-1}),$$

is $O(2)$ -equivalent to G .

We now prove a proposition which determines the set of T admissible in Definition 2.2. Let $M(4)$ be the set of all the \mathbf{R} -linear mappings on \mathbf{C}^2 .

PROPOSITION 2.3. The set of all the $M(4)$ -valued mappings on $\mathbf{R} \times \mathbf{C}^2$ which satisfy (2.13) but not necessarily nonsingular at the origin, is a finitely generated \mathcal{E} -module generated by the following 16-elements:

$$T_1: (H_1, H_2) \longrightarrow (H_1, 0), \quad T_2: (H_1, H_2) \longrightarrow (\xi^2 \bar{H}_1, 0),$$

$$T_3: (H_1, H_2) \longrightarrow (\xi \bar{\zeta} H_2, 0), \quad T_4: (H_1, H_2) \longrightarrow (\xi \bar{\zeta} \bar{H}_2, 0),$$

$$T_5: (H_1, H_2) \longrightarrow (0, \bar{\xi} \zeta H_1), \quad T_6: (H_1, H_2) \longrightarrow (0, \xi \zeta \bar{H}_1),$$

$$T_7: (H_1, H_2) \longrightarrow (0, \zeta^2 \overline{H_2}), \quad T_8: (H_1, H_2) \longrightarrow (0, H_2),$$

$$T_{j+8} = \sqrt{-1} \operatorname{Im} [\xi^{n'} \bar{\zeta}^{m'}] T_j \quad (1 \leq j \leq 8).$$

Proof. We represent T as follows :

$$\begin{aligned} & T(\lambda, \xi, \zeta)(H_1, H_2) \\ &= (t_{11}H_1 + t_{12}\overline{H_1} + t_{13}H_2 + t_{14}\overline{H_2}, t_{21}H_1 + t_{22}\overline{H_1} + t_{23}H_2 + t_{24}\overline{H_2}). \end{aligned}$$

By (2.13) we have

$$\begin{aligned} (2.15) \quad & T(\lambda, e^{im\alpha} \xi, e^{in\alpha} \zeta)(e^{im\alpha} H_1, e^{in\alpha} H_2) \\ &= (e^{im\alpha}(t_{11}H_1 + t_{12}\overline{H_1} + t_{13}H_2 + t_{14}\overline{H_2}), e^{in\alpha}(t_{21}H_1 + t_{22}\overline{H_1} + t_{23}H_2 + t_{24}\overline{H_2})). \end{aligned}$$

for all $\alpha \in [0, 2\pi)$ and

$$(2.16) \quad \overline{T(\lambda, \bar{\xi}, \bar{\zeta})(\overline{H_1}, \overline{H_2})} = T(\lambda, \xi, \zeta)(H_1, H_2).$$

If put $\hat{t}_{kj} = t_{kj}(\lambda, e^{im\alpha} \xi, e^{in\alpha} \zeta)$, then (2.15, 16) are equivalently written as follows :

$$\begin{aligned} (2.17) \quad & \hat{t}_{11} = t_{11}, \quad \hat{t}_{12} = e^{2im\alpha} t_{12}, \quad \hat{t}_{13} = e^{i(m-n)\alpha} t_{13}, \quad \hat{t}_{14} = e^{i(m+n)\alpha} t_{14}, \\ & \hat{t}_{21} = e^{i(-m+n)\alpha} t_{21}, \quad \hat{t}_{22} = e^{i(m+n)\alpha} t_{22}, \quad \hat{t}_{23} = t_{23}, \quad \hat{t}_{24} = e^{2in\alpha} t_{24}, \end{aligned}$$

$$(2.18) \quad t_{kj}(\lambda, \bar{\xi}, \bar{\zeta}) = \overline{t_{kj}(\lambda, \xi, \zeta)} \quad (k=1, 2 \quad j=1, 2, 3, 4).$$

Denoting the real part of $\xi^{n'} \bar{\zeta}^{m'}$ and the imaginary part by r and s , we prove that

$$\begin{aligned} (2.19) \quad & t_{11} = g_1 + isg_2, \quad t_{12} = \xi^2 g_3 + is\xi^2 g_4, \quad t_{13} = \xi \bar{\zeta} g_5 + is\xi \bar{\zeta} g_6, \\ & t_{14} = \xi \bar{\zeta} g_7 + is\xi \bar{\zeta} g_8, \quad t_{21} = \bar{\xi} \zeta g_9 + is\bar{\xi} \zeta g_{10}, \quad t_{22} = \xi \zeta g_{11} + is\xi \zeta g_{12}, \\ & t_{23} = g_{13} + isg_{14}, \quad t_{24} = \zeta^2 g_{15} + is\zeta^2 g_{16}, \end{aligned}$$

where $g_j \in \mathcal{E}$. The equalities in (2.19) prove the present proposition.

Among the equalities in (2.19) we present details of the proof of the second one. The others are proved similarly. We expand t_{12} as

$$t_{12} = \sum_{pqkl} a_{pqkl}(\lambda) \xi^p \bar{\xi}^q \zeta^k \bar{\zeta}^l$$

As in [6], we may, without loss of generality, assume that this is a finite sum. By (2.18), a_{pqkl} must be real-valued. By $\hat{t}_{12} = e^{2im\alpha} t_{12}$, we have

$$\sum_{pqkl} a_{pqkl} e^{im(p-q)\alpha + in(k-l)\alpha} \xi^p \bar{\xi}^q \zeta^k \bar{\zeta}^l = \sum_{pqkl} a_{pqkl} e^{2im\alpha} \xi^p \bar{\xi}^q \zeta^k \bar{\zeta}^l$$

which means $e^{im(p-q-2)\alpha + in(k-l)\alpha} = 1$ for all $\alpha \in [0, 2\pi)$ if $a_{pqkl} \neq 0$. Therefore $m'(p-q-2) = -n'(k-l)$ if $a_{pqkl} \neq 0$. This means that $k-l$ must be a multiple of m' .

We put $l-k=jm'$ ($j \in \mathbf{Z}$). Therefore t_{12} is represented as a finite sum of the following functions with C^∞ -functions of λ as coefficients:

$$\xi^{q+2+jn'} \bar{\xi}^q \zeta^k \bar{\zeta}^l$$

This function is equal to $u^q v^k (r+is)^j \xi^2$ if $j \geq 0$, and $u^{q+jn'} v^{k+jm'} (r-is)^{-j} \xi^2$ if $j < 0$. By the binomial theorem and $r^2+s^2=u^{n'} v^{m'}$, we can represent t_{12} by an \mathcal{E} -combination of ξ^2 and $is\xi^2$.

COROLLARY 2.4. *If smooth functions $h_1, h_2: \mathbf{R}^4 \longrightarrow \mathbf{R}$ satisfy*

$$h_j(0; 0, 0, 0) \neq 0 \quad (j=1, 2),$$

and if $\alpha_j, \beta_j, \gamma_j \in \mathcal{E}$, then $H=(H_1, H_2)$ given by

$$\begin{aligned} H_1(\lambda, \xi, \zeta) &= h_1(\lambda, u, v, r) G_1(\lambda, \xi, \zeta) + \alpha_1 \xi^2 \bar{G}_1 \\ &\quad + \beta_1 \xi (\bar{\xi} G_2 + \zeta \bar{G}_2) + \gamma_1 \xi (\bar{\xi} G_2 - \zeta \bar{G}_2), \\ H_2(\lambda, \xi, \zeta) &= h_2(\lambda, u, v, r) G_2(\lambda, \xi, \zeta) + \alpha_2 \zeta^2 \bar{G}_2 \\ &\quad + \beta_2 \zeta (\bar{\xi} G_1 + \xi \bar{G}_1) + \gamma_2 \zeta (\bar{\xi} G_1 - \xi \bar{G}_1), \end{aligned}$$

is $O(2)$ -equivalent to G .

In later sections we will transform a given G to a simpler form by some changes of coordinates which are allowed by Definition 2.2. From now on, we assume that $n/m=2$. Therefore we have $n'=2, m'=1$. What we consider is, therefore,

$$(2.20) \quad G_1 = f_1 \xi + f_2 \bar{\xi} \zeta$$

$$(2.21) \quad G_2 = f_3 \zeta + f_4 \xi^2,$$

where $f_j (j=1, 2, 3, 4)$ are in \mathcal{E} . We fix this G .

We finally state a theorem on which our analysis is based. This theorem is borrowed from [6].

NOTATION. \hat{E} is the set of all C^∞ -mappings $X: \mathbf{R} \times \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ such that $X(\lambda, \gamma(\xi, \zeta)) = \gamma X(\lambda, \xi, \zeta)$ for all $\gamma \in O(2)$.

$\tilde{I}G = \{dG(X); X \in \hat{E}\} + \{TG; T \text{ satisfies (2.13) but may be a singular matrix}\}$, where dG is the Jacobi matrix of G .

$$\mathcal{E}_1 = \{A; A(\lambda) \text{ is a } C^\infty\text{-function of } \lambda\},$$

$$\Gamma G = \tilde{I}G + \mathcal{E}_1 \frac{\partial G}{\partial \lambda}.$$

THEOREM 2.1 ([6]). *Suppose that $\tilde{I}(G+tP) = \tilde{I}G$ for all $t \in [0, 1]$, then $G+P$ is $O(2)$ -equivalent to G . If ΓG is an \mathbf{R} -linear subspace of finite codimension of*

E and $q_1, q_2, \dots, q_k \in \tilde{E}$ are such that

$$\Gamma G \oplus Rq_1 \oplus Rq_2 \cdots \oplus Rq_k = \tilde{E},$$

then

$$F(\alpha_1, \alpha_2, \dots, \alpha_k, \lambda; \xi, \zeta) = G\lambda, \xi, \zeta + \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_k q_k$$

is a universal unfolding of G .

REMARK 2.3. By a universal unfolding we mean that all small perturbation of G is O(2)-equivalent to F with an appropriate choice of the parameters α_j , see [6] for the precise definition.

Our main results are loosely stated in the following way.

A) Under a certain generic condition on coefficients of the Taylor expansion of G , given are normal forms of (2.20, 21) in the case of (I), (II) and (III) by, respectively,

$$(N.1) \quad ((\epsilon\lambda + bv)\xi + \bar{\xi}\zeta, (\delta\lambda + \bar{b}v)\zeta + \xi^2),$$

$$(N.2) \quad ((\epsilon\lambda + au + bv + cr)\xi + ev\bar{\xi}\zeta, (\delta\lambda + \hat{a}u + \bar{b}v)\zeta + \xi^2),$$

$$(N.3) \quad ((\epsilon\lambda + au + bv)\xi + \bar{\xi}\zeta, (\delta\lambda + \hat{a}u + \bar{b}v + \hat{c}r)\zeta + du\xi^2),$$

respectively, where $\epsilon, \delta, a, b, c, \hat{a}, \hat{b}, \hat{c}, d$ and e are real constants.

B) Universal unfoldings of the above mappings are, respectively,

$$(U.1) \quad ((\epsilon\lambda + \alpha + (b + s_1)v)\xi + \bar{\xi}\zeta, (\delta\lambda + \bar{b}v)\zeta + \xi^2),$$

$$(U.2) \quad ((\epsilon\lambda + \alpha + (a + s_1)u + (b + s_2)v + (c + s_3)r)\xi + ((e + s_4)v + \beta)\bar{\xi}\zeta, \\ (\delta\lambda + \hat{a}u + \bar{b}v)\zeta + \xi^2),$$

$$(U.3) \quad ((\epsilon\lambda + \alpha + au + bv)\xi + \bar{\xi}\zeta, \\ (\delta\lambda + (\hat{a} + s_1)u + (\bar{b} + s_2)v + (\hat{c} + s_3)r)\zeta + ((\hat{d} + s_4)u + \beta)\xi^2),$$

where α, β, s_j are unfolding parameters. Among these unfolding parameters, only α and β are essential parameters and s_j are modal parameters. Namely, changes of s_j do not cause any topological change of the bifurcation diagrams, while changes of α and β cause qualitative differences of the diagrams.

§ 3. Computation of ΓG : Case (I)

In this section we consider the case (I) and compute ΓG . We first simplify G by considering $(G_1/f_2, G_2/f_4)$. This is $O(2)$ -equivalent to the original G by Corollary 2.4. To make the presentation compact, we follow a method of [6] and use the following algebraic notation:

$$\mathcal{M} = \{f \in \mathcal{E}; f(0; 0, 0, 0) = 0\},$$

which is a maximal ideal of \mathcal{E} . For $g \in \mathcal{E}$, we put $\langle g \rangle = \{fg; f \in \mathcal{E}\}$, which is a principal ideal in \mathcal{E} generated by g . We also use $\langle g_1, g_2 \rangle = \{f_1 g_1 + f_2 g_2; f_1, f_2 \in \mathcal{E}\}$, etc. Now we may assume without loss of generality that

$$(3.1) \quad G_1 = (\epsilon\lambda + au + bv + g_1)\xi + \bar{\xi}\zeta$$

$$(3.2) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + g_2)\zeta + \xi^2,$$

where $\epsilon, \delta, a, b, \hat{a}, \hat{b}$ are real constants and g_1, g_2 belong to the ideal $\mathcal{M}^2 + \langle r \rangle$.

PROPOSITION 3.1. *Let G be given by (3.1, 2). Then G is $O(2)$ -equivalent to*

$$(3.1') \quad G'_1 = (\epsilon\lambda + b_1v + g'_1)\xi + \bar{\xi}\zeta$$

$$(3.2') \quad G'_2 = (\delta\lambda + \hat{b}v + g'_2)\zeta + \xi^2,$$

where $b_1 = b - \hat{a}/2$ and $g'_1, g'_2 \in \mathcal{M}^2 + \langle r \rangle$.

Proof. We change the variables: $(\xi, \zeta) \longrightarrow (\xi', \zeta')$, where $\xi = \xi' + \alpha \bar{\xi}' \zeta'$, $\zeta = \zeta' + \beta \xi'^2$ with real constants α, β (see Proposition 2.2). We put

$$H_1(\lambda, \xi', \zeta') = G_1(\lambda, \xi' + \alpha \bar{\xi}' \zeta', \zeta' + \beta \xi'^2),$$

$$H_2(\lambda, \xi', \zeta') = G_2(\lambda, \xi' + \alpha \bar{\xi}' \zeta', \zeta' + \beta \xi'^2).$$

If we drop the primes, we have

$$H_1 = (\epsilon\lambda + (a + \beta)u + (b + \alpha)v + g_3)\xi + (1 + g_4)\bar{\xi}\zeta,$$

where $g_3 \in \mathcal{M}^2 + \langle r \rangle$ and $g_4 \in \mathcal{M}$. Therefore

$$H_1/(1 + g_4) = (\epsilon\lambda + (a + \beta)u + (b + \alpha)v + g_5)\xi + \bar{\xi}\zeta,$$

where $g_5 \in \mathcal{M}^2 + \langle r \rangle$. Similarly we have

$$H_2 = (\delta\lambda + (\hat{a} + 2\alpha)u + \hat{b}v + g_6)\zeta + (1 + g_7)\xi^2,$$

where $g_6 \in \mathcal{M}^2 + \langle r \rangle, g_7 \in \mathcal{M}$. If we choose α and β so that $2\alpha + \hat{a} = 0$ and $a + \beta = 0$, then G is $O(2)$ -equivalent to

$$H_1/(1+g_4)=(\epsilon\lambda+(b-\hat{a}/2)v+g_5)\xi+\bar{\xi}\zeta,$$

$$H_2/(1+g_7)=(\delta\lambda+\hat{b}v+g_8)\zeta+\xi^2,$$

where $g_5, g_8 \in \mathcal{M}^2 + \langle r \rangle$. ■

We now put

$$(3.3) \quad G_1 = (\epsilon\lambda + bv + g_1)\xi + \bar{\xi}\zeta$$

$$(3.4) \quad G_2 = (\delta\lambda + \hat{b}v + g_2)\zeta + \xi^2,$$

where $g_1, g_2 \in \mathcal{M}^2 + \langle r \rangle$.

PROPOSITION 3.2. *If $\epsilon\delta \neq 0$, then we can assume that in (3.3, 4)*

$$(3.5) \quad \epsilon = \pm 1, \quad \delta = \pm 1,$$

and that

$$(3.6) \quad g_1(\lambda, 0, 0, 0) \equiv 0, \quad g_2(\lambda, 0, 0, 0) \equiv 0$$

Proof. Consider the following transformation:

$$(G_1, G_2) \longrightarrow \left(\gamma^{-1} G_1 \left(\frac{\lambda}{|\epsilon|}, r\xi, \zeta \right), \gamma^{-2} G_2 \left(\frac{\lambda}{|\epsilon|}, r\xi, \zeta \right) \right)$$

with $\gamma = (|\delta|/|\epsilon|)^{1/2}$. Then the absolute values of the coefficients of $\lambda\xi$ and $\lambda\zeta$ are one. In order to get a transformation which allows (3.6), we use the following maximal ideal of \mathcal{E}_1 :

$$\mathcal{M}_1 = \{A \in \mathcal{E}_1; A(0) = 0\}.$$

We use the following change of variables:

$$(\xi, \zeta) \longrightarrow (\xi, (1 + \phi(\lambda))\zeta),$$

where $\phi \in \mathcal{M}_1$. We put $g_j(\lambda, 0, 0, 0) = \eta_j(\lambda)$ ($j=1, 2$). Note that $\eta_j \in \mathcal{M}_1^2$. Then we have

$$G_1(\lambda, \xi(1 + \phi(\lambda)), \zeta(1 + \phi(\lambda))^{-1}) = (1 + \phi(\lambda))^{-1}(\epsilon\lambda + bv + \eta_1(\lambda) + g_3)\xi + \bar{\xi}\zeta,$$

$$G_2(\lambda, \xi, (1 + \phi(\lambda))\zeta) = (1 + \phi(\lambda))(\delta\lambda + \hat{b}v + \eta_2(\lambda) + g_4)\zeta + \xi^2,$$

where $g_j \in \mathcal{M}^2 + \langle r \rangle$, $g_j(\lambda, 0, 0, 0) \equiv 0$ ($j=3, 4$). Since $\eta_j \in \mathcal{M}_1^2$, we can choose ϕ so that

$$\frac{\lambda + \epsilon^{-1}\eta_1(\lambda)}{1 + \phi(\lambda)} = (\lambda + \delta^{-1}\eta_2(\lambda))(1 + \phi(\lambda)).$$

Let $A(\lambda)$ denote this quantity. Then $A(\lambda)$ is a diffeomorphism near 0 and satisfy $A'(0) > 0$. Therefore we get to $H = (H_1, H_2)$ with

$$H_1(A, \xi, \zeta) = (\epsilon A + bv + g_s)\xi + \bar{\xi}\zeta,$$

$$H_2(A, \xi, \zeta) = (\delta A + \bar{b}v + g_s)\zeta + \xi^2,$$

where $g_j(A, 0, 0, 0) \equiv 0$ ($j=5, 6$). Thus the proof is completed. ■

From now on we consider

$$(3.7) \quad G_1 = (\epsilon\lambda + bv + p)\xi + \bar{\xi}\zeta,$$

$$(3.8) \quad G_2 = (\delta\lambda + bv + q)\zeta + \xi^2,$$

under the condition (3.5) and

$$(3.9) \quad p, q \in \langle \lambda u, \lambda v, u^2, uv, v^2, r \rangle.$$

In the remaining part of this section we compute ΓG concretely. We first consider $dG(X)$. We put $X_1 = (\xi, 0)$, $X_2 = (\bar{\xi}\zeta, 0)$, $X_3 = (0, \zeta)$, $X_4 = (0, \xi^2)$. Then $\{dG(X); X \in \bar{E}\}$ is an \mathcal{E} -module generated by $dG(X_1)$, $dG(X_2)$, $dG(X_3)$ and $dG(X_4)$. As for the Jacobi matrix, we use the following notation:

$$dG(f, g) = \left(\frac{\partial G_1}{\partial \xi} f + \frac{\partial G_1}{\partial \bar{\xi}} \bar{f} + \frac{\partial G_1}{\partial \zeta} g + \frac{\partial G_1}{\partial \zeta} \bar{g}, \frac{\partial G_2}{\partial \xi} f + \frac{\partial G_2}{\partial \bar{\xi}} \bar{f} + \frac{\partial G_2}{\partial \zeta} g + \frac{\partial G_2}{\partial \zeta} \bar{g} \right).$$

We compute the coefficients of dG as follows:

$$\begin{aligned} \frac{\partial G_1}{\partial \xi} &= \epsilon\lambda + bv + p + up_u + p_r \xi \frac{\partial r}{\partial \xi}, & \frac{\partial G_1}{\partial \bar{\xi}} &= \zeta + p_u \xi^2 + p_r \bar{\xi} \frac{\partial r}{\partial \bar{\xi}}, \\ \frac{\partial G_1}{\partial \zeta} &= \bar{\xi} + \left(b\bar{\xi} + p_v \bar{\xi} + p_r \frac{\partial r}{\partial \zeta} \right) \xi, & \frac{\partial G_1}{\partial \zeta} &= \left(b\zeta + p_v \zeta + p_r \frac{\partial r}{\partial \zeta} \right) \bar{\xi}, \\ \frac{\partial G_2}{\partial \xi} &= 2\xi + \left(qu\bar{\xi} + q_r \frac{\partial r}{\partial \xi} \right) \zeta, & \frac{\partial G_2}{\partial \bar{\xi}} &= \left(qu\xi + q_r \frac{\partial r}{\partial \bar{\xi}} \right) \zeta, \\ \frac{\partial G_2}{\partial \zeta} &= \delta\lambda + 2\bar{b}v + q + vq_v + q_r \zeta \frac{\partial r}{\partial \zeta}, & \frac{\partial G_2}{\partial \zeta} &= \left(\bar{b}\zeta + q_v \zeta + q_r \frac{\partial r}{\partial \zeta} \right) \xi, \end{aligned}$$

where subscripts mean differentiations. We also have the following formulas:

$$(3.10) \quad \begin{aligned} \xi \frac{\partial r}{\partial \xi} + \bar{\xi} \frac{\partial r}{\partial \bar{\xi}} &= 2r, & \bar{\xi}\zeta \frac{\partial r}{\partial \xi} + \xi\zeta^2 \frac{\partial r}{\partial \bar{\xi}} &= 2uv, & \zeta \frac{\partial r}{\partial \zeta} + \xi \frac{\partial r}{\partial \zeta} &= r, \\ \xi^2 \frac{\partial r}{\partial \zeta} + \bar{\xi}^2 \frac{\partial r}{\partial \bar{\zeta}} &= u^2. \end{aligned}$$

Using these formulas, we compute $dG(X_j)$ ($j=1, 2, 3, 4$) to have:

$$\begin{aligned} dG(X_1) &= (\epsilon\lambda + bv + p + 2up_u + 2rp_r)X_1 + X_2 + (2uq_u + 2rq_r)X_3 + 2X_4, \\ dG(X_2) &= (v + 2rp_u + 2uvp_r)X_1 + (\epsilon\lambda + bv + p)X_2 + (2u + 2rq_u + 2uvq_r)X_3, \end{aligned}$$

$$dG(X_3) = (2bv + 2vp_v + rp_r)X_1 + X_2 + (\delta\lambda + 3\hat{b}v + q + 2vq_v + rq_r)X_3,$$

$$dG(X_4) = (u + 2br + 2rp_v + u^2p_r)X_1 + (2\hat{b}r + 2rq_v + u^2q_r)X_3 + (\delta\lambda + \hat{b}v + q)X_4.$$

We now consider the set $\{TG; T \text{ satisfies (2.13) but may be singular}\}$. This set is an \mathcal{E} -module generated by the following 16 elements:

$$T_1G = (\epsilon\lambda + bv + p)X_1 + X_2,$$

$$T_2G = [u(\epsilon\lambda + bv + p) + 2r]X_1 - uX_2,$$

$$T_3G + T_4G = [2v(\delta\lambda + \hat{b}v + q) + 2r]X_1,$$

$$T_3G - T_4G = 2rX_1 - 2uX_2,$$

$$T_5G + T_6G = [2u(\epsilon\lambda + bv + p) + 2r]X_3,$$

$$T_5G - T_6G = 2rX_3 - 2vX_4,$$

$$T_7G = [v(\delta\lambda + \hat{b}v + q) + 2r]X_3 - vX_4,$$

$$T_8G = (\delta\lambda + \hat{b}v + q)X_3 + X_4,$$

and remaining 8 elements made by multiplying these 8 elements by $\sqrt{-1}$ s. Note that $T_2G = uT_1G + T_3G - T_4G$. Therefore we can omit T_2G from the generators. Similarly we can omit T_7G . Since we have

$$(3.11) \quad \begin{aligned} isX_1 &= rX_1 - uX_2, & isX_2 &= uvX_1 - rX_2, & isX_3 &= -rX_3 + vX_4, \\ & & isX_4 &= -u^2X_3 + rX_4, \end{aligned}$$

the \mathcal{E} -module in question is generated by the following 8 elements:

$$(\epsilon\lambda + bv + p)X_2 + X_2, \quad [v(\delta\lambda + \hat{b}v + q) + r]X_1, \quad rX_1 - uX_2,$$

$$[u(\epsilon\lambda + bv + p) + r]X_3, \quad rX_3 - vX_4, \quad (\delta\lambda + \hat{b}v + q)X_3 + X_4,$$

$$uvX_1 - rX_2, \quad u^2X_3 - rX_4.$$

Using T_1G and T_8G to eliminate first order terms involving λ in X_1 and X_3 , we have now proved

PROPOSITION 3.3. *\mathcal{E} -module $\tilde{\Gamma}G$ is generated by the following 12 elements:*

$$W_1 = (2up_u + 2rp_r)X_1 + (2uq_u + 2rq_r)X_3 + 2X_4,$$

$$W_2 = (v + 2rp_u + 2uvp_r)X_1 + (\epsilon\lambda + bv + p)X_2 + (2u + 2rq_r + 2uvq_r)X_3,$$

$$W_3 = (2bv + 2vp_v + rp_r)X_1 + X_2 + (2\hat{b}v + 2vq_v + rq_r)X_3 - X_4,$$

$$W_4 = (u + 2br + 2rp_v + u^2p_r)X_1 + (2\hat{b}r + 2rq_v + u^2q_r)X_3 + (\delta\lambda + \hat{b}v + q)X_4,$$

$$\begin{aligned}
W_5 &= (\epsilon\lambda + bv + p)X_1 + X_2, \\
W_6 &= [v(\hat{b} - \epsilon\delta\hat{b})v + q - \epsilon\delta p + r]X_1 - \epsilon\delta vX_2, \\
W_7 &= rX_1 - uX_2, \\
W_8 &= [u((b - \epsilon\delta\hat{b})v + p - \epsilon\delta q) + r]X_3 - \epsilon\delta uX_4, \\
W_9 &= rX_3 - vX_4, \\
W_{10} &= (\delta\lambda + \hat{b}v + q)X_3 + X_4, \\
W_{11} &= uvX_1 - rX_2, \\
W_{12} &= u^2X_3 - rX_4.
\end{aligned}$$

Before going further, we remark that, in what follows, the conditions of propositions are supposed to hold in later propositions. Namely, the conditions in Proposition 3.K is assumed in Proposition 3.J if $K < J$. This convention is used in each section, although it is effective only in the same section (e.g., the assumptions in §3 are not assumed in §§4 and 5). We now choose 8 elements Y_1, \dots, Y_8 from $\tilde{I}G$ which contain only X_1 and X_3 :

$$\begin{aligned}
Y_1 &\equiv \frac{1}{2} \epsilon\delta u W_1 + W_8 \\
&= \epsilon\delta(u^2 p_u + ru p_r)X_1 + [\epsilon\delta(u^2 q_u + ru q_r) + r + uv(b - \epsilon\delta\hat{b}) + u(p - \epsilon\delta q)]X_3, \\
Y_2 &\equiv \frac{1}{2} v W_1 + W_9 = (uv p_u + rv p_r)X_1 + [uv q_u + rv q_r + r]X_3, \\
Y_3 &\equiv u W_3 + W_7 - \epsilon\delta W_8 \\
&= [2buv + 2uv p_v + ru p_r + r]X_1 \\
&\quad + [2\hat{b}uv + 2uv q_v + ru q_r - \epsilon\delta r - (\epsilon\delta\hat{b} - \hat{b})uv - u(\epsilon\delta p - q)]X_3, \\
Y_4 &\equiv v W_3 + \epsilon\delta W_6 - W_9 \\
&= [2bv^2 + 2v^2 p_v + rv p_r + \epsilon\delta r + (\epsilon\delta\hat{b} - b)v^2 + (\epsilon\delta q - p)v]X_1 \\
&\quad + [2\hat{b}v^2 + 2v^2 q_v + rv q_r - r]X_3, \\
Y_5 &\equiv \frac{1}{2} r W_1 + W_{12} = [ru p_u + r^2 p_r]X_1 + [ru q_u + r^2 q_r + u^2]X_3, \\
Y_6 &\equiv r W_3 + W_{11} - W_{12} \\
&= [2brv + 2rv p_v + r^2 p_r + uv]X_1 + [2\hat{b}rv + 2rv q_v + r^2 q_r - u^2]X_3, \\
Y_7 &\equiv v W_2 + \epsilon\delta(\epsilon\lambda + bv + p)W_6 - (\epsilon\delta r + (\epsilon\delta\hat{b} - b)v^2 + v(\epsilon\delta q - p))W_5
\end{aligned}$$

$$\begin{aligned}
& -\epsilon\delta W_{11}-(v(\hat{b}-\epsilon\delta b)+q-\epsilon\delta p)W_6 \\
& =\{v^2+2rvp_u+2uv^2p_r-\epsilon\delta uv \\
& \quad -[v(\hat{b}-\epsilon\delta b)+q-\epsilon\delta p][r+(\hat{b}-\epsilon\delta b)v^2+(q-\epsilon\delta p)v]\}X_1 \\
& \quad +[2uv+2rvq_u+2uv^2q_r]X_3, \\
Y_8 & \equiv uW_4+\epsilon\delta(\delta\lambda+\hat{b}v+q)W_8-(\epsilon\delta r+(\epsilon\delta b-\hat{b})uv+u(\epsilon\delta p-q))W_{10} \\
& \quad -\epsilon\delta W_{12}-(v(b-\epsilon\delta\hat{b})+p-\epsilon\delta q)W_8 \\
& = (u^2+2bru+2rup_v+u^3p_r)X_1+\{2bru+2ruq_v+2u^3q_r-\epsilon\delta u^2 \\
& \quad -[v(b-\epsilon\delta\hat{b})+p-\epsilon\delta q][r+(b-\epsilon\delta\hat{b})uv+u(p-\epsilon\delta q)]\}X_3.
\end{aligned}$$

We now prove

PROPOSITION 3.4. *If $\hat{b}\neq 0$, $\hat{b}\neq\epsilon\delta b$, then*

$$u^2X_j, \quad uvX_j, \quad v^2X_j, \quad rX_j \quad (j=1, 3)$$

belong to $\tilde{F}G$.

Proof. We put

$$V_j=u^{3-j}v^{j-1}X_1 \quad (j=1, 2, 3), \quad V_{j+3}=u^{3-j}v^{j-1}X_3 \quad (j=1, 2, 3),$$

$V_7=rX_1$ and $V_8=rX_3$. By the assumption on p and q and the fact that $Y_1\in\tilde{F}G$, we have

$$V_8+(b-\epsilon\delta\hat{b})V_6\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle.$$

Similarly by Y_2, \dots, Y_8 , we have

$$\begin{aligned}
& V_8\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle, \\
& 2bV_2+V_7+(3\hat{b}-\epsilon\delta b)V_5-\epsilon\delta V_8\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle, \\
& (b+\epsilon\delta\hat{b})V_3+\epsilon\delta V_7+2\hat{b}V_6-V_8\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle, \\
& V_4\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle, \\
& V_2-V_4\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle, \\
& V_3-\epsilon\delta V_2+2V_5\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle, \\
& V_1-\epsilon\delta V_4\in\tilde{F}G+\mathcal{M}\langle V_1, V_2, \dots, V_8\rangle,
\end{aligned}$$

respectively. These relations are rewritten as follows:

$$AV\in(\tilde{F}G)^3.$$

where $V=(V_1, V_2, \dots, V_8)$, and A is a 8×8 matrix whose entries are in \mathcal{E} . The matrix A reduces to an R -matrix at $(\lambda, u, v, r)=(0, 0, 0, 0)$ which we denote by A_0 . A_0 is given by

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & b - \epsilon \delta \bar{b} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2b & 0 & 0 & 3\bar{b} - \epsilon \delta b & 0 & 1 & -\epsilon \delta \\ 0 & 0 & b + \epsilon \delta \bar{b} & 0 & 0 & 2\bar{b} & \epsilon \delta & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\epsilon \delta & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\epsilon \delta & 0 & 0 & 0 & 0 \end{pmatrix}$$

In order to prove the present proposition, it is sufficient to show that $\det A_0 \neq 0$ under the condition stated in the proposition. This is indeed the case, since we have $\det A_0 = 2\bar{b}(b - \epsilon \delta \bar{b})$ by a direct computation. ■

PROPOSITION 3.5. *Under the same assumption as in the previous proposition, the following elements belong to $\tilde{F}G$:*

$$\mathcal{M}^2 X_1, \mathcal{M}^2 X_3, \mathcal{M} X_2, \mathcal{M} X_4, rX_1, rX_3.$$

Proof. Since $\tilde{F}G$ is an \mathcal{E} -module, we have only to show that

$$\lambda^2 X_j, \lambda u X_j, \lambda v X_j, (j=1, 3), \lambda X_k, u X_k, v X_k, r X_k, (k=2, 4)$$

belong to $\tilde{F}G$. Considering uW_1, vW_1 and rW_1 , we see that uX_4, vX_4 and rW_4 belong to $\tilde{F}G$. Then uX_2, vX_2 and rX_2 belong to $\tilde{F}G$ by uW_3, vW_3 and rW_3 . By uW_5 and vW_5 , we see that $\lambda u X_1$ and $\lambda v X_1$ belong to $\tilde{F}G$. Similarly, $\lambda u X_3, \lambda v X_3 \in \tilde{F}G$ by W_{10} . By λW_1 and λW_3 , the elements λX_2 and λX_4 belong to $\tilde{F}G$. Finally λW_5 and λW_{10} prove that $\lambda^2 X_1, \lambda^2 X_3$ belong to $\tilde{F}G$. ■

This proposition simplifies the characterization of $\tilde{F}G$: it is equal to the \mathcal{E} -module generated by the submodules given in Proposition 3.5 and the following 6 elements.

$$Z_1 = X_4, \quad Z_2 = vX_1 + 2uX_3, \quad Z_3 = 2bvX_1 + X_2 + 2\bar{b}X_3 - X_4,$$

$$Z_4 = uX_1, \quad Z_5 = (\epsilon\lambda + bv)X_1 + X_2, \quad Z_6 = (\delta\lambda + \bar{b}v)X_3 + X_4.$$

Therefore $\tilde{F}G$ is independent of p and q . This fact and Theorem 2.1 immediately prove the following

THEOREM 3.1. *If $\epsilon\delta \neq 0$, $\bar{b} \neq 0$, $\bar{b} \neq \epsilon\delta b$, then the mapping G given by (3.7, 8) is $O(2)$ -equivalent to*

$$(3.12) \quad (\epsilon\lambda + bv)\xi + \bar{\xi}\zeta,$$

$$(3.13) \quad (\delta\lambda + \hat{b}v)\zeta + \xi^2,$$

where $\epsilon = \pm 1$, $\delta = \pm 1$, $\hat{b} = \pm 1$.

Proof. By Theorem 2.1, the mapping (3.7, 8) is O(2)-equivalent to (3.12, 13). We consider the following O(2)-equivariant transformation :

$$H_1 = \beta^{-1}\gamma^{-1}G_1(\alpha\lambda, \beta\xi, \gamma\zeta), \quad H_2 = \beta^{-2}G_2(\alpha\lambda, \beta\xi, \gamma\zeta)$$

where α, β, γ are positive constants. By choosing $\alpha = \beta = \gamma = 1/|\hat{b}|$, the absolute values of the coefficients of $\lambda\xi$ in H_1 and $\lambda\zeta, v\zeta$ in H_2 are equal to one. This completes the proof. ■

Summing up Proposition 3.1, 2 and Theorem 3.1, we have

THEOREM 3.2. *If a mapping G given by (3.1, 2) satisfies $\epsilon\delta \neq 0$, $\hat{b} \neq 0$, $\epsilon\hat{b} \neq \delta(b - \hat{a}/2)$, then it is O(2)-equivalent to*

$$G_1 = \left(\frac{\epsilon}{|\epsilon|} \lambda + \frac{(b - \hat{a}/2)}{|\hat{b}|} v \right) \xi + \bar{\xi}\zeta,$$

$$G_2 = \left(\frac{\delta}{|\delta|} \lambda + \frac{\hat{b}}{|\hat{b}|} v \right) \zeta + \xi^2,$$

We finally have the following

THEOREM 3.3. *O(2)-codimension of the mapping G given by (3.12, 13) is two. The following F is a universal unfolding of G :*

$$(3.14) \quad F(\alpha, s_1, \lambda; \xi, \zeta) = ((\epsilon\lambda + \alpha + (b + s_1)v)\xi + \bar{\xi}\zeta, (\delta\lambda + \hat{b}v)\zeta + \xi^2).$$

Proof. Let G be given by (3.12, 13). Since $\mathcal{M}^2 X_1, \mathcal{M}^2 X_3$ are submodules of $\tilde{I}G$, we have

$$\Gamma G = \tilde{I}G + \mathbf{R} \frac{\partial G}{\partial \lambda} + \mathbf{R} \lambda \frac{\partial G}{\partial \lambda} = \tilde{I}G + \mathbf{R}(\epsilon X_1 + \delta X_3) + \mathbf{R}(\epsilon \lambda X_1 + \delta \lambda X_3).$$

By Z_1, \dots, Z_6 , the following elements belong to $\tilde{I}G$:

$$(3.15) \quad X_4, \quad uX_1, \quad vX_1 + 2uX_3, \quad 2bvX_1 + X_2 + 2\hat{b}vX_3, \quad (\epsilon\lambda + bv)X_1 + X_2, \quad (\delta\lambda + \hat{b}v)X_3.$$

The last two and the fact that $\epsilon\lambda X_1 + \delta\lambda X_3 \in \Gamma G$ imply that $bvX_1 + X_2 + \hat{b}vX_3 \in \Gamma G$. Since this element and the fourth element in (3.15) belong to ΓG , we see that X_2 and $bvX_1 + \hat{b}vX_3$ belong to ΓG . Then it is easy to verify that

$$\Gamma G \oplus \mathbf{R}X_1 \oplus \mathbf{R}vX_1 = \tilde{E}.$$

Theorem 2.1 completes the proof. ■

§ 4. Computation of ΓG : Case (II)

In this section we consider the case (II) and compute ΓG . We first transform G to $(G_1, G_2/f_4)$. Note that we can write it as

$$(4.1) \quad G_1 = (\epsilon\lambda + p_1 + q_1)\xi + (\eta\lambda + p_2 + q_2)\bar{\xi}\zeta$$

$$(4.2) \quad G_2 = (\delta\lambda + p_3 + q_3)\zeta + \xi^2,$$

where ϵ, δ, η are real constants, p_1, p_2, p_3 belong to \mathcal{M}_2^2 and $q_1, q_2, q_3 \in \langle u, v, r \rangle$.

PROPOSITION 4.1. *Assume that $\epsilon\delta \neq 0$. Then in (4.1, 2) we may assume, without loss of generality, that $\eta=0, \epsilon=\pm 1, \delta=\pm 1$.*

Proof. We use the following change of variables:

$$(\xi, \zeta) \longrightarrow (\xi + \alpha\bar{\xi}\zeta, \zeta),$$

where α is a real parameter. Then G_1 and G_2 are transformed to

$$H_1 = (\epsilon\lambda + p_1 + q_1)(\xi + \alpha\bar{\xi}\zeta) + (\eta\lambda + p_2 + q_2)(\bar{\xi} + \alpha\xi\bar{\zeta})\zeta,$$

$$H_2 = (\delta\lambda + p_3 + q_3)\zeta + (\xi + \alpha\bar{\xi}\zeta)^2,$$

respectively. Here and hereafter $q_j \in \langle u, v, r \rangle$. We can write these quantities as follows:

$$H_1 = (\epsilon\lambda + p_1 + q_1)\xi + (\eta\lambda + p_2 + \epsilon\alpha\lambda + \alpha p_1 + q_2)\bar{\xi}\zeta,$$

$$H_2 = (\delta\lambda + p_3 + q_3)\zeta + (1 - \alpha^2 v)\xi^2.$$

Therefore

$$H'_2 \equiv H_2 / (1 - \alpha^2 v) = (\delta\lambda + p_3 + q_3)\zeta + \xi^2.$$

We now choose α so that $\eta + \epsilon\alpha = 0$. Then H_1 is of the form (4.1) with $\eta=0$. By

$$(H_1, H'_2) \longrightarrow \left(\frac{|\delta|}{|\epsilon|} H_1 \left(\frac{\lambda}{|\delta|}, \xi, \zeta \right), H'_2 \left(\frac{\lambda}{|\delta|}, \xi, \zeta \right) \right)$$

we can assume that $\epsilon=\pm 1, \delta=\pm 1$. ■

Henceforth we consider

$$(4.3) \quad G_1 = (\epsilon\lambda + au + bv + cr + f)\xi + (du + ev + kr + g)\bar{\xi}\zeta$$

$$(4.4) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + \hat{h})\zeta + \xi^2,$$

where f , g and h belong to \mathcal{M}^2 , and $\epsilon = \pm 1$, $\delta = \pm 1$.

PROPOSITION 4.2. *If $\epsilon\delta \neq 0$, and $e \neq 0$, then we can assume in (4.3, 4) that $\hat{d} = \hat{c} = k = 0$.*

Proof. Consider the following transformation :

$$\begin{aligned} H_1(\lambda, \xi, \zeta) &= G_1(\lambda, \xi, \zeta) + \gamma_1 \xi (\bar{\zeta} G_2 - \zeta \overline{G_2}), \\ H_2(\lambda, \xi, \zeta) &= G_2(\lambda, \xi, \zeta) + \alpha_2 \zeta^2 \overline{G_2}, \end{aligned}$$

where α_2 and γ_1 are real parameters. Then we have

$$\begin{aligned} (4.5) \quad H_1 &= (\epsilon\lambda + au + bv + (c + 2\gamma_1)r + g_1)\xi + ((d - 2\gamma_1)u + ev + kr + g_2)\bar{\xi}\zeta, \\ H_2 &= (\delta\lambda + \hat{a}u + \hat{b}v + (\hat{c} + 2\alpha_2)r + g_3)\zeta + (1 - \alpha_2 v)\bar{\xi}^2, \end{aligned}$$

where g_1 , g_2 and $g_3 \in \mathcal{M}^2$. Hereafter g_j is an element of \mathcal{M}^2 . H_2 is transformed further to

$$(4.6) \quad H_2/(1 - \alpha_2 v) = (\delta\lambda + \hat{a}u + \hat{b}v + (\hat{c} + 2\alpha_2)r + g_4)\zeta + \xi^2.$$

We transform (4.5, 6) by the following change of variables: $(\xi, \zeta) \longrightarrow (\xi, \zeta + \alpha_0 \xi^2)$, where α_0 is a real parameter. Note that v is replaced by $v + 2\alpha_0 r + \alpha_0^2 u^2$. Now (4.5, 6) are transformed, respectively, to

$$H'_1 = (\epsilon\lambda + au + bv + (c + 2\gamma_1 + 2\alpha_0 b)r + g_5)\xi + ((d - 2\gamma_1)u + ev + (k + 2\alpha_0 e)r + g_6)\bar{\xi}\zeta$$

and

$$H'_2 = (\delta\lambda + \hat{a}u + \hat{b}v + (\hat{c} + 2\alpha_2 + 2\alpha_0 \hat{b})r + g_7)\zeta + (1 + q_1)\xi^2$$

where $q_1 \in \mathcal{M}$. Thereby we have

$$H'_2/(1 + q_1) = (\delta\lambda + \hat{a}u + \hat{b}v + (\hat{c} + 2\alpha_2 + 2\alpha_0 \hat{b})r + g_8)\zeta + \xi^2$$

Choosing γ_1 , α_0 and α_2 so that

$$k + 2\alpha_0 e = 0, \quad d - 2\gamma_1 = 0, \quad \hat{c} + 2\alpha_2 + 2\alpha_0 \hat{b} = 0,$$

the proof is completed. ■

Hereafter we consider

$$(4.7) \quad G_1 = (\epsilon\lambda + au + bv + cr + f)\xi + (ev + g)\bar{\xi}\zeta$$

$$(4.8) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + h)\zeta + \xi^2,$$

where f , g and h belong to \mathcal{M}^2 .

PROPOSITION 4.3. *Suppose that $\epsilon\delta a b e \hat{b} \neq 0$, $\hat{b} - \epsilon\delta b \neq 0$, and*

$$6e\hat{b} + b^2(a - \epsilon\delta\hat{a}) + 2a\hat{b}(\hat{b} - \epsilon\delta b) + \hat{b}(a\hat{b} - \hat{a}b) \neq 0.$$

Then we can assume in (4.7, 8) the following three conditions:

- i) $f \in \mathcal{M}^3 + r\mathcal{M}$. In other words, the coefficients of λ^2 , λu , λv , u^2 , uv and v^2 of f vanish,
- ii) the coefficients of λ^2 , λu , λv , uv and v^2 of h vanish,
- iii) the coefficients of λ^2 , λv , and v^2 of g vanish.

Proof. We transform G_1 and G_2 to

$$H_1(\lambda, \xi, \zeta) = (1 + \alpha_0\lambda + \alpha_1u + \alpha_2v)G_1(\lambda, \xi', \zeta'),$$

$$H_2(\lambda, \xi, \zeta) = (1 + \beta_0\lambda + \beta_1u + \beta_2v)^{-2}G_2(\lambda, \xi', \zeta'),$$

respectively, where

$$\xi' = (1 + \beta_0\lambda + \beta_1u + \beta_2v)\xi, \quad \zeta' = (1 + \gamma_0\lambda + \gamma_1u + \gamma_2v)\zeta,$$

with $\alpha_j, \beta_j, \gamma_j$ real constants. Let f_0, g_0 and h_0 denote, respectively, the r -independent quadratic parts of f, g and h . Therefore we can write $f = f_0 + f_1$, $g = g_0 + g_1$ and $h = h_0 + h_1$, where $f_1, g_1, h_1 \in \mathcal{M}^3 + r\mathcal{M}$. From now on until the end of the proof of the proposition, f_j, g_j, h_j ($1 \leq j$) denote elements of $\mathcal{M}^3 + r\mathcal{M}$. We have

$$\begin{aligned} H_1 = & [\epsilon\lambda + au + bv + cr + f_0 + \epsilon(\alpha_0 + \beta_0)\lambda^2 + (3a\beta_0 + \epsilon\beta_1 + \epsilon\alpha_1 + a\alpha_0)\lambda u \\ & + (2b\gamma_0 + b\beta_0 + \epsilon\beta_2 + \epsilon\alpha_2 + b\alpha_0)\lambda v + (3a\beta_1 + a\alpha_1)u^2 \\ & + (3a\beta_2 + 2b\gamma_1 + b\beta_1 + b\alpha_1 + a\alpha_2)uv + (2b\gamma_2 + b\beta_2 + b\alpha_2)v^2 + f_2]\xi \\ & + [ev + g_0 + e(\alpha_0 + \beta_0 + 3\gamma_0)\lambda v + e(\alpha_1 + \beta_1 + 3\gamma_1)uv + e(\alpha_2 + \beta_2 + 3\gamma_2)v^2 + g_2]\bar{\xi}\zeta, \end{aligned}$$

and

$$\begin{aligned} H_2 = & [\delta\lambda + \hat{a}u + \hat{b}v + h_0 + \delta(\gamma_0 - 2\beta_0)\lambda^2 + (\hat{a}\gamma_0 + \delta\gamma_1 - 2\delta\beta_1)\lambda u \\ & + (3\hat{b}\gamma_0 + \delta\gamma_2 - 2\hat{b}\beta_0 - 2\delta\beta_2)\lambda v + \hat{a}\gamma_1u^2 + (3\hat{b}\gamma_1 + \hat{a}\gamma_2 - 2\hat{b}\beta_1)uv \\ & + (3\hat{b}\gamma_2 - 2\hat{b}\beta_2)v^2 + h_2]\zeta + \xi^2. \end{aligned}$$

We change λ to $\lambda + \gamma_1\lambda^2$ with $\gamma_1 \in \mathbf{R}$. We then transform H_1 and H_2 as follows:

$$K_1 = H_1 + \eta_2\xi(\bar{\xi}H_2 + \zeta\bar{H}_2),$$

$$K_2 = H_2 + \eta_3\zeta(\bar{\xi}H_1 + \xi\bar{H}_1),$$

where $\eta_2, \eta_3 \in \mathbf{R}$. We have

$$\begin{aligned} K_1 = & [\epsilon\lambda + au + bv + (c + 2\eta_2)r + f_0 + \epsilon(\alpha_0 + \beta_0 + \gamma_1)\lambda^2 \\ & + (3a\beta_0 + \epsilon\beta_1 + \epsilon\alpha_1 + a\alpha_0)\lambda u + (2b\gamma_0 + b\beta_0 + \epsilon\beta_2 + \epsilon\alpha_2 + b\alpha_0 + 2\delta\eta_2)\lambda v + (3a\beta_1 + a\alpha_1)u^2 \\ & + (3a\beta_2 + 2b\gamma_1 + b\beta_1 + b\alpha_1 + a\alpha_2 + 2\hat{a}\eta_2)uv + (2b\gamma_2 + b\beta_2 + b\alpha_2 + 2\hat{b}\eta_2)v^2 + f_2]\xi \end{aligned}$$

$$\begin{aligned}
& +[ev + g_0 + e(\alpha_0 + \beta_0 + 3\gamma_0)\lambda v + e(\alpha_1 + \beta_1 + 3\gamma_1)v + e(\alpha_2 + \beta_2 + 3\gamma_2)v^2 + g_3]\bar{\xi}\zeta, \\
K_2 = & [\delta\lambda + \hat{a}u + \hat{b}v + h_0 + \delta(\gamma_0 - 2\beta_0 + \eta_1)\lambda^2 + (\hat{a}\gamma_0 + \delta\gamma_1 - 2\delta\beta_1 + 2\epsilon\eta_3)\lambda u \\
& + (3\hat{b}\gamma_0 + \delta\gamma_2 - 2\hat{b}\beta_0 - 2\delta\beta_2)\lambda v + (\hat{a}\gamma_1 + 2a\eta_3)u^2 + (3\hat{b}\gamma_1 + \hat{a}\gamma_2 - 2\hat{b}\beta_1 + 2b\eta_3)uv \\
& + (3\hat{b}\gamma_2 - 2\hat{b}\beta_2)v^2 + h_3]\zeta + \xi^2.
\end{aligned}$$

Finally we change ξ to $\xi + \eta_4\lambda\bar{\xi}\zeta + \eta_5v\bar{\xi}\zeta$. Then K_1 and K_2 are transformed to the following K'_1 and K'_2 , respectively:

$$\begin{aligned}
K'_1 = & [\epsilon\lambda + au + bv + (c + 2\eta_2)r + f_0 + \epsilon(\alpha_0 + \beta_0 + \eta_1)\lambda^2 + (3a\beta_0 + \epsilon\beta_1 + \epsilon\alpha_1 + a\alpha_0)\lambda u \\
& + (2b\gamma_0 + b\beta_0 + \epsilon\beta_2 + \epsilon\alpha_2 + b\alpha_0 + 2\delta\eta_2)\lambda v + (3a\beta_1 + a\alpha_1)u^2 \\
& + (3a\beta_2 + 2b\gamma_1 + b\beta_1 + b\alpha_1 + a\alpha_2 + 2\hat{a}\eta_2)uv + (2b\gamma_2 + \hat{b}\beta_2 + b\alpha_2 + 2\hat{b}\eta_2)v^2 + f_4]\bar{\xi} \\
& + [ev + g_0 + \epsilon\eta_4\lambda^2 + a\eta_4\lambda u + (e(\alpha_0 + \beta_0 + 3\gamma_0) + b\eta_4 + \epsilon\eta_5)\lambda v \\
& + (e(\alpha_1 + \beta_1 + 3\gamma_1) + a\eta_5)uv + (e(\alpha_2 + \beta_2 + 3\gamma_2) + b\eta_5)v^2 + g_4]\bar{\xi}\zeta, \\
K'_2 = & [\delta\lambda + \hat{a}u + \hat{b}v + h_0 + \delta(\gamma_0 - 2\beta_0 + \eta_1)\lambda^2 + (\hat{a}\gamma_0 + \delta\gamma_1 - 2\delta\beta_1 + 2\epsilon\eta_3 + 2\eta_4)\lambda u \\
& + (3\hat{b}\gamma_0 + \delta\gamma_2 - 2\hat{b}\beta_0 - 2\delta\beta_2)\lambda v + (\hat{a}\gamma_1 + 2a\eta_3)u^2 + (3\hat{b}\gamma_1 + \hat{a}\gamma_2 - 2\hat{b}\beta_1 + 2b\eta_3 + 2\eta_5)uv \\
& + (3\hat{b}\gamma_2 - 2\hat{b}\beta_2)v^2 + h_4]\zeta + (1 - v(\eta_4\lambda + \eta_5v))^2\xi^2.
\end{aligned}$$

Dividing K'_2 by $1 - v(\eta_4\lambda + \eta_5v)^2$, we have

$$\begin{aligned}
K'_2/(1 - v(\eta_4\lambda + \eta_5v)^2) = & [\delta\lambda + \hat{a}u + \hat{b}v + h_0 + \delta(\gamma_0 - 2\beta_0 + \eta_1)\lambda^2 \\
& + (\hat{a}\gamma_0 + \delta\gamma_1 - 2\delta\beta_1 + 2\epsilon\eta_3 + 2\eta_4)\lambda u + (3\hat{b}\gamma_0 + \delta\gamma_2 - 2\hat{b}\beta_0 - 2\delta\beta_2)\lambda v + (\hat{a}\gamma_1 + 2a\eta_3)u^2 \\
& + (3\hat{b}\gamma_1 + \hat{a}\gamma_2 - 2\hat{b}\beta_1 + 2b\eta_3 + 2\eta_5)uv + (3\hat{b}\gamma_2 - 2\hat{b}\beta_2)v^2 + h_5]\zeta + \xi^2.
\end{aligned}$$

We now wish to choose α_j etc. so that the terms listed in i), ii) and iii) vanish. This is possible, if the following 14×14 matrix is nonsingular:

$$\begin{pmatrix}
\epsilon & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 \\
a & \epsilon & 0 & 3a & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & \epsilon & b & 0 & \epsilon & 2b & 0 & 0 & 0 & 2\delta & 0 & 0 & 0 \\
0 & a & 0 & 0 & 3a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & a & 0 & b & 3a & 0 & 2b & 0 & 0 & 2\hat{a} & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & b & 0 & 0 & 2b & 0 & 2\hat{b} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 \\
e & 0 & 0 & e & 0 & 0 & 3e & 0 & 0 & 0 & 0 & 0 & b & \epsilon
\end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & e & 0 & 0 & e & 0 & 0 & 3e & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & -2\delta & 0 & 0 & \delta & 0 & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\delta & 0 & \hat{a} & \delta & 0 & 0 & 0 & 2\epsilon & 2 & 0 \\ 0 & 0 & 0 & -2\hat{b} & 0 & -2\delta & 3\hat{b} & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\hat{b} & 0 & 0 & 3\hat{b} & \hat{a} & 0 & 0 & 2\hat{b} & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -2\hat{b} & 0 & 0 & 3\hat{b} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix is indeed nonsingular under the assumption of the present proposition, which completes the proof. ■

From now on we consider

$$(4.9) \quad G_1 = (\epsilon\lambda + au + bv + cr + f)\xi + (ev + g)\bar{\xi}\zeta,$$

$$(4.10) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + h)\zeta + \xi^2,$$

where f , g and h are of the following form:

$$f \in \mathcal{M}^8 + r\mathcal{M},$$

$$h = d_1 u^2 + h_1 \quad (d_1 \in \mathbf{R}, h_1 \in \mathcal{M}^8 + r\mathcal{M}),$$

$$g = d_2 \lambda u + d_3 u^2 + d_4 uv + g_1 \quad (d_2, d_3, d_4 \in \mathbf{R}, g_1 \in \mathcal{M}^8 + r\mathcal{M}),$$

PROPOSITION 4.4. *Assume that $\epsilon\delta \neq 0$. Then we can assume without loss of generality that f , g and h in (4.9, 10) satisfy*

$$f(\lambda, 0, 0, 0) \equiv g(\lambda, 0, 0, 0) \equiv h(\lambda, 0, 0, 0) \equiv 0.$$

Proof. We write $f = p_1 + f_1$, $g = p_2 + g_1$ and $h = p_3 + h_1$, where

$$p_1 = f(\lambda, 0, 0, 0), \quad p_2 = g(\lambda, 0, 0, 0), \quad p_3 = h(\lambda, 0, 0, 0).$$

These three functions belong to \mathcal{M}^3 . We use the following change of variables:

$$(\xi, \zeta) \longrightarrow (\xi + \phi(\lambda)\zeta, \zeta + \phi(\lambda)\bar{\xi}\zeta),$$

where $\phi, \phi \in \mathcal{M}^3$. Then G is transformed to $H = (H_1, H_2)$ where

$$H_1 = (\epsilon\lambda + p_1 + au + bv + cr + f_1)(\xi + \phi\bar{\xi}\zeta) + (ev + p_2 + g_1)(\bar{\xi} + \phi\xi\bar{\zeta})\zeta(1 + \phi),$$

$$H_2 = (\delta\lambda + p_3 + \hat{a}u + \hat{b}v + h_1)\zeta(1 + \phi) + (\xi + \phi\bar{\xi}\zeta)^2.$$

Hereafter until the end of the proof, the symbols f_j , g_j and h_j ($1 \leq j$) denote some functions satisfying

$$f_j \in (\mathcal{M}^8 + r\mathcal{M}) \cap \langle u, v, r \rangle,$$

$$g_j - (d_2 \lambda u + d_3 u^2 + d_4 uv) \in (\mathcal{M}^8 + r\mathcal{M}) \cap \langle u, v, r \rangle,$$

$$h_j - d_1 u^2 \in (\mathcal{M}^3 + r\mathcal{M}) \cap \langle u, v, r \rangle.$$

Now H_1 and H_2 are written as

$$\begin{aligned} H_1 &= (\epsilon\lambda + p_1 + au + bv + cr + f_3)\xi + (ev + p_2 + p_3\phi + \epsilon\phi\lambda + \phi p_1 + g_3)\bar{\xi}\zeta, \\ H_2 &= (\delta\lambda + p_3 + \delta\phi\lambda + p_3\phi + \hat{a}u + \hat{b}v + h_3)\zeta + (1 - v\phi^2)\xi^2. \end{aligned}$$

We transform the second equation to

$$H_2/(1 - v\phi^2) = (\delta\lambda + p_3 + \delta\phi\lambda + p_3\phi + \hat{a}u + \hat{b}v + h_3)\zeta + \xi^2.$$

To complete the proof, it is enough to show that we can choose ϕ and ψ so that

$$\lambda + \epsilon^{-1}p_1 \equiv \lambda + \delta^{-1}p_3 + \phi\lambda + \delta^{-1}p_3\phi, \text{ and } p_2(1 + \phi) + \epsilon\phi\lambda + \phi p_1 \equiv 0.$$

The former yields

$$\phi = (\epsilon^{-1}p_1 - \delta^{-1}p_3)/(\lambda + \delta^{-1}p_3),$$

which certainly belongs to \mathcal{M}_2^3 since $p_j \in \mathcal{M}_2^3$. The latter yields

$$\phi = -(p_2 + p_3\phi)/(\epsilon\lambda + p_1),$$

which belongs to \mathcal{M}_2^3 , since $p_j \in \mathcal{M}_2^3$ and $\epsilon \neq 0$. Now we have only to put $A = \lambda + \epsilon^{-1}p_1$. ■

We have now transformed G to (4.9, 10) where $\epsilon = \pm 1$, $\delta = \pm 1$ and f, g, h are of the following form:

$$\begin{aligned} f_j &\in (\mathcal{M}^3 + r\mathcal{M}) \cap \langle u, v, r \rangle, \\ g_j - (d_2\lambda u + d_3u^2 + d_4uv) &\in (\mathcal{M}^3 + r\mathcal{M}) \cap \langle u, v, r \rangle, \\ h_j - d_1u^2 &\in (\mathcal{M}^3 + r\mathcal{M}) \cap \langle u, v, r \rangle, \\ f(\lambda, 0, 0, 0) &\equiv g(\lambda, 0, 0, 0) \equiv h(\lambda, 0, 0, 0) \equiv 0. \end{aligned}$$

We are now in a position to compute $dG(X)$. We have

$$\begin{aligned} \frac{\partial G_1}{\partial \xi} &= \epsilon\lambda + 2au + bv + cr + f + uf_u + (c + f_r)\xi \frac{\partial r}{\partial \xi} + g_u \bar{\xi}^2 \zeta + g_r \bar{\xi} \zeta \frac{\partial r}{\partial \xi}, \\ \frac{\partial G_1}{\partial \bar{\xi}} &= (a + f_u)\xi^2 + (c + f_r)\xi \frac{\partial r}{\partial \bar{\xi}} + g_r \bar{\xi} \zeta \frac{\partial r}{\partial \bar{\xi}} + (ev + g + ug_u)\zeta, \\ \frac{\partial G_1}{\partial \zeta} &= \left[(b + f_v)\xi + (c + f_r) \frac{\partial r}{\partial \zeta} \right] \xi + \left(2ev + g + vg_v + g_r \zeta \frac{\partial r}{\partial \zeta} \right) \bar{\xi}, \\ \frac{\partial G_1}{\partial \zeta} &= \left[(b + f_v)\zeta + (c + f_r) \frac{\partial r}{\partial \zeta} \right] \xi + \left[(e + g_v)\zeta + g_r \frac{\partial r}{\partial \zeta} \right] \bar{\xi} \zeta, \end{aligned}$$

$$\begin{aligned}\frac{\partial G_2}{\partial \xi} &= 2\xi + \left[(\hat{a} + h_u)\bar{\xi} + h_r \frac{\partial r}{\partial \xi} \right] \zeta, & \frac{\partial G_2}{\partial \bar{\xi}} &= \left[(\hat{a} + h_u)\xi + h_r \frac{\partial r}{\partial \bar{\xi}} \right] \zeta, \\ \frac{\partial G_2}{\partial \zeta} &= \delta\lambda + \hat{a}u + 2\hat{b}v + h + v h_v + h_r \zeta \frac{\partial r}{\partial \zeta}, \\ \frac{\partial G_2}{\partial \bar{\zeta}} &= \left[(\hat{b} + h_v)\zeta + h_r \frac{\partial r}{\partial \bar{\zeta}} \right] \zeta,\end{aligned}$$

where the subscripts mean differentiations. By these equalities we obtain

$$\begin{aligned}dG(X_1) &= [\epsilon\lambda + 3au + bv + cr + f + 2uf_u + 2r(c + f_r)]X_1 \\ &\quad + [ev + g + 2ug_u + 2rg_r]X_2 + [2\hat{a}u + 2uh_u + 2rh_r]X_3 + 2X_4, \\ dG(X_2) &= [2r(a + f_u) + 2uv(c + f_r) + ev^2 + vg]X_1 \\ &\quad + [\epsilon\lambda + au + bv + cr + f + 2uv g_r + 2rg_u]X_2 + [2u + 2\hat{a}r + 2rh_u + 2uvh_r]X_3, \\ dG(X_3) &= [2v(b + f_v) + r(c + f_r)]X_1 + [3ev + g + 2vg_v + rg_r]X_2 \\ &\quad + [\delta\lambda + \hat{a}u + 3\hat{b}v + h + 2vh_v + rh_r]X_3, \\ dG(X_4) &= [2r(b + f_v) + u^2(c + f_r) + evu + ug]X_1 + [2r(e + g_v) + u^2g_r]X_2 \\ &\quad + [2r(\hat{b} + h_v) + u^2h_r]X_3 + [\delta\lambda + \hat{a}u + \hat{b}v + h]X_4.\end{aligned}$$

Our next task is to compute $T_j G$ ($j=1, 2, \dots, 8$), where T_j are given in Proposition 2.3. We have

$$\begin{aligned}T_1 G &= (\epsilon\lambda + au + bv + cr + f)X_1 + (ev + g)X_2, \\ T_2 G &= [u(\epsilon\lambda + au + bv + cr + f) + 2r(ev + g)]X_1 - u(ev + g)X_2, \\ T_3 G + T_4 G &= [2v(\delta\lambda + \hat{a}u + \hat{b}v + h) + 2r]X_1, \\ T_3 G - T_4 G &= 2rX_1 - 2uX_2, \\ T_5 G + T_6 G &= [2u(\epsilon\lambda + au + bv + cr + f) + 2r(ev + g)]X_3, \\ T_5 G - T_6 G &= (ev + g)(2rX_3 - 2vX_4), \\ T_7 G &= [v(\delta\lambda + \hat{a}u + \hat{b}v + h) + 2r]X_3 - vX_4, \\ T_8 G &= (\delta\lambda + \hat{a}u + \hat{b}v + h)X_3 + X_4,\end{aligned}$$

Note that $T_2 G$ and $T_5 G - T_6 G$ are \mathcal{E} -combinations of the remaining 6 elements. In fact, $T_2 G = uT_1 G + (ev + g)(T_3 G - T_4 G)$, $T_5 G - T_6 G = (ev + g)(T_7 G - vT_8 G)$. We have to compute *is*-multiples of above elements. But this is easily done by (3.11). The following proposition holds now:

PROPOSITION 4.5. \mathcal{E} -module $\tilde{F}G$ is generated by the following 13 elements. In the expressions of W_6 and W_8 , we put $p=a-\epsilon\delta\hat{a}$, and $q=b-\epsilon\delta\hat{b}$ for the sake of the convenience:

$$\begin{aligned}
W_1 &= [2u(a+f_u)+2r(c+f_r)]X_1 + [2ug_u+2rg_r]X_2 + [2u(\hat{a}+h_u)+2rh_r]X_3 + 2X_4, \\
W_2 &= [2r(a+f_u)+2uv(c+f_r)+ev^2+vg]X_1 \\
&\quad + [\epsilon\lambda+au+bv+cr+f+2uvgr+2rg_u]X_2 + [2u+2r(\hat{a}+h_u)+2uvh_r]X_3, \\
W_3 &= [2v(b+f_v)+r(c+f_r)]X_1 + [3ev+g+2vg_v+rg_r]X_2 + [2v(\hat{b}+h_v)+rh_r]X_3 - X_4, \\
W_4 &= [2r(b+f_v)+u^2(c+f_r)+euv+ug]X_1 + [2r(e+g_v)+u^2g_r]X_2 \\
&\quad + [2r(\hat{b}+h_v)+u^2h_r]X_3 + (\delta\lambda+\hat{a}u+\hat{b}v+h)X_4, \\
W_5 &= (\epsilon\lambda+au+bv+cr+f)X_1 + (ev+g)X_2, \\
W_6 &= [v(-\epsilon\delta pu-\epsilon\delta qv+h-\epsilon\delta f)+r-\epsilon\delta crv]X_1 - \epsilon\delta v(ev+g)X_2, \\
W_7 &= rX_1 - uX_2, \\
W_8 &= [u(pu+qv+f-\epsilon\delta h)+r(cu+ev+g)]X_3 - \epsilon\delta uX_4, \\
W_9 &= rX_3 - vX_4, \\
W_{10} &= (\delta\lambda+\hat{a}u+\hat{b}v+h)X_3 + X_4, \\
W_{11} &= (ev+g)(uvX_1 - rX_2), \\
W_{12} &= u(uvX_1 - rX_2), \\
W_{13} &= u^2X_3 - rX_4.
\end{aligned}$$

Proof is similar to that of Proposition 3.3. From $\tilde{F}G$ we now choose elements which contain X_1 and X_3 only. We have

$$\begin{aligned}
Y_1 &\equiv \frac{1}{2}uW_1 + (ug_u+rg_r)W_7 + \epsilon\delta W_8 = [au^2+u^2f_u+ru(c+f_r)+r(ug_u+rg_r)]X_1 \\
&\quad + [\hat{a}u^2+u^2h_u+ruh_r+\epsilon\delta pu^2+\epsilon\delta quv+(\epsilon\delta f-h)u+\epsilon\delta r(cu+ev+g)]X_3, \\
Y_2 &\equiv \frac{1}{2}v(ev+g)W_1 + \epsilon\delta(ug_u+rg_r)W_6 + (ev+g)W_9 \\
&= [v(ev+g)(u(a+f_u)+r(c+f_r))+(ug_u+rg_r)(-puv-qv^2+v(\epsilon\delta h-f) \\
&\quad +\epsilon\delta r-crv)]X_1 + (ev+g)[uv(\hat{a}+h_u)+rvh_r+r]X_3, \\
Y_3 &\equiv uW_3 + (3ev+g+2vg_v+rg_r)W_7 - \epsilon\delta W_8
\end{aligned}$$

$$\begin{aligned}
&= [2uv(b+f_v)+ru(c+f_r)+r(3ev+g+2vg_v+rg_r)]X_1 \\
&\quad + [-\epsilon\delta pu^2+(3\hat{b}-\epsilon\delta\hat{b})uv+2uvh_v+ruh_r-u(\epsilon\delta f-h)-\epsilon\delta r(cu+ev+g)]X_3, \\
Y_4 &\equiv v(ev+g)W_3+\epsilon\delta(3ev+g+2vg_v+rg_r)W_8-(ev+g)W_9 \\
&= [v(ev+g)(2bv+2vf_v+r(c+f_r))+(3ev+g+2vg_v+rg_r)(-puv-qv^2 \\
&\quad +v(\epsilon\delta h-f)+\epsilon\delta r-crv)]X_1+(ev+g)[2\hat{b}v^2+2v^2h_v+rvh_r-r]X_3, \\
Y_5 &\equiv r(ev+g)W_3+(3ev+g+2vg_v+rg_r)W_{11}-(ev+g)W_{13} \\
&= (ev+g)[2rv(b+f_v)+r^2(c+f_r)+uv(3ev+g+2vg_v+rg_r)]X_1 \\
&\quad + (ev+g)[2rv(\hat{b}+h_v)+r^2h_r-u^2]X_3, \\
Y_6 &\equiv \frac{ru}{2}W_1+r(ug_u+rg_r)W_7+uW_{13} \\
&= [ru^2(a+f_u)+r^2u(c+f_r))+r^2(ug_u+rg_r)]X_1+[ru(\hat{a}u+uh_u+rh_r)+u^3]X_3 \\
Y_7 &\equiv ruW_3+r(3ev+g+2vg_v+rg_r)W_7-uW_{13} \\
&= [ru(2bv+2vf_v+r(c+f_r))+r^2(3ev+g+2vg_v+rg_r)]X_1 \\
&\quad + [ru(2\hat{b}v+2vh_v+rh_r)-u^3]X_3, \\
Y_8 &\equiv uW_2+(\epsilon\lambda+au+bv+cr+f+2uvgr+2rg_u)W_7-rW_5-W_{11} \\
&= [2ru(a+f_u)+2u^2v(c+f_r)+2ruvgr+2r^2g_u]X_1 \\
&\quad + [2u^2+2ru(\hat{a}+h_u)+2u^2vh_r]X_3, \\
Y_9 &\equiv v(ev+g)W_2+\epsilon\delta(\epsilon\lambda+au+bv+cr+f+2uvgr+2rg_u)W_8-(\epsilon\delta-cv)W_{11} \\
&\quad - [-puv-qv^2+(\epsilon\delta h-f)v+\epsilon\delta r-crv]W_5-(-pu-qv+\epsilon\delta h-f)\epsilon\delta W_6 \\
&= [v(ev+g)(2r(a+f_u)+2uv(c+f_r)+v(ev+g))-(\epsilon\delta-cv)uv(ev+g) \\
&\quad + (-puv-qv^2+v(\epsilon\delta h-f)+\epsilon\delta r-crv)(pu+qv-\epsilon\delta h+f+2uvgr+2rg_u)]X_1 \\
&\quad + [2u+2\hat{a}r+2rh_u+2uvh_r]v(ev+g)X_3, \\
Y_{10} &\equiv uW_4+(2r(e+g_v)+u^2g_r)W_7+\epsilon\delta(\delta\lambda+\hat{a}u+\hat{b}v+h)W_8-\epsilon\delta(cu+ev+g)W_{13} \\
&\quad - \epsilon\delta[pu^2+quv+(f-\epsilon\delta h)u+r(cu+ev+g)]W_{10}-[pu+qv+f-\epsilon\delta h]W_8 \\
&= [2ru(b+f_v)+u^3(c+f_r)+(ev+g)u^2+2r^2(e+g_v)+ru^2g_r]X_1 \\
&\quad + [2ru(\hat{b}+h_v)+u^3h_r-\epsilon\delta u^2(cu+ev+g) \\
&\quad - (pu+qv+f-\epsilon\delta h)(pu^2+quv+(f-\epsilon\delta h)u+r(cu+ev+g))]X_3.
\end{aligned}$$

where $\bar{p}=\hat{a}-\epsilon\delta a$, $\bar{q}=\hat{b}-\epsilon\delta b$, $\theta=2\bar{b}+\bar{q}$ and $\Theta=e(2b-3q)$. In order to see how we obtain this matrix, let us consider uY_1 , for instance. We can write it as

$$\begin{aligned} uY_1 = & (a+\mu_1)V_1 + (c+\mu_2)V_5 + (a\epsilon\delta+\mu_3)V_8 + (\epsilon\delta q+\mu_4)V_9 \\ & + (\epsilon\delta c+\mu_5)V_{12} + (\epsilon\delta e+\mu_6)V_{13} + \mu_7 r^2 X_1 + \mu_8 r^2 X_3, \end{aligned}$$

where $\mu_j \in \mathcal{M}$ ($1 \leq j \leq 8$). On the other hand, we have

$$(4.11) \quad rW_7 - W_{12} = (r^2 - u^2 v)X_1, \quad rW_9 - vW_{13} = (r^2 - u^2 v)X_3.$$

Therefore we have

$$\begin{aligned} uY_1 - \mu_7(rW_7 - W_{12}) - \mu_8(rW_9 - vW_{13}) = & (a+\mu_1)V_1 + \mu_7 V_2 + (c+\mu_2)V_5 \\ & + (a\epsilon\delta+\mu_3)V_8 + (\epsilon\delta q+\mu_4+\mu_8)V_9 + (\epsilon\delta c+\mu_5)V_{12} + (\epsilon\delta e+\mu_6)V_{13}. \end{aligned}$$

This leads to the first low of the above matrix. Other lows are similarly obtained. It is elementary to compute the determinant of A_0 and we have

$$\det A_0 = -64\epsilon\delta ab^4\bar{b}e^4q^3(a\bar{b}-b\hat{a}).$$

Therefore, under the assumption of the proposition, all the components of V belong to $\tilde{F}G$. ■

From this proposition we can further deduce that some elements which are of lower order than those given in the proposition belong to $\tilde{F}G$. For example, it follows from (4.11) that $r^2 X_1, r^2 X_3 \in \tilde{F}G$. If we consider uW_8 , then we have that $u^2 X_4 \in \tilde{F}G$. We give diagrams below which tell us that the right hand side belong to $\tilde{F}G$ by the left hand side.

$$\begin{aligned} (4.11) \longrightarrow & r^2 X_1, r^2 X_3, \quad uW_8, vW_8, rW_8 \longrightarrow u^2 X_4, uvX_4, ruX_4, \\ v^2 W_9, rW_9 \longrightarrow & v^3 X_4, rvX_4, \quad uW_9 \longrightarrow ruX_3, \\ u^2 W_7, uvW_7, v^2 W_7, rW_7 \longrightarrow & u^3 X_2, u^2 vX_2, uv^2 X_2, ruX_2 \\ rW_{13} \longrightarrow & r^2 X_4, \quad rvW_4 \longrightarrow r^2 vX_2, \quad rvW_8 \longrightarrow rv^2 X_2, \\ v^3 W_3 \longrightarrow & v^4 X_2, \quad v^2 W_1 \longrightarrow v^2 X_4, \quad vW_9 \longrightarrow rvX_3, \\ uW_6 \longrightarrow & ruX_1, \quad uW_7 \longrightarrow u^2 X_2, \quad Y_8 \longrightarrow u^2 X_3, \\ W_{13} \longrightarrow & rX_4, \quad rW_4 \longrightarrow r^2 X_2, \quad W_{11} \longrightarrow rvX_2, \\ rW_{10} \longrightarrow & \lambda rX_3, \quad \lambda W_9 \longrightarrow \lambda vX_4, \quad vW_4 \longrightarrow rvX_1, \\ Y_9 \longrightarrow & v^3 X_1, \quad W_6 \longrightarrow v^3 X_2, \quad v^2 W_3 \longrightarrow v^3 X_3, \\ rW_6 \longrightarrow & \lambda rX_1, \quad \lambda W_7 \longrightarrow \lambda uX_2, \quad uW_2 \longrightarrow uvX_2, \\ u^2 W_5, uvW_5, v^2 W_5 \longrightarrow & \lambda u^2 X_1, \lambda uvX_1, \lambda v^2 X_1, \end{aligned}$$

$$\begin{aligned}
u^2 W_{10}, uv W_{10}, v^2 W_{10} &\longrightarrow \lambda u^2 X_3, \lambda uv X_3, \lambda v^2 X_3, \\
u W_2, v^2 W_2, r W_2 &\longrightarrow \lambda u X_2, \lambda v^2 X_2, \lambda r X_2, \\
\lambda u W_1, \lambda v W_1 &\longrightarrow \lambda u X_4, \lambda v X_4, \\
W_5 &\longrightarrow \lambda^2 u X_1, \lambda^2 v X_1, & W_{10} &\longrightarrow \lambda^2 u X_3, \lambda^2 v X_3, & W_2 &\longrightarrow \lambda^2 v X_2, \\
W_4 &\longrightarrow \lambda^2 X_4, & W_6 &\longrightarrow \lambda^3 X_1, & W_{10} &\longrightarrow \lambda^3 X_3, \\
W_2 &\longrightarrow \lambda^3 X_2.
\end{aligned}$$

By these diagrams we have proved

PROPOSITION 4.7. *Under the same assumption as in the previous proposition, $\tilde{f}G$ contains the followings:*

$$(4.12) \quad \mathcal{M}^3 X_j, r \mathcal{M} X_j \ (j=1, 2, 3), \ \mathcal{M}^2 X_4, r X_4, u^2 X_2, uv X_2, \lambda u X_2, u^2 X_3$$

Propositions 4.3, 4.5 and 4.7 ensures that the elements in (4.12) and the following elements generate $\tilde{f}G$:

$$\begin{aligned}
Z_1 &= (2au + 2cr)X_1 + 2\hat{a}uX_3 + 2X_4, \\
Z_2 &= (2ar + 2cuv + ev^2)X_1 + (\epsilon\lambda + au + bv + cr)X_2 + (2u + 2\hat{a}r)X_3, \\
Z_3 &= (2bv + cr)X_1 + 3evX_2 + 2\hat{b}vX_3 - X_4, \\
Z_4 &= (2br + cu^2 + euv)X_1 + 2erX_2 + 2\hat{b}rX_3 + (\delta\lambda + \hat{a}u + \hat{b}v)X_4, \\
Z_5 &= (\epsilon\lambda + au + bv + cr)X_1 + evX_2, \\
Z_6 &= (-\epsilon\delta p u w - \epsilon\delta q v^2 + r)X_1 - \epsilon\delta ev^2 X_2, \\
Z_7 &= rX_1 - uX_2, \\
Z_8 &= quvX_3 - \epsilon\delta uX_4, \\
Z_9 &= rX_3 - vX_4, \\
Z_{10} &= (\delta\lambda + \hat{a}u + \hat{b}v)X_3 + X_4.
\end{aligned}$$

Therefore ΓG is independent of f, g, h . Note that Proposition 4.3 is fully used here. Now we have proved

THEOREM 4.1. *If $\epsilon\delta ab\hat{a}\hat{b}e(b - \epsilon\delta\hat{b})(a\hat{b} - b\hat{a}) \neq 0$ and the assumption in Proposition 4.3 is satisfied, then (4.9, 10) is O(2)-equivalent to the following (4.13, 14):*

$$(4.13) \quad G_1 = (\epsilon\lambda + au + bv + cr)\xi + ev\bar{\xi}\zeta,$$

$$(4.14) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v)\zeta + \xi^2,$$

where $|\epsilon| = |\delta| = |\hat{a}| = |\hat{b}| = 1$.

Proof. Since $\tilde{I}G$ is independent of f, g, h , this follows from Theorem 2.1. The condition on $\epsilon, \delta, \hat{a}, \hat{b}$ is allowed by

$$(G_1, G_2) \longrightarrow (\alpha G_1(\beta\lambda, \gamma_1\xi, \gamma_2\zeta), \gamma_1^{-2}G_2(\beta\lambda, \gamma_1\xi, \gamma_2\zeta))$$

with appropriate positive constants α, β, γ_1 and γ_2 . ■

We now compute a universal unfolding of (4.13, 14). This is given by

THEOREM 4.2. *Let G be given by (4.13, 14). Under the same assumption as before, the following $F=(F_1, F_2)$ is a universal unfolding of G :*

$$(4.15) \quad F_1 = (\epsilon\lambda + \alpha + (a+s_1)u + (b+s_1)v + (c+s_3)\xi + (\beta + (e+s_4)v)\bar{\xi}\zeta$$

$$(4.16) \quad F_2 = (\delta\lambda + \hat{a}u + \hat{b}v)\zeta + \xi^2,$$

where $\alpha, \beta, s_1, s_2, s_3$ and s_4 are unfolding parameters. In particular, $O(2)$ -codimension of G is 6.

Proof. By Theorem 2.1 we have only to prove that

$$IG \oplus \mathbf{R}X_1 \oplus \mathbf{R}X_2 \oplus \mathbf{R}uX_1 \oplus \mathbf{R}vX_1 \oplus \mathbf{R}rX_1 \oplus \mathbf{R}vX_2 = \hat{E}$$

Since $\langle \lambda^3 \rangle$ is a submodule of $\tilde{I}G$ by Proposition 4.7, it clearly holds that

$$(4.17) \quad \begin{aligned} IG &= \tilde{I}G \oplus \mathbf{R} \frac{\partial G}{\partial \lambda} \oplus \mathbf{R} \lambda \frac{\partial G}{\partial \lambda} \oplus \mathbf{R} \lambda^2 \frac{\partial G}{\partial \lambda} \\ &= \tilde{I}G \oplus \mathbf{R}(\epsilon X_1 + \delta X_3) \oplus \mathbf{R} \lambda(\epsilon X_1 + \delta X_3) \oplus \mathbf{R} \lambda^2(\epsilon X_1 + \delta X_3). \end{aligned}$$

Let N denote the following \mathbf{R} -linear space:

$$(4.18) \quad \tilde{I}G + \mathbf{R}X_1 + \mathbf{R}X_2 + \mathbf{R}uX_1 + \mathbf{R}vX_1 + \mathbf{R}rX_1 + \mathbf{R}vX_2.$$

We will prove that $N = \hat{E}$. From (4.17, 18) and Z_5 , it follows that

$$X_1, X_2, X_3, \lambda X_1, \lambda X_3, uX_1, vX_1, rX_1, vX_2$$

belong to N . By Z_1, Z_3 and Z_{10} the elements

$$\hat{a}uX_3 + X_4, \quad 2\hat{b}vX_3 - X_4, \quad (\hat{a}u + \hat{b}v)X_3 + X_4$$

belong to N . Therefore uX_3, vX_3, X_4 belong to N if $\hat{a}\hat{b} \neq 0$. By considering

$$\lambda Z_1, uZ_1, vZ_1, vZ_2, \lambda Z_3, uZ_3, vZ_3, \lambda Z_5, uZ_5, vZ_5, -\epsilon\delta Z_6, Z_8, Z_9, \lambda Z_{10}, uZ_{10}, vZ_{10},$$

we see that the following 16 elements belong to $\tilde{I}G$:

$$\begin{aligned} &a\lambda uX_1 + \hat{a}\lambda uX_3 + \lambda X_4, \quad au^2X_1 + uX_4, \quad auvX_1 + \hat{a}uvX_3 + vX_4, \\ &(\epsilon\lambda v + \hat{b}v^2)X_2 + 2uvX_3, \quad 2b\lambda vX_1 + 3e\lambda vX_2 + 2\hat{b}\lambda vX_3 - \lambda X_4, \quad 2buvX_1 + 2\hat{b}uvX_3 - uX_4, \end{aligned}$$

$$\begin{aligned}
& 2bv^2X_1 + 3ev^2X_2 + 2\hat{b}v^2X_3 - vX_4, \quad (\epsilon\lambda^2 + a\lambda u + b\lambda v)X_1 + e\lambda vX_2, \\
& (\epsilon\lambda u + au^2 + buv)X_1, \quad (\epsilon\lambda v + auv + bv^2)X_1 + ev^2X_2, \quad (puv + qv^2 - \epsilon\delta r)X_1 + ev^2X_2, \\
& quvX_3 - \epsilon\delta uX_4, \quad rX_3 - vX_4, \quad (\delta\lambda^2 + \hat{a}\lambda u + \hat{b}\lambda v)X_3 + \lambda X_4, \\
& (\delta\lambda u + buv)X_3 + uX_4, \quad (\delta\lambda v + \hat{a}uv + bv^2)X_3 + vX_4.
\end{aligned}$$

In addition to this we know that $\epsilon\lambda^2X_1 + \delta\lambda^2X_3$ belongs to ΓG . We now define

$$\begin{aligned}
V = & (\lambda^2X_1, \lambda uX_1, \lambda vX_1, u^2X_1, uvX_1, v^2X_1, \lambda^2X_3, \lambda uX_3, \\
& \lambda vX_3, uvX_3, v^2X_3, rX_3, \lambda vX_2, v^2X_2, \lambda X_4, uX_4, vX_4).
\end{aligned}$$

Then above 17 relations are written as follows:

$$AV = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \epsilon\delta rX_1, 0, 0, 0, 0, 0, 0),$$

where A is a 17×17 real matrix given by

$$\begin{pmatrix}
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \hat{a} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & \hat{a} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & \epsilon & b & 0 & 0 & 0 \\
0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 2\hat{b} & 0 & 0 & 0 & 3e & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 2\hat{b} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 2\hat{b} & 0 & 0 & 3e & 0 & 0 & -1 & 0 \\
\epsilon & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\
0 & \epsilon & 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & -\epsilon\delta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & \hat{a} & \hat{b} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & \hat{b} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & \hat{a} & \hat{b} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\epsilon & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Since this matrix is nonsingular under the same condition as above, all the

component of V belong to N . We now have the following diagram:

$$\begin{array}{ll} \lambda Z_2 \longrightarrow \lambda^2 X_2, & Z_4 \longrightarrow r X_2, \\ Z_7 \longrightarrow u X_2, & Z_2 \longrightarrow \lambda X_2, \end{array}$$

which means that the right hand sides belong to N . This completes the proof that $N = \hat{E}$. There remains to show that the sum in (4.18) is a direct sum. Let W denote the \mathcal{E} -module generated by the elements given in Proposition 4.6. Then we can check in an elementary way that \mathbf{R} -linear space \hat{E}/W is of dimension 33. On the other hand, we have only 24 relations which is nontrivial modulo W . These and the fact that

$$\epsilon X_1 + \delta X_3, \quad \epsilon \lambda X_1 + \delta \lambda X_3, \quad \epsilon \lambda^2 X_1 + \delta \lambda^2 X_3 \in N$$

are the only nontrivial relations in \hat{E}/W . Thus we see that $\hat{E}/\Gamma G$ is of dimension 6, which completes the proof of the present proposition. ■

REMARK. The condition in Theorems 4.1 and 4.2 is a sufficient condition. Remained is a possibility to find weaker and simpler conditions. Since formidable computations are expected, we do not aim at best conditions.

§ 5. Computation of ΓG : Case (III)

In this section we consider the case (III) and compute ΓG . We first transform G to $(G_1/f_2, G_2)$. We write it as:

$$(5.1) \quad G_1 = (\epsilon \lambda + p_1 + q_1) \xi + \bar{\xi} \zeta$$

$$(5.2) \quad G_2 = (\delta \lambda + p_2 + q_2) \zeta + (\eta \lambda + p_3 + q_3) \xi^2,$$

where ϵ, δ, η are real constants, $p_1, p_2, p_3 \in \mathcal{M}_\lambda^2$, and $q_1, q_2, q_3 \in \langle u, v, r \rangle$.

PROPOSITION 5.1. Assume that $\epsilon \delta \neq 0$. Then $G = (G_1, G_2)$ is $O(2)$ -equivalent to (5.1, 2) with $\epsilon = \pm 1$, $\delta = \pm 1$, $\eta = 0$ and $p_3 \equiv 0$, although numerical values of other coefficients may change.

Proof. We use the following change of coordinates:

$$(\xi, \zeta) \longrightarrow (\xi, \zeta + \beta \xi^2),$$

where $\beta \in \mathcal{E}_\lambda$. Choosing β so that $(\delta \lambda + p_2) \beta + \eta \lambda + p_3 \equiv 0$, we can make η, p_3 vanish. This choice is possible, since $\delta \neq 0$. The conditions on ϵ and δ are fulfilled by the following transform:

$$\frac{1}{\gamma^2} G_1(\gamma_1 \lambda, \xi, \gamma_2 \zeta), \quad G_2(\gamma_1 \lambda, \xi, \gamma_2 \zeta)$$

with

$$\gamma_1 = \sqrt{1/|\epsilon\delta|}, \quad \gamma_2 = \sqrt{|\epsilon|/|\delta|}. \blacksquare$$

We now write G as

$$(5.3) \quad G_1 = (\epsilon\lambda + au + bv + cr + f)\xi + \bar{\xi}\zeta,$$

$$(5.4) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)\zeta + (du + ev + kr + h)\xi^2,$$

where $f, g, h \in \mathcal{M}^2$ and $\epsilon, \delta, a, b, c, \hat{a}, \hat{b}, \hat{c}, d, e, k$ are real constants, and $\epsilon = \pm 1, \delta = \pm 1, h(\lambda, 0, 0, 0) \equiv 0$.

PROPOSITION 5.2. *Assume that $d \neq 0$. Then $G = (G_1, G_2)$ is $O(2)$ -equivalent to (5.3, 4) with $c = e = k = 0$ and $h(\lambda, 0, 0, 0) \equiv 0$, although numerical values of other coefficients may change.*

Proof. We use the following change of coordinates:

$$(\xi, \zeta) \longrightarrow (\xi + \alpha \bar{\xi} \zeta, \zeta),$$

where α is a real parameter. G_1 and G_2 are transformed to

$$H_1 = (\epsilon\lambda + au + (b + \alpha)v + (c + 2a\alpha)r + f_1)\xi + (1 + \mu)\bar{\xi}\zeta,$$

$$H_2 = (\delta\lambda + \hat{a}u + \hat{b}v + (\hat{c} + 2\hat{a}\alpha)r + g_1)\zeta + (du + ev + (k + 2d\alpha)r + h_1)\xi^2,$$

respectively, where $\mu \in \mathcal{M}, f_j, g_j, h_j \in \mathcal{M}^2$. Choosing α so that $k + 2d\alpha = 0$ we can make k vanish. We next consider the following transformation:

$$K_1 = H'_1(\lambda, \xi, \zeta) + \gamma_1 \xi^2 \bar{H}'_1,$$

$$K_2 = H_2(\lambda, \xi, \zeta) + \gamma_2 \zeta (\bar{\xi} H'_1 - \xi \bar{H}'_1),$$

where $H'_1 = H_1/(1 + \mu)$. We note that the coefficient of ξ^2 in K_2 belongs to $\langle u, v, r \rangle$ after these operation. Choosing γ_1 and γ_2 so that $e - 2\gamma_2 = 0, c + 2a\alpha + 2\gamma_1 = 0$, and dividing K_1 by $1 - \gamma_1 u$, the proof is completed. \blacksquare

Henceforth we consider

$$(5.5) \quad G_1 = (\epsilon\lambda + au + bv + f)\xi + \bar{\xi}\zeta,$$

$$(5.6) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)\zeta + (du + h)\xi^2,$$

where f, g and h belong to \mathcal{M}^2 and $\epsilon = \pm 1, \delta = \pm 1, h(\lambda, 0, 0, 0) \equiv 0$.

PROPOSITION 5.3. *If $\epsilon\delta d \neq 0$, and*

$$2d(\hat{a} - 2\epsilon\delta\alpha) + \alpha(\hat{a} - \epsilon\delta\alpha)(\epsilon\delta\alpha - 2\hat{a}) \neq 0,$$

then G given by (5.5, 6) is $O(2)$ -equivalent to (5.5, 6) in which the coefficients of $\lambda^2, \lambda u$ and u^2 in f, g, h vanish, and

$$f(\lambda, 0, 0, 0) \equiv g(\lambda, 0, 0, 0) \equiv h(\lambda, 0, 0, 0) \equiv 0.$$

Proof. We transform G_1 and G_2 to

$$H_1(\lambda, \xi, \zeta) = (1 + \beta_0\lambda + \beta_1u)^{-1}(1 + \gamma_1u)^{-1}G_1(\lambda, (1 + \beta_0\lambda + \beta_1u)\xi, (1 + \gamma_1u)\zeta),$$

$$H_2(\lambda, \xi, \zeta) = (1 + \alpha_0\lambda + \alpha_1u)G_2(\lambda, (1 + \beta_0\lambda + \beta_1u)\xi, (1 + \gamma_1u)\zeta),$$

respectively. As in §4, let f_0, g_0, h_0 be the r -independent quadratic parts of f, g, h , respectively. Therefore we can write $f = f_0 + f_1$, $g = g_0 + g_1$, $h = h_0 + h_1$, where $f_1, g_1, h_1 \in \mathcal{M}^3 + r\mathcal{M}$. Then we have

$$H_1 = [\epsilon\lambda + au + bv + f_0 + (2a\beta_0 - \epsilon\gamma_1)\lambda u + (2a\beta_1 - a\gamma_1)u^2 + b\gamma_1uv + f_2]\xi + \bar{\xi}\zeta$$

and

$$\begin{aligned} H_2 = & [\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g_0 + \delta\alpha_0\lambda^2 + (\hat{a}(\alpha_0 + 2\beta_0)) + \delta(\alpha_1 + \gamma_1))\lambda u \\ & + \hat{b}\alpha_0\lambda v + \hat{a}(\alpha_1 + 2\beta_1 + \gamma_1)u^2 + \hat{b}(\alpha_1 + 3\gamma_1)uv + g_2]\zeta \\ & + [du + h_0 + d(\alpha_0 + 4\beta_0)\lambda u + d(\alpha_1 + 4\beta_1)u^2 + h_2]\xi^2. \end{aligned}$$

Here and hereafter f_j, g_j, h_j ($1 \leq j$) denote some elements of $\mathcal{M}^3 + r\mathcal{M}$. We next change λ to $\lambda + \eta_0\lambda^2$ ($\eta_0 \in \mathbf{R}$). Then H_1 and H_2 are transformed to the following H'_1 and H'_2 , respectively:

$$H'_1 = [\epsilon\lambda + au + bv + f_0 + \epsilon\eta_0\lambda^2 + (2a\beta_0 - \epsilon\gamma_1)\lambda u + (2a\beta_1 - a\gamma_1)u^2 + b\gamma_1uv + f_3]\xi + \bar{\xi}\zeta$$

and

$$\begin{aligned} H'_2 = & [\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g_0 + \delta(\alpha_0 + \eta_0)\lambda^2 + (\hat{a}(\alpha_0 + 2\beta_0)) + \delta(\alpha_1 + \gamma_1))\lambda u \\ & + \hat{b}\alpha_0\lambda v + \hat{a}(\alpha_1 + 2\beta_1 + \gamma_1)u^2 + \hat{b}(\alpha_1 + 3\gamma_1)uv + g_3]\zeta \\ & + [du + h_0 + d(\alpha_0 + 4\beta_0)\lambda u + d(\alpha_1 + 4\beta_1)u^2 + h_3]\xi^2. \end{aligned}$$

Note that $h_0(\lambda, 0, 0, 0) \equiv h_3(\lambda, 0, 0, 0) \equiv 0$. We now transform H'_2 to $K_2 = H'_2 + \eta_1\zeta(\bar{\xi}H'_1 + \bar{\xi}\bar{H}'_1)$ and change ζ to $\zeta + \eta_2u\xi^2$, where $\eta_1, \eta_2 \in \mathbf{R}$. Now H'_1 and H'_2 are transformed to

$$K_1 = [\epsilon\lambda + au + bv + f_0 + \epsilon\eta_0\lambda^2 + (2a\beta_0 - \epsilon\gamma_1)\lambda u + (2a\beta_1 - a\gamma_1 + \eta_2)u^2 + b\gamma_1uv + f_4]\xi + \bar{\xi}\zeta$$

and

$$\begin{aligned} K_2 = & [\delta\lambda + \hat{a}u + \hat{b}v + (\hat{c} + 2\eta_1)r + g_0 + \delta(\alpha_0 + \eta_0)\lambda^2 + (\hat{a}(\alpha_0 + 2\beta_0) + \delta(\alpha_1 + \gamma_1) + 2\epsilon\eta_1)\lambda u \\ & + \hat{b}\alpha_0\lambda v + (\hat{a}(\alpha_1 + 2\beta_1 + \gamma_1) + 2a\eta_1)u^2 + (\hat{b}(\alpha_1 + 3\gamma_1) + 2b\eta_1)uv + g_4]\zeta \\ & + [du + h_0 + (d(\alpha_0 + 4\beta_0) + \delta\eta_2)\lambda u + (d(\alpha_1 + 4\beta_1) + \hat{a}\eta_2)u^2 + \hat{b}\eta_2uv + h_4]\xi^2. \end{aligned}$$

Note that h_4 satisfy $h_4(\lambda, 0, 0, 0) \equiv 0$. In order to show that the coefficients listed in the proposition can be put zero, we have only to prove that the following 8×8 matrix is nonsingular under the assumption of the proposition.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 2a & 0 & -\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a & -a & 0 & 0 & 1 \\ \delta & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\ \hat{a} & \delta & 2\hat{a} & 0 & \delta & 0 & 2\epsilon & 0 \\ 0 & \hat{a} & 0 & 2\hat{a} & \hat{a} & 0 & 2a & 0 \\ d & 0 & 4d & 0 & 0 & 0 & 0 & \delta \\ 0 & d & 0 & 4d & 0 & 0 & 0 & \hat{a} \end{pmatrix}$$

We, however, do not give the computation of the determinant, since it is straightforward.

To make $f(\lambda, 0, 0, 0)$ and $g(\lambda, 0, 0, 0)$ identically zero, we put

$$t(\lambda) \equiv f(\lambda, 0, 0, 0), \quad w(\lambda) \equiv g(\lambda, 0, 0, 0)$$

and define $f_1 = f - t$ and $g_1 = g - w$. We apply

$$(\xi, \zeta) \longrightarrow (\xi, (1 + \phi(\lambda))\zeta),$$

where $\phi \in \mathcal{M}_1$ and divide the first equation by $1 + \phi$. Then G given by (5.5, 6) is transformed to the following $L = (L_1, L_2)$.

$$L_1 = \left(\frac{\epsilon\lambda + t}{1 + \phi} + au + bv + f_2 \right) \xi + \bar{\xi}\zeta,$$

$$L_2 = [(\delta\lambda + w)(1 + \phi) + \hat{a}u + \hat{b}v + \hat{c}r + g_1]\zeta + (du + h_1)\xi^2,$$

where $f_j, g_j, h_j \in \mathcal{M}\langle u, v, r \rangle$. We choose ϕ so that the λ -parts of H_1 and H_2 coincide. Namely

$$\frac{\lambda + \epsilon^{-1}t}{1 + \phi} = (\lambda + \delta^{-1}w)(1 + \phi).$$

Letting λ denote this quantity, we can make $f(\lambda, 0, 0, 0)$ and $g(\lambda, 0, 0, 0)$ identically zero. ■

From now on we consider (5.5, 6) in which the coefficients of λ^2 , λu and u^2 in the Taylor expansion of f, g, h vanish and $\epsilon = \pm 1$, $\delta = \pm 1$, $f(\lambda, 0, 0, 0) \equiv g(\lambda, 0, 0, 0) \equiv h(\lambda, 0, 0, 0) \equiv 0$.

We now compute dG :

$$\frac{\partial G_1}{\partial \xi} = \epsilon\lambda + 2au + bv + f + uf_u + f_r \bar{\xi} \frac{\partial r}{\partial \bar{\xi}}$$

$$\begin{aligned}
\frac{\partial G_1}{\partial \bar{\xi}} &= (a + f_u)\xi^2 + f_r \xi \frac{\partial r}{\partial \bar{\xi}} + \zeta, \\
\frac{\partial G_1}{\partial \bar{\zeta}} &= \left[(b + f_v)\bar{\zeta} + f_r \frac{\partial r}{\partial \bar{\zeta}} \right] \xi + \bar{\xi} \\
\frac{\partial G_1}{\partial \bar{\zeta}} &= \left[(b + f_v)\bar{\zeta} + f_r \frac{\partial r}{\partial \bar{\zeta}} \right] \xi, \\
\frac{\partial G_2}{\partial \xi} &= 2\xi(du + h) + \xi^2 \left[(d + h_u)\bar{\xi} + h_r \frac{\partial r}{\partial \xi} \right] + \left[(\hat{a} + g_u)\bar{\xi} + (\hat{c} + g_r) \frac{\partial r}{\partial \xi} \right] \zeta, \\
\frac{\partial G_2}{\partial \xi} &= \left[(\hat{a} + g_u)\xi + (\hat{c} + g_r) \frac{\partial r}{\partial \xi} \right] \zeta + \left((d + h_u)\xi + h_r \frac{\partial r}{\partial \xi} \right) \xi^2, \\
\frac{\partial G_2}{\partial \zeta} &= \delta\lambda + \hat{a}u + 2\hat{b}v + \hat{c}r + g + vg_v + (\hat{c} + g_r)\zeta \frac{\partial r}{\partial \zeta} + \left(h_v\bar{\zeta} + h_r \frac{\partial r}{\partial \zeta} \right) \xi^2, \\
\frac{\partial G_2}{\partial \zeta} &= \left[(\hat{b} + g_v)\zeta + (\hat{c} + g_r) \frac{\partial r}{\partial \zeta} \right] \zeta + \left(h_v\zeta + h_r \frac{\partial r}{\partial \zeta} \right) \xi^2,
\end{aligned}$$

where the subscripts mean differentiations. By these formulas we obtain

$$\begin{aligned}
dG(X_1) &= [\epsilon\lambda + 3au + bv + f + 2uf_u + 2rf_r]X_1 + X_2 \\
&\quad + [2\hat{a}u + 2ug_u + 2r(\hat{c} + g_r)]X_3 + [4du + 2h + 2uh_u + 2rh_r]X_4, \\
dG(X_2) &= [v + 2r(a + f_u) + 2uvf_r]X_1 + [\epsilon\lambda + au + bv + f]X_2 \\
&\quad + [2u(du + h) + 2\hat{a}r + 2rg_u + 2uv(\hat{c} + g_r)]X_3 + [2r(d + h_u) + 2uvh_r]X_4, \\
dG(X_3) &= [2v(b + f_v) + rf_r]X_1 + X_2 \\
&\quad + [\delta\lambda + \hat{a}u + 3\hat{b}v + \hat{c}r + g + 2vg_v + r(\hat{c} + g_r)]X_3 + (rh_r + 2vh_v)X_4, \\
dG(X_4) &= [2r(b + f_v) + u^2f_r + u]X_1 + [2r(\hat{b} + g_v) + u^2(\hat{c} + g_r)]X_3 \\
&\quad + [\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + 2rh_v + u^2h_r]X_4.
\end{aligned}$$

$T_j G$ ($j=1, 2, \dots, 8$) are given by

$$\begin{aligned}
T_1 G &= (\epsilon\lambda + au + bv + f)X_1 + X_2, \\
T_2 G &= [u(\epsilon\lambda + au + bv + f) + 2r]X_1 - uX_2, \\
T_3 G + T_4 G &= [2v(\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g) + 2r(du + h)]X_1, \\
T_3 G - T_4 G &= 2(du + h)(rX_1 - uX_2), \\
T_5 G + T_6 G &= [2u(\epsilon\lambda + au + bv + f) + 2r]X_3,
\end{aligned}$$

$$T_5G - T_6G = 2rX_3 - 2vX_4,$$

$$T_7G = [v(\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g) + 2r(du + h)]X_3 - v(du + h)X_4,$$

$$T_8G = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)X_3 + (du + h)X_4.$$

Note that $T_3G - T_4G$ and T_7G are \mathcal{E} -combinations of the remaining 6 elements. In a way similar to Proposition 4.5, we have

PROPOSITION 5.4. *The \mathcal{E} -module $\tilde{T}G$ is generated by the following 13 elements:*

$$W_1 = [2u(a + f_u) + 2rf_r]X_1 + [2u(\hat{a} + g_u) + 2r(\hat{c} + g_r)]X_3$$

$$+ [4du + 2h + 2uh_u + 2rh_r]X_4,$$

$$W_2 = [v + 2r(a + f_u) + 2uvf_r]X_1 + [\epsilon\lambda + au + bv + f]X_2$$

$$+ [2u(du + h) + 2\hat{a}r + 2rg_u + 2uv(\hat{c} + g_r)]X_3 + [2r(d + h_u) + 2uvh_r]X_4,$$

$$W_3 = [2v(b + f_v) + rf_r]X_1 + X_2 + [2v(\hat{b} + g_v) + r(\hat{c} + g_r)]X_3$$

$$+ [rh_r + 2vh_v - du - h]X_4,$$

$$W_4 = [2r(b + f_v) + u^2f_r + u]X_1 + [2r(\hat{b} + g_v) + u^2(\hat{c} + g_r)]X_3$$

$$+ [\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + u^2h_r + 2rh_v]X_4,$$

$$W_5 = (\epsilon\lambda + au + bv + f)X_1 + X_2,$$

$$W_6 = [-\epsilon\delta v(pu + qv + f - \epsilon\delta g) + r(du + \hat{c}v + h)]X_1 - \epsilon\delta vX_2,$$

$$W_7 = rX_1 - uX_2,$$

$$W_8 = [u(pu + qv + f - \epsilon\delta g) + r - \epsilon\delta\hat{c}ru]X_3 - \epsilon\delta u(du + h)X_4,$$

$$W_9 = rX_3 - vX_4,$$

$$W_{10} = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g)X_3 + (du + h)X_4,$$

$$W_{11} = uvX_1 - rX_2,$$

$$W_{12} = v(u^2X_3 - rX_4),$$

$$W_{13} = (du + h)(u^2X_3 - rX_4),$$

where $p = a - \epsilon\delta\hat{a}$, $q = b - \epsilon\delta\hat{b}$.

From $\tilde{T}G$ we now choose elements which contain X_1 and X_3 only. We have

$$Y_1 \equiv \frac{1}{2} u(du + h)W_1 + \epsilon\delta(2du + h + uh_u + rh_r)W_8$$

$$\begin{aligned}
&= u(du+h)[(a+f_u)u+rf_r]X_1 + [u(du+h)(\hat{a}u+ug_u+r(\hat{c}+g_r)) \\
&\quad + \epsilon\delta(2du+h+uh_u+rh_r)(pu^2+quv+(f-\epsilon\delta g)u+r-\epsilon\delta\hat{c}ru)]X_3, \\
Y_2 &\equiv \frac{1}{2}vW_1 + (2du+h+uh_u+rh_r)W_9 \\
&= [uv(a+f_u)+rvf_r]X_1 + [uv(\hat{a}+g_u)+rv(\hat{c}+g_r)+r(2du+h+uh_u+rh_r)]X_3, \\
Y_3 &\equiv u(du+h)W_3 + (du+h)W_7 + \epsilon\delta(rh_r+2vh_v-du-h)W_8 \\
&= [u(du+h)(2bv+2vf_v+rf_r)+r(du+h)]X_1 \\
&\quad + [u(du+h)(2\hat{b}v+2vg_v+r(\hat{c}+g_r))+\epsilon\delta(rh_r+2vh_v-du-h) \\
&\quad \times (pu^2+quv+(f-\epsilon\delta g)u+r-\epsilon\delta\hat{c}ru)]X_3, \\
Y_4 &\equiv vW_3 + \epsilon\delta W_6 + (rh_r+2vh_v-du-h)W_9 \\
&= [v(2bv+2vf_v+rf_r)-puv-qv^2+v(\epsilon\delta g-f)+\epsilon\delta r(du+\hat{c}v+h)]X_1 \\
&\quad + [2\hat{b}v^2+2v^2g_v+rv(\hat{c}+g_r)+r(rh_r+2vh_v-du-h)]X_3, \\
Y_5 &\equiv \frac{r}{2}(du+h)W_1 + (2du+h+uh_u+rh_r)W_{13} \\
&= r(du+h)(au+uf_u+rf_r)X_1 \\
&\quad + (du+h)[ru(\hat{a}+g_u)+r^2(\hat{c}+g_r)+u^2(2du+h+uh_u+rh_r)]X_3, \\
Y_6 &\equiv r(du+h)W_3 + (du+h)W_{11} + (rh_r+2vh_v-du-h)W_{13} \\
&= (du+h)[2brv+2rvf_v+r^2f_r+uv]X_1 \\
&\quad + (du+h)[2\hat{b}rv+2rvg_v+r^2(\hat{c}+g_r)+u^2(rh_r+2vh_v-du-h)]X_3, \\
Y_7 &\equiv rvW_3 + vW_{11} + (rh_r+2vh_v-du-h)W_{12} \\
&= v[2brv+2rvf_v+r^2f_r+uv]X_1 \\
&\quad + v[2\hat{b}rv+2rvg_v+r^2(\hat{c}+g_r)+u^2(rh_r+2vh_v-du-h)]X_3, \\
Y_8 &\equiv (du+h)(uW_2 + (\epsilon\lambda+au+bv+f)W_7 - rW_5 - W_{11}) \\
&\quad + \epsilon\delta(2r(d+h_u)+2uvh_r)W_8 \\
&= (du+h)[2ru(a+f_u)+2u^2vf_r]X_1 \\
&\quad + [(du+h)(2u^2(du+h)+2ru(\hat{a}+g_u)+2u^2v(\hat{c}+g_r)) \\
&\quad + \epsilon\delta(2r(d+h_u)+2uvh_r)(pu^2+quv+(f-\epsilon\delta g)u+r-\epsilon\delta\hat{c}ru)]X_3, \\
Y_9 &\equiv vW_2 + \epsilon\delta(\epsilon\lambda+au+bv+f)W_6
\end{aligned}$$

$$\begin{aligned}
& -[-puv - qv^2 + (\epsilon\delta g - f)v + \epsilon\delta r(du + \hat{c}v + h)]W_6 \\
& + (2r(d + h_u) + 2uvh_r)W_9 - \epsilon\delta(du + \hat{c}v + h)W_{11} - \epsilon\delta(-pu - qv + \epsilon\delta g - f)W_6 \\
& = [v^2 + 2rv(a + f_u) + 2uv^2f_r - \epsilon\delta(du + \hat{c}v + h)uv \\
& - (pu + qv + f - \epsilon\delta g)(v(pu + qv + f - \epsilon\delta g) - \epsilon\delta r(du + \hat{c}v + h))]X_1 \\
& + [2uv(du + h) + 2\hat{a}rv + 2rvq_u + 2uv^2(\hat{c} + g_r) + 2r^2(d + h_u) + 2ruvh_r]X_3, \\
Y_{10} & \equiv u(du + h)W_4 + \epsilon\delta(\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r + g + 2rh_v + u^2h_r)W_8 - (\epsilon\delta - \hat{c}u)W_{13} \\
& - \epsilon\delta(pu^2 + quv + u(f - \epsilon\delta g) + r - \epsilon\delta\hat{c}ru)W_{10} - [pu + qv + f - \epsilon\delta g]W_8 \\
& = u(du + h)(2r(b + f_v) + u^2f_r + u)X_1 \\
& + [(2r(\hat{b} + g_v) + u^2(\hat{c} + g_r))u(du + h) - (\epsilon\delta - \hat{c}u)u^2(du + h) \\
& + (pu^2 + quv + (f - \epsilon\delta g)u + r - \epsilon\delta\hat{c}ru) \\
& \times (-pu - qv - (f - \epsilon\delta g) + \epsilon\delta(2rh_v + u^2h_r))]X_3.
\end{aligned}$$

We now prove

PROPOSITION 5.5. *If $\epsilon\delta b\hat{a}\hat{b}\hat{d}(b - \epsilon\delta\hat{b})(d - a\hat{a})(-4\epsilon\delta\hat{d} - (a - \epsilon\delta\hat{a})^2) \neq 0$, then the following elements ($j=1, 3$) belong to $\tilde{\Gamma}G$:*

$$u^4X_j, \quad u^2vX_j, \quad uv^2X_j, \quad v^3X_j, \quad ru^2X_j, \quad ruvX_j, \quad rv^2X_j.$$

Proof. We note that we consider u^4X_j , not u^3X_j . We define V_j ($j=1, 2, \dots, 14$) by

$$\begin{aligned}
V_1 &= u^4X_1, & V_{j+1} &= u^{4-j}v^{j-1}X_1 \quad (j=1, 2, 3), \\
V_{j+4} &= ru^{3-j}v^{j-1}X_1 \quad (j=1, 2, 3), \\
V_8 &= u^4X_3, & V_{8+j} &= u^{4-j}v^jX_3, \quad (j=1, 2, 3), \\
V_{j+11} &= ru^{3-j}v^{j-1}X_3 \quad (j=1, 2, 3),
\end{aligned}$$

The remaining part of the proof is similar to that of Proposition 4.6. We consider

$$vY_1, uY_2, vY_2, uY_3, vY_3, vY_4, Y_5, Y_6, Y_7, Y_8, uY_9, vY_9, -uY_{10}, -vY_{10}.$$

We have the following 14×14 matrix and the proof is completed by the fact that the determinant of this matrix is nonzero under the condition given above:

$$\begin{pmatrix}
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sigma & 0 \\
 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \hat{a} & 0 & 0 & 2d & \hat{c} & 0 \\
 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \hat{a} & 0 & 0 & 2d & \hat{c} \\
 0 & 0 & 0 & 0 & d & 0 & 0 & -\sigma p & 0 & 0 & 0 & -\sigma & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma & 0 \\
 0 & 0 & -p & \theta & 0 & \sigma & \epsilon \delta \hat{c} & 0 & 0 & 0 & 2\hat{b} & 0 & -d & \hat{c} \\
 0 & 0 & 0 & 0 & ad & 0 & 0 & 2d^2 & 0 & 0 & 0 & \hat{a}d & 0 & 0 \\
 0 & d & 0 & 0 & 0 & 2bd & 0 & -d^2 & 0 & 0 & 0 & 0 & 2\hat{b}d & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 0 & 0 & 0 & 2\hat{b} \\
 0 & 0 & 0 & 0 & 2ad & 0 & 0 & 2d^2 & 2\sigma & 0 & 0 & 2\sigma a & 2\sigma q & 0 \\
 0 & 0 & 1 & 0 & 0 & 2a & 0 & 0 & 0 & 0 & 0 & 0 & 2\hat{a} & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 2a & 0 & 0 & 0 & 0 & 0 & 0 & 2\hat{a} \\
 -d & 0 & 0 & 0 & 0 & 0 & 0 & \Theta & 0 & 0 & 0 & p & q & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & q
 \end{pmatrix}$$

where we have put $\theta = b + \epsilon \delta \hat{b}$, $\Theta = \epsilon \delta d + p^2$, $\sigma = \epsilon \delta d$. We omit the details of the computation. ■

As in §4, we procede in the following way. The diagram below shows that if we consider the left hand side multiplied by λ , u , v , r or 1, then the right hand side is proved to belong to $\tilde{I}G$.

$$\begin{array}{lll}
 rW_7 - uW_{11} \longrightarrow r^2X_1, & rW_9 - W_{12} \longrightarrow r^2X_3, & \\
 W_7 \longrightarrow u^3X_2, & W_6 \longrightarrow uvX_2, v^2X_2, & W_9 \longrightarrow u^2vX_4, uv^2X_4, v^3X_4, \\
 W_{12} \longrightarrow rvX_4, & W_{11} \longrightarrow ruX_2, rvX_2, r^2X_2, & W_7 \longrightarrow rvX_1, \\
 ruW_2 \longrightarrow r^2uX_4, & uW_{13} \longrightarrow ru^2X_4, & u^3W_1 \longrightarrow u^4X_4, \\
 W_8 \longrightarrow rvX_3, & W_9 \longrightarrow v^2X_4, & Y_9 \longrightarrow v^2X_1, \\
 uvW_5 \longrightarrow \lambda uvX_1, & \lambda W_{11} \longrightarrow \lambda rX_2, & rW_2 \longrightarrow r^2X_4, \\
 uW_{13} \longrightarrow ru^2X_4, & uvW_{10} \longrightarrow \lambda uvX_3, & u^3W_{10} \longrightarrow \lambda u^3W_3, \\
 \lambda W_{13} \longrightarrow \lambda ruX_4, & u^3W_4 \longrightarrow \lambda u^3X_4, & W_4 \longrightarrow \lambda uvX_4.
 \end{array}$$

We now consider $\frac{u^2}{2}W_1$, W_{13} , u^2W_3 , rW_3 , u^2W_4 , u^2W_{10} , $\frac{uv}{2}W_1$, u^2W_6 , λW_8 , rW_{10} , uW_7 and uW_8 . The fact that these 12 elements belong to $\tilde{I}G$ is represented as $BV \in (\tilde{I}G)^{12}$, where V is a vector given by

$$V = (u^3 X_1, \lambda u^2 X_1, u^3 X_3, \lambda u^2 X_3, u^3 X_4, \lambda u^2 X_4, u^2 X_2, \lambda r X_3, r X_2, ru X_1, ru X_3, ru X_4)$$

and B is a matrix whose entries are in \mathcal{E} and expressed as follows when $(\lambda, u, v, r) = (0, 0, 0, 0)$:

$$\begin{pmatrix} a & 0 & \hat{a} & 0 & 2d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d \\ 0 & 0 & 0 & 0 & -d & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -d \\ 1 & 0 & 0 & 0 & \hat{a} & \delta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{a} & \delta & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & \hat{a} & 0 & 2d & 0 & 0 & 0 & 0 & 0 & 0 \\ a & \epsilon & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & -\epsilon\delta d & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & \hat{a} & d \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 & -\epsilon\delta d & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This matrix is nonsingular if the condition in Proposition 5.5 are satisfied.

We now have

PROPOSITION 5.6. *If*

$$\epsilon\delta b\hat{a}bd(b-\epsilon\delta\hat{b})(d-a\hat{a})(4\epsilon\delta d+(a-\epsilon\delta\hat{a})^2)\neq 0,$$

then the following elements belong to $\tilde{F}G$:

$$(5.7) \quad \mathcal{M}^3 X_j, r\mathcal{M}X_j \ (j=1, 3, 4), \mathcal{M}^2 X_2, \\ rX_2, uvX_1, v^2 X_1, uvX_3, v^2 X_3, \lambda vX_4, uvX_4, v^2 X_4, \lambda vX_1, \lambda vX_3, vX_2.$$

Proof. Since the matrix B above is nonsingular under this condition, all the components of V belong to $\tilde{F}G$. We have the following diagram:

$$\begin{array}{lll} W_{11} \longrightarrow uvX_1, & Y_2 \longrightarrow uvX_3, & Y_4 \longrightarrow v^2 X_3, \\ rW_5 \longrightarrow \lambda rX_1, & rW_{10} \longrightarrow \lambda rX_3, & vW_{10} \longrightarrow \lambda vX_3, \\ W_6 \longrightarrow vX_2, & vW_6 \longrightarrow \lambda vX_1, & Y_4 \longrightarrow v^2 X_3. \end{array}$$

Remaining part of the proof is easy. ■

By this proposition and the condition imposed on f, g, h , the \mathcal{E} -module $\tilde{F}G$

is generated by those in (5.7) and the following 9 elements :

$$\begin{aligned}
Z_1 &= auX_1 + (\hat{a}u + 2\hat{c}r)X_3 + 2duX_4, \\
Z_2 &= (v + 2ar)X_1 + (\epsilon\lambda + au)X_2 + (2du^2 + 2\hat{a}r)X_3 + 2drX_4, \\
Z_3 &= 2bvX_1 + X_2 + (2\hat{b}v + \hat{c}r)X_3 - duX_4, \\
Z_4 &= (2br + u)X_1 + (2\hat{b}r + u^2\hat{c})X_3 + (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r)X_4, \\
Z_5 &= (\epsilon\lambda + au + bv)X_1 + X_2, \\
Z_6 &= rX_1 - uX_2, \\
Z_7 &= (pu^2 + r)X_3 - \epsilon\delta du^2X_4, \\
Z_8 &= rX_3 - vX_4, \\
Z_9 &= (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r)X_3 + duX_4.
\end{aligned}$$

Therefore $\tilde{\Gamma}G$ is independent of f, g, h . Now we have proved

THEOREM 5.1. *If we assume the conditions in Propositions 5.3 and 5.6, then the bifurcation equation (5.5, 6) are $O(2)$ -equivalent to $G = (G_1, G_2)$ with*

$$(5.8) \quad G_1 = (\epsilon\lambda + au + bv)\xi + \bar{\xi}\zeta,$$

$$(5.9) \quad G_2 = (\delta\lambda + \hat{a}u + \hat{b}v + \hat{c}r)\zeta + du\xi^2,$$

By the computations similar to that of Theorem 4.2 we obtain

THEOREM 5.2. *Let G be given by (5.8, 9). In addition to the assumption in Theorem 5.1, we assume that $a \neq 0$. Then the $O(2)$ -codimension of G is 6. The following $F = (F_1, F_2)$ is a universal unfolding of G :*

$$F_1 = (\epsilon\lambda + \alpha + au + bu)\xi + \bar{\xi}\zeta$$

$$F_2 = (\delta\lambda + (\hat{a} + s_1)u + (\hat{b} + s_2)v + (\hat{c} + s_3)r)\zeta + (\beta + (d + s_4)u)\xi^2,$$

where $\alpha, \beta, s_1, s_2, s_3$ and s_4 are unfolding parameters.

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