Fundamental Solutions and Eigenfunction Expansions for Schrödinger Operators III. Complex Potentials

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Abstract

A simplified proof of the eigenfunction expansion theorems for Schrödinger operators is given. These results also include a generalization to complex potentials of the results obtained for real potentials in the previous parts [8] and [4].

Introduction

The purpose of the present paper, Part III, is twofold. The first and main purpose is to give a simplified proof of eigenfunction expansion theorems discussed in the previous Part II [4]. Namely, in constructing eigenfunctions we utilize Riesz lemma in this Part instead of the concrete expression of the fundamental solutions constructed in Part I [8]. In this sense our new proof is purely of functional analytic nature and hopefully is more transparent than before. The second purpose is to extend our results in Part II to include complex valued $\mathscr{B}^\infty$ potentials $V(\omega; x)$ depending analytically on a complex parameter $\omega \in B$ with B a domain in the complex plane $\boldsymbol{C}=\{x+iy|x, y\in R^1\}$, $i=\sqrt{-1}$. The analytical dependency on $\omega\in B$ is included here in order to discuss an analytic extension of eigenfunctions in the case of dilation analytic potentials elsewhere.

We consider the Schrödinger operator

$$(1) \qquad H(\omega)=H_0+V(\omega; x), \quad H_0=-\frac{1}{2}\sum_{j=1}^n \partial^2/\partial x_j^2, \quad n\geqq 1,$$

and assume that $V(\omega; x)$ belongs to $\mathscr{B}^\infty=\mathscr{B}^\infty(R^n_x)$ locally uniformly in $\omega\in B$ and is analytic in $\omega\in B$ for each fixed $x\in R^n$. Then $H(\omega)$ is an anlytic family of type (A) with domain $\mathscr{D}(H(\omega))=H^2(R^n)$, the Sobolev space of order two. We further assume the existence and intertwining property of some wave operators and the related properties of the fundamental solution $U(\omega; t)=\exp(-itH(\omega))$ for $H(\omega)$. These assumptions will be given in a general form in Sect. 1. Under these assumptions, we construct eigenfunctions which expand the range of wave operators using the Riesz lemma instead of the results in Part I. We then apply these results in Sect. 2 to the weakly coupled N-body systems ($N\geqq 2$) with very short-range complex potentials and to the systems with real N-body potentials plus complex 2-body short-range potentials.

We use the same notation and conventions as in Part I and Part II. We quote the sections, formulae, theorems, and references in these Parts as Sect. II. 1 for sections, (I.1.5) for formulae, Theorem II.1 for theorems, and [I.1] for references.

§ 1. General theory

Our first assumption on $V(\omega; x)$ is an analytic version of Assumption I in Part I:

ASSUMPTION V. i) For any compact set K of B and multi-index α,

$$M(\alpha, K)\equiv \sup_{\omega\in K, x\in R^n} |\partial_x^\alpha V(\omega; x)|<\infty .$$

ii) For each $x\in R^n$, $V(\omega; x)$ is analytic in $\omega\in B$.

Under this assumption, Theorem I.1 guarantees the unique existence of the fundamental solution $U(\omega; t)=\exp(-itH(\omega))$ of

$$(1.1)\qquad (D_t+H(\omega))U(\omega; t)=0, \qquad U(\omega; 0)=I,$$

and gives a representation (I.1.4-1.5) of $U(\omega; t)$ as a Fourier integral operator, which is bounded in $B(\mathcal{H})$, $\mathcal{H}\equiv L^2(R^n)$, locally uniformly in $\omega\in B$ for each $t\in R^1$ and forms a group of operators in $B(\mathcal{H})$ with respect to t by (I.2.25). (We should state here a correction that the bar over $V(t, x)$ in the integrands of (I.2.4-2.5) and in the definition of the symbol of $\Lambda(t, s, y)$ in Sect. I.2 must be deleted). Further, since in this case $U(\omega; t)$ can also be expressed as

$$(1.2)\qquad U(\omega; t)=e^{-itH_0}(I+\sum_{\nu=1}^{\infty}\int_0^t dt_1 \cdots \int_0^{t_{\nu-1}} dt_\nu V(\omega; t_1)\cdots V(\omega; t_\nu)),$$

where $V(\omega; t)=e^{itH_0}V(\omega)e^{-itH_0}$, $V(\omega)=V(\omega; x)\cdot$, one sees by Assumption V-i) that for $\omega\in K$ and $t\in R^1$

$$(1.3)\qquad \|U(\omega; t)\|_{B(\mathcal{H})}\leqq e^{M(0,K)t}.$$

Thus $z\in \boldsymbol{C}$ belongs to the resolvent set $\rho(H(\omega))$ of $H(\omega)$ if $|\mathrm{Im}\, z|>M(0, K)$ and $\omega\in K$. Since $\mathcal{D}(H(\omega))=H^2(R^n)$ independent of $\omega\in B$ and $V(\omega)$ is analytic in $\omega\in B$ in $B(\mathcal{H})$ by Assumption V, $\{H(\omega)\}_{\omega\in B}$ therefore forms an analytic family of type (A) (for the definition and related properties, see [7, VII.2] and [11, XII.2]). We also note that Assumption V and the expansion (1.2) imply that $\exp(-itH(\omega))$ is analytic in $B(\mathcal{H})$ with respect to $\omega\in B$.

We now consider the same situation as in Sect. II.1. Namely, the variable $x \in R^n$, $n \geqq 1$, is decomposed as $x=(x_1, x_2)$, $x_j \in R^{n_j}$ ($n_1 \geqq 1$, $n_2 \geqq 0$, $n=n_1+n_2$), and the conjugate variables of x, x_j are denoted by ξ, ξ_j, respectively. Accordingly, $\mathcal{H}=L^2(R^n)$ is decomposed as a tensor product:

$$\mathcal{H}=\mathcal{H}_1 \otimes \mathcal{H}_2, \quad \mathcal{H}_j \equiv L^2(R^{n_j}_{x_j}) \quad (j=1, 2) \quad (\mathcal{H}_2=\boldsymbol{C} \text{ if } n_2=0). \tag{1.4}$$

Fix one $\varphi \in H^2(R^{n_2}_{x_2})$ with $\|\varphi\|_{\mathcal{H}_2}=1$ and define $P_2 f=(f, \varphi)\varphi$ for $f \in \mathcal{H}_2$. Let $W(t, \xi_1) \in C^\infty(R^{n_1}-\{0\})$ $(t \in R^1)$ and $T(\xi_1) \in C^\infty(R^{n_1})$ be real-valued with $|\partial^\alpha_{\xi_1} T(\xi_1)| \leqq C_\alpha \langle \xi_1 \rangle^2$ for all α, and define $W(t)=\mathcal{F}_1^{-1}[W(t, \xi_1)\cdot]\mathcal{F}_1$ and $T(D_{x_1})=\mathcal{F}_1^{-1}[T(\xi_1)\cdot]\mathcal{F}_1$, where $\mathcal{F}_j$ and $\mathcal{F}$ are Fourier transformations from $\mathcal{H}_j$ to $\hat{\mathcal{H}}_j \equiv L^2(R^{n_j}_{\xi_j})$ and from $\mathcal{H}$ to $\hat{\mathcal{H}} \equiv L^2(R^n_\xi)$, respectively. Let $E_1(\Delta)$ be the spectral measure for the self-adjoint operator $T(D_{x_1})$ in $\mathcal{H}_1$, and let $\lambda \in R^1$. Under these circumstances, we make the following assumption which is a version of Assumptions III$_\pm$ and IV$_\pm$ in Part II.

ASSUMPTION VI. There exist a Borel set Δ of R^1 and a family $\{P_\pm(\omega; \Delta)\}_{\omega \in B}$ of bounded (not necessarily orthogonal) projections on $\mathcal{H}$ satisfying the following conditions:

i) $P_\pm(\omega; \Delta)$ is analytic in $B(\mathcal{H})$ with respect to $\omega \in B$ and commutes with $H(\omega)$: $P_\pm(\omega; \Delta)H(\omega) \subset H(\omega)P_\pm(\omega; \Delta)$.

ii) For any compact subset K of B,

$$\sup_{\omega \in K,\, t \in R^1} \|\exp(-itH(\omega))P_\pm(\omega; \Delta)\|_{B(\mathcal{H})} < \infty. \tag{1.5}$$

iii) For any $\omega \in B$,

$$\lim_{\omega' \to \omega,\, \omega' \in B} \sup_{t \in R^1} \|e^{-itH(\omega')} P_\pm(\omega'; \Delta) - e^{-itH(\omega)} P_\pm(\omega; \Delta)\|_{B(\mathcal{H})} = 0. \tag{1.6}$$

iv)$_\pm$ For each $\omega \in B$, there exist the strong limits on $\mathcal{H}$:

$$W_\pm(\omega; \Delta) = \operatorname*{s-lim}_{t \to \pm\infty} P_\pm(\omega; \Delta) e^{itH(\omega)} (E_1(\Delta) \otimes P_2)(e^{-iW(t)} \otimes e^{-it\lambda} I_2), \tag{1.7}$$

$$Y_\pm(\omega; \Delta) = \operatorname*{s-lim}_{t \to \pm\infty} P_\pm(\omega; \Delta)^* e^{itH(\omega)^*} (E_1(\Delta) \otimes P_2)(e^{-iW(t)} \otimes e^{-it\lambda} I_2). \tag{1.8}$$

v)$_\pm$ For each $\omega \in B$, one has on $\mathcal{D}(T(D_{x_1}) \otimes I_2 + \lambda I_1 \otimes I_2)$,

$$H(\omega) W_\pm(\omega; \Delta) = W_\pm(\omega; \Delta)(T(D_{x_1}) \otimes I_2 + \lambda I_1 \otimes I_2), \tag{1.9}$$

$$H(\omega)^* Y_\pm(\omega; \Delta) = Y_\pm(\omega; \Delta)(T(D_{x_1}) \otimes I_2 + \lambda I_1 \otimes I_2). \tag{1.10}$$

vi)$_\pm$ For each $\omega \in B$,

$$P_\pm(\omega; \Delta) = W_\pm(\omega; \Delta) Y_\pm(\omega; \Delta)^*, \tag{1.11}$$

$$E_1(\Delta) \otimes P_2 = Y_\pm(\omega; \Delta)^* W_\pm(\omega; \Delta). \tag{1.12}$$

We call $W_\pm(\omega; \Delta)$ the (modified) wave operators, which are not necessarily partial isometries on $\mathcal{H}$. But $W_\pm(\omega; \Delta)|_{(E_1(\Delta) \otimes P_2)\mathcal{H}}$ has inverse $Y_\pm(\omega; \Delta)^*|_{P_\pm(\omega; \Delta)\mathcal{H}}$

by (1.11) and (1.12), hence the range $\mathcal{R}(W_\pm(\omega\,;\,\Delta)|_{(E_1(\Delta)\otimes P_2)\mathcal{H}})$ equals $P_\pm(\omega\,;\,\Delta)\mathcal{H}$.

We now define $\mathcal{F}_\pm(\omega\,;\,\Delta)=\mathcal{F}Y_\pm(\omega\,;\,\Delta)^*$ and $\tilde{\mathcal{F}}_\pm(\omega\,;\,\Delta)=W_\pm(\omega\,;\,\Delta)\mathcal{F}^{-1}$. Then

$$(1.13)\qquad \tilde{\mathcal{F}}_\pm(\omega\,;\,\Delta)\mathcal{F}_\pm(\omega\,;\,\Delta)=P_\pm(\omega\,;\,\Delta)\,,$$
$$\mathcal{F}_\pm(\omega\,;\,\Delta)\tilde{\mathcal{F}}_\pm(\omega\,;\,\Delta)=\hat{E}_1(\Delta)\otimes\hat{P}_2\,,$$

by (1.11) and (1.12), where $\hat{E}_1(\Delta)=\mathcal{F}_1E_1(\Delta)\mathcal{F}_1^{-1}$ and $\hat{P}_2=\mathcal{F}_2P_2\mathcal{F}_2^{-1}$. We set $\mathcal{H}_1(\Delta)=\hat{E}_1(\Delta)\mathcal{H}_1=L^2(\Gamma_1(\Delta))$, where $\Gamma_1(\Delta)=\{\xi_1\in R^{n_1}|\,T(\xi_1)\in\Delta\}$. Our main result of this section is the following

THEOREM 1. *Let Assumptions V and VI be satisfied. Then for each* $\omega\in B$, *there exist functions* $\tilde{\phi}_\pm(\omega\,;\,x,\,\xi_1)$, $\phi_\pm(\omega\,;\,x,\,\xi_1)\in H^k_{-s}(R^n_x)\otimes L^2_{-m_0-k}(\Gamma_1(\Delta))$ *for all real* $s>n/2$, *integer* $k\geqq 0$, *and even integers* $m_0>n/2$, *hence* $\in C^\infty(R^n_x,\,\mathcal{H}_{1,\mathrm{loc}}(\Delta))$, *which satisfy the following properties.*

i) *For* $f,\,g\in\mathcal{S}=\mathcal{S}(R^n)$

$$(1.14)\qquad (\tilde{\mathcal{F}}_\pm(\omega\,;\,\Delta)f,\,g)$$
$$=(2\pi)^{-n/2}<(\tilde{\phi}_\pm(\omega\,;\,x,\,\xi_1)\otimes\overline{\varphi(\xi_2)},\,\overline{g(x)}\otimes(\hat{E}_1(\Delta)\otimes I_2)f(\xi)>\,,$$
$$(1.15)\qquad (f,\,\mathcal{F}_\pm(\omega\,;\,\Delta)g)$$
$$=(2\pi)^{-n/2}<\overline{\phi_\pm(\omega\,;\,x,\,\xi_1)}\otimes\overline{\varphi(\xi_2)},\,\overline{g(x)}\otimes(\hat{E}_1(\Delta)\otimes I_2)f(\xi)>\,.$$

The conditions (1.14) *and* (1.15) *determine* $\tilde{\phi}_\pm$ *and* $\phi_\pm$ *uniquely, respectively.*

ii) *The functions* $\tilde{\phi}_\pm(\omega\,;\,x,\,\xi_1)$ *and* $\phi_\pm(\omega\,;\,x,\,\xi_1)$ *are analytic with respect to* $\omega\in B$ *in* $C^\infty(R^n_x,\,\mathcal{H}_{1,\mathrm{loc}}(\Delta))$, *and satisfy*

$$(1.16)\qquad (H(\omega)-(T(\xi_1)+\lambda))\tilde{\phi}_\pm(\omega\,;\,x,\,\xi_1)=0,$$
$$(H(\omega)-(T(\xi_1)+\lambda))\phi_\pm(\omega\,;\,x,\,\xi_1)=0.$$

Here the analyticity in $C^\infty(R^n_x,\,\mathcal{H}_{1,\mathrm{loc}}(\Delta))$ *means the analyticity in* $H^k_{-s}(R^n_x)\otimes L^2_{-m_0-k}(\Gamma_1(\Delta))$ *for any real* $s>n/2$, *integer* $k\geqq 0$, *and even integer* $m_0>n/2$.

iii) *If* $V(\omega\,;\,x)$ *is real and* $P_\pm(\omega\,;\,\Delta)$ *is self-adjoint for real* $\omega\in B$, *then* $\tilde{\phi}_\pm(\omega\,;\,x,\,\xi_1)$ *and* $\overline{\phi_\pm(\bar{\omega}\,;\,x,\,\xi_1)}$ *coincide with each other for* ω, $\bar{\omega}\in B$ *in* $C^\infty(R^n_x,\,\mathcal{H}_{1,\mathrm{loc}}(\Delta))$.

REMARKS. 1° By (1.13), this theorem gives an eigenfunction expansion on $\mathcal{R}(W_\pm(\omega\,;\,\Delta)|_{(E_1(\Delta)\otimes P_2)\mathcal{H}})=P_\pm(\omega\,;\,\Delta)\mathcal{H}$.

2° When $V(x)$ is real-valued, this theorem includes Theorems II.1 and II.2 as a special case with $\Delta=R^1$, $P_\pm(\omega\,;\,\Delta)=I$, and $V(\omega)$ a constant self-adjoint operator $V(x)\cdot$ except for the pointwise bound result (II.2.3). Notice that we have adopted here in (1.15) the complex conjugate of $\phi_\pm(x,\,\xi_1)$ in (II.1.6) as eigenfunctions $\phi_\pm(\omega\,;\,x,\,\xi_1)$.

Proof of Theorem 1. We first prove i) and ii) only for $\tilde{\mathcal{F}}_+(\omega\,;\,\Delta)$. The other cases can be treated similarly. For $f\in\mathscr{H}$, we have by Assumption VI-iv)$_+$-(1.7)

$$\tilde{\mathscr{F}}_+(\omega\,;\,\Delta)\mathscr{F}f(x)=W_+(\omega\,;\,\Delta)f(x)=\langle D_x\rangle^{m_0}\operatorname*{s-lim}_{t\to\infty}Z(\omega,\ t)f(x) \tag{1.17}$$

in $\mathscr{H}$. Here m_0 is an even integer greater than $n/2$, s-lim means the strong limit in $H^{m_0}(R_x^n)\subset\mathscr{H}$, and with $\chi_0(\xi)=\langle\xi\rangle^{-m_0}\in\hat{\mathscr{H}}$

$$\begin{aligned}&Z(\omega,\ t)f(x)\\&=\chi_0(D_x)e^{itH(\omega)}P_+(\omega\,;\,\Delta)(E_1(\Delta)\otimes P_2)(e^{-iW(t)}\otimes e^{-it\lambda})f(x).\end{aligned} \tag{1.18}$$

Since $\chi_0\in\hat{\mathscr{H}}$ and

$$\chi_0(D_x)g(x)=(2\pi)^{-n/2}\int e^{ix\xi}\,\chi_0(\xi)(\mathscr{F}g)(\xi)d\xi \tag{1.19}$$

for $g\in\mathscr{H}$, we have from (1.18)

$$\begin{aligned}&Z(\omega,\ t)f(x)\\&=(2\pi)^{-n/2}(\mathscr{F}e^{itH(\omega)}P_+(\omega\,;\,\Delta)(E_1(\Delta)\otimes P_2)(e^{-iW(t)}\otimes e^{-it\lambda})f,\ e^{-ix\xi}\,\chi_0(\xi))_{\hat{\mathscr{H}}}.\end{aligned} \tag{1.20}$$

Since $e^{itH(\omega)}$ is analytic in $B(\mathscr{H})$ with respect to $\omega\in B$ as remarked after (1.3), we have using Assumption VI-i) and (1.20) that

(1.21) $Z(\omega,\ t)f(x)$ is analytic in $\omega\in B$ for each $x\in R^n$ and $t\in R^1$.

On the other hand, from (1.20) and Assumption VI-ii), it easily follows that

$$\sup_{\omega\in K,\,t,\,x}|Z(\omega,\ t)f(x)|\leqq C_K\|\chi_0\|\,\|(E_1(\Delta)\otimes P_2)f\|, \tag{1.22}$$

$$\sup_{\omega\in K,\,t,\,x}|Z(\omega,\ t)f(x)-Z(\omega,\ t)h(x)|\leqq C_K\|\chi_0\|\,\|f-h\|, \tag{1.23}$$

$$\begin{aligned}&\sup_{t,\,x}|Z(\omega',\ t)f(x)-Z(\omega,\ t)f(x)|\\&\leqq C\|\chi_0\|\,\|f\|\sup_t\|e^{itH(\omega')}P_+(\omega'\,;\,\Delta)-e^{itH(\omega)}P_+(\omega\,;\,\Delta)\|_{B(\mathscr{H})},\end{aligned} \tag{1.24}$$

for some constants C_K and $C>0$ with C independent of $\omega,\ \omega'\in B$.

Given a countable dense subset $\tilde{B}$ of B, $f\in\mathscr{H}$, and a sequence $\{t_l\}$ tending to infinity as $l\to\infty$, we can choose, by Assumption VI-iv)$_+$-(1.7) and a diagonal argument, a subsequence $t_k=t_k(f)$ of $\{t_l\}$ and a null set $N=N(f)$ of R^n such that, for all $\omega\in\tilde{B}$ and $x\in R^n-N(f)$, we have the existence of the following limit and the relation

$$\lim_{k\to\infty}Z(\omega,\ t_k)f(x)=\chi_0(D_x)W_+(\omega\,;\,\Delta)f(x). \tag{1.25}$$

Now given $\varepsilon>0$ and $\omega\in B$, we can take $\omega_0\in\tilde{B}$ by (1.24) and Assumption VI-iii) so that

$$\sup_{t,x}|Z(\omega, t)f(x)-Z(\omega_0, t)f(x)|<\varepsilon/3. \tag{1.26}$$

Thus one has for all $x\in R^n-N$ and integers k, $m>1$

$$\begin{aligned}&|Z(\omega, t_k)f(x)-Z(\omega, t_m)f(x)|\\&<2\varepsilon/3+|Z(\omega_0, t_k)f(x)-Z(\omega_0, t_m)f(x)|.\end{aligned} \tag{1.27}$$

The right hand side is asymptotically less than ε as $k, m\to\infty$, since $\omega_0\in\tilde{B}$. We have so far proved that for any $f\in\mathcal{H}$ and any sequence $t_l\to\infty$ (as $l\to\infty$) there are a subsequence $\{t_k(f)\}$ of $\{t_l\}$ and a null set $N(f)$ of R^n such that for all $x\in R^n-N(f)$ and $\omega\in B$, there exists the limit

$$\lim_{k\to\infty} Z(\omega, t_k(f))f(x), \tag{1.28}$$

which equals

$$\chi_0(D_x)W_+(\omega; \Delta)f(x) \tag{1.29}$$

for a.e. $x\in R^n$.

Now using (1.23) and arguing quite similarly to the proof of Theorem II.1, we can prove that there exist a null set N_+ of R^n and a sequence $t_k^+\to\infty$ (as $k\to\infty$) such that, for all $x\in R^n-N_+$, $\omega\in B$ and $f\in\mathcal{H}$, the limit

$$\lim_{k\to\infty} Z(\omega, t_k^+)f(x) \tag{1.30}$$

exists, and is equal for a.e. $x\in R^n$ to

$$\chi_0(D_x)W_+(\omega; \Delta)f(x). \tag{1.31}$$

Since (1.30) is continuous in $\omega\in B$ by (1.24) and Assumption VI-iii), we see by (1.21), (1.22) and Morera's theorem that the limit (1.30) is analytic in $\omega\in B$ for each $x\in R^n-N_+$ and $f\in\mathcal{H}$.

Taking $f=f_1\otimes f_2$ ($f_j\in\mathcal{H}_j$) in (1.20), we see by (1.22) and Riesz lemma that for each $t\in R^1$, $x\in R^n$ and $\omega\in B$ there exists a function $b_{\chi_0}(\omega; t, x, \xi_1)\in\hat{\mathcal{H}}_1(\Delta)$ such that

$$\begin{aligned}&Z(\omega, t)(f_1\otimes f_2)(x)\\&=(2\pi)^{-n/2}\int\hat{E}_1(\Delta)\hat{f}_1(\xi_1)b_{\chi_0}(\omega; t, x, \xi_1)d\xi_1\times(f_2, \varphi)\end{aligned} \tag{1.32}$$

and

$$\sup_{\omega\in K, t, x}\|b_{\chi_0}(\omega; t, x, \cdot)\|_{\mathcal{H}_1(\Delta)}\leqq C_K\|\chi_0\|. \tag{1.33}$$

Then, the existence of the limit (1.30) implies that of the following weak limit in $\hat{\mathcal{H}}_1(\Delta)$ for all $x \in R^n - N_+$ and $\omega \in B$:

$$(1.34) \qquad b_{\chi_0}^+(\omega\,;\, x,\, \xi_1) \equiv \text{w-}\lim_{k\to\infty} b_{\chi_0}(\omega\,;\, t_k^+,\, x,\, \xi_1).$$

By (1.33) we have for $x \in R^n - N_+$ and $\omega \in K$

$$(1.35) \qquad \|b_{\chi_0}^+(\omega\,;\, x,\, \cdot)\|_{\hat{\mathcal{H}}_1(\Delta)} \leqq C_K \|\chi_0\|.$$

Namely, $b_{\chi_0}^+(\omega\,;\, x,\, \xi_1) \in L^\infty(R_x^n,\, \hat{\mathcal{H}}_1(\Delta)) \subset H_{-s}^0(R_x^n) \otimes \hat{\mathcal{H}}_1(\Delta)$ $(s > n/2)$ locally uniformly in $\omega \in B$. Thus, since (1.30) is analytic in $\omega \in B$ for all $x \in R^n - N_+$ and $f \in \mathcal{H}$, so is (1.34) as an $\hat{\mathcal{H}}_1(\Delta)$-valued function of $\omega \in B$ for each $x \in R^n - N_+$ as well as an $H_{-s}^0(R_x^n) \otimes \hat{\mathcal{H}}_1(\Delta)$-valued function of $\omega \in B$, where $s > n/2$ and we have used Morera's theorem. Further from (1.30), (1.31), (1.32) and (1.34), we have for a.e. $x \in R^n$ and $f_j \in \mathcal{H}_j$

$$(1.36) \qquad \chi_0(D_x) W_+(\omega\,;\, \Delta)(f_1 \otimes f_2)(x)$$
$$= (2\pi)^{-n/2} \int b_{\chi_0}^+(\omega\,;\, x,\, \xi_1) \hat{E}_1(\Delta) \hat{f}_1(\xi_1) d\xi_1 \times (\hat{f}_2,\, \hat{\varphi}).$$

Thus we have constructed a distribution $\tilde{\phi}_+(\omega\,;\, x,\, \xi_1) = \langle D_x \rangle^{m_0} b_{\chi_0}^+(\omega\,;\, x,\, \xi_1) \in H_{-s}^{-m_0}(R_x^n) \otimes \hat{\mathcal{H}}_1(\Delta)$ $(s > n/2)$ which satisfies (1.14). The relation (1.16) is proved in quite the same way as in the proof of Theorem II.1 by using (1.14), the definition of $\tilde{\mathcal{F}}_\pm(\omega\,;\, \Delta)$, Assumptions V-i) and VI-v)$_+$ $-$(1.9).

By (1.3), there exists $\lambda_0 > 0$ such that $i\lambda_0 \in \rho(H(\omega))$ for all $\omega \in K$. We can therefore make an argument quite similar to the proof of Theorem II.2. Namely, similarly to (II.2.8) we get for $\omega \in K$ and even integers $k \geqq 0$

$$(1.37) \qquad \tilde{\phi}_+(\omega\,;\, x,\, \xi_1)$$
$$= (T(\xi_1) + \lambda - i\lambda_0)^{(m_0+k)/2} (H(\omega) - i\lambda_0)^{-(m_0+k)/2} \langle D_x \rangle^{m_0} b_{\chi_0}^+(\omega\,;\, x,\, \xi_1),$$

where we have used Assumption V-i) (see the proof of Theorem II.2). Thus $\tilde{\phi}_+(\omega\,;\, x,\, \xi_1) \in H_{-s}^k(R_x^n) \otimes L_{-m_0-k}^2(\Gamma_1(\Delta))$ for $\omega \in K$ and all even integers $k \geqq 0$, $m_0 > n/2$, and real $s > n/2$, and in particular $\tilde{\phi}_+(\omega\,;\, x,\, \xi_1) \in C^\infty(R_x^n,\, \hat{\mathcal{H}}_{1,\mathrm{loc}}(\Delta))$ for all $\omega \in B$. Now (1.37) and the fact that $\{H(\omega)\}_{\omega \in B}$ is an analytic family of type (A) and $b_{\chi_0}^+(\omega\,;\, x,\, \xi_1)$ is analytic in $H_{-s}^0(R_x^n) \otimes \hat{\mathcal{H}}_1(\Delta)$ imply that $\tilde{\phi}_+(\omega\,;\, x,\, \xi_1)$ is analytic with respect to $\omega \in B$ in $C^\infty(R_x^n,\, \hat{\mathcal{H}}_{1,\mathrm{loc}}(\Delta))$.

We finally prove iii). Under the assumption of iii), $H(\omega)^* = H(\omega)$ and $P_\pm(\omega\,;\, \Delta)^* = P_\pm(\omega\,;\, \Delta)$ for real $\omega \in B$. Hence for real $\omega \in B$

$$(1.38) \qquad \mathcal{F}_\pm(\omega\,;\, \Delta)^* = \tilde{\mathcal{F}}_\pm(\omega\,;\, \Delta).$$

Thus by the uniqueness result in i), $\overline{\phi_\pm(\bar{\omega}\,;\, x,\, \xi_1)} = \tilde{\phi}_\pm(\omega\,;\, x,\, \xi_1)$ for real $\omega \in B$, which together with the analyticity of $\phi_\pm(\omega\,;\, x,\, \xi_1)$ and $\tilde{\phi}_\pm(\omega\,;\, x,\, \xi_1)$ in $\omega \in B$ proves iii). The proof of Theorem 1 is complete.

REMARKS. 1° In the present case, $b_{\chi_0}(\omega\,;\, t,\, x,\, \xi_1)$ in (1.32) is given explicitly by

(1.39)
$$b_{\chi_0}(\omega;\,t,\,x,\,\xi_1)$$
$$=\int e^{-i[W(t,\,\xi_1)+t\lambda]}\overline{\hat{P}_+(\omega;\,\Delta)^*[e^{-ix\xi-it\xi^2/2}\,a^*_{\chi_0}(\omega;\,t,\,\xi,\,x)]}\,\hat{\varphi}(\xi_2)d\xi_2\,,$$

which and (1.34) give an extended version of (II.1.10). Here $\hat{P}_+(\omega;\,\Delta)=\mathcal{F}P_+(\omega;\,\Delta)\mathcal{F}^{-1}$; $\hat{\varphi}=\mathcal{F}_2\varphi$; and $a^*_{\chi_0}(\omega;\,t,\,\xi,\,x)$ is defined by (I.3.1) with $V(x)=\overline{V(\omega;\,x)}$ and $\chi(\xi,\,y)=\chi_0(\xi)\in\mathcal{H}$, and belongs to $\mathcal{H}$ for each ω, t, x by (I.3.1) and the L^2-boundedness of $E(t,\,0,\,x)$ in (I.2.19) (cf. Sect. I.3). (1.39) follows from (1.18) by noting that

(1.40)
$$\chi_0(D_x)e^{itH(\omega)}=(e^{-itH(\omega)^*}\chi_0(D_x))^*$$
$$=(e^{-itH_0}\,a^*_{\chi_0}(\omega;\,t,\,D_x,\,X'))^*,$$

which is seen by (I.3.4) and (I.3.2).

2° In Assumption VI-iv)$_\pm$-(1.7), (1.8), the factor $E_1(\Delta)e^{-iW(t)}=e^{-iW(t)}E_1(\Delta)$ can be replaced by any other bounded modified free propagators, e.g., by $J_1e^{-itT(D_{x_1})}E_1(\Delta)$ with $J_1\in B(\mathcal{H}_1)$. Also in this situation the proof of Theorem 1 works well once the assumptions are satisfied. Some examples of this type of modified propagators are found in e.g. [2], [3], [II.13].

§ 2. Complex short-range potentials

In this section, we treat the two examples of application of Theorem 1 as stated in the introduction.

2.1. *Weakly coupled N-body systems* ($N\geqq 2$) We consider the potentials of the form

(2.1)
$$\kappa V(\omega;\,X)=\kappa\sum_{i<j}V_{ij}(\omega;\,r_i-r_j)\,,\qquad X=(r_1,\,\cdots,\,r_N)\,,$$

where $i,\,j=1,\cdots,N$, $r_i\in R^\nu$ ($\nu\geqq 3$), $\kappa\in\boldsymbol{C}$, and $V_{ij}(\omega;\,r)$ ($r\in R^\nu$) satisfies Assumption V with $x=r$ and $n=\nu$ there. In addition, we assume

ASSUMPTION VII. $V_{ij}(\omega;\,r)$ $(1\leqq i<j\leqq N)$ satisfies

(2.2)
$$|V_{ij}(\omega;\,r)|\leqq C\langle r\rangle^{-2\varepsilon}$$

for $\omega\in B$ and $r\in R^\nu$ with some constants C and $\varepsilon>0$.

Choosing Jacobi coordinates $x=(x_1,\,\cdots,\,x_{N-1})\in R^n$, $x_j\in R^\nu$, $n=\nu(N-1)$, one can rewrite (2.1) as

(2.3)
$$\kappa V(\omega;\,x)=\kappa\sum_{i<j}V_{ij}(\omega;\,x^{(ij)})\,,$$

where $x^{(ij)}$ denotes the representation of $r_i - r_j$ by our Jacobi coordinates x. Then our Schrödinger operator is

$$H_\kappa(\omega) = H_0 + \kappa V(\omega\,;\, x), \qquad H_0 = -\Delta_x/2. \tag{2.4}$$

With this situation, we take in Assumption VI

$$\begin{gathered}\Delta = R^1,\ P_\pm(\omega\,;\, R^1) = I,\ n_1 = n,\ n_2 = 0,\ E_1(R^1) = I,\\ W(t, \xi) = tT(\xi),\ T(\xi) = \xi^2/2,\ \lambda = 0.\end{gathered} \tag{2.5}$$

Then

LEMMA. *There exists a constant $k = k(B) > 0$ such that, for $|\kappa| < k$, Assumption VI holds good.*

Proof. Assumption VI-i) is clearly satisfied. Set

$$C_l(\omega) = |V_l(\omega\,;\, x^l)|^{1/2} \tag{2.6}$$

for $\omega \in B$ and $l = (i, j)$, $1 \leqq i < j \leqq N$. Then by Assumption VII and Iorio-O'Carroll estimates ([1], see also [11, Theorem XIII. 27])

$$a_{lk}(\omega) \equiv \sup_{Z \in C - R^1} \|C_l(\omega)(H_0 - z)^{-1} C_k(\omega)\|_{B(\mathcal{H})} < \infty \tag{2.7}$$

for all $\omega \in B$ and $l = (i, j)$, $k = (i', j')$, $1 \leqq i < j \leqq N$, $1 \leqq i' < j' \leqq N$ with $a_{lk}(\omega)$ uniformly bounded in $\omega \in B$. Namely, all assumptions in Kato [6, Theorem 1.5] are satisfied with $T = H_0$, $\mathcal{H} = L^2(R^n)$, $\mathcal{H}' = \otimes_{l=1}^{L} \mathcal{H}$ $(L = \binom{N}{2})$, $A = (C_l(\omega))_l$, $B = ((\mathrm{sgn}\, V_l(\omega\,;\, x^l)) C_l(\omega))_l$, and $N = N(\omega) =$ norm of the matrix $(a_{lk}(\omega))_{lk}$, where $\mathrm{sgn}\, a = a/|a|$ $(a \neq 0)$, $= 0$ $(a = 0)$. Thus for $|\kappa| < k(B) \equiv \inf_{\omega \in B} \{N(\omega)^{-1}\}$, our Assumption VI-ii), iv)$_\pm$, v)$_\pm$ and vi)$_\pm$ hold by [6, Theorems 1.5, 3.9 and 4.1]. Assumption VI-iii) is proved as follows. We write

$$\begin{aligned}&((e^{-itH_\kappa(\omega')} - e^{-itH_\kappa(\omega)})f,\ g)\\ &= -i\kappa \sum_l \left(\int_0^t e^{-i(t-\tau)H_\kappa(\omega')} \{V_l(\omega'\,;\, x^l) - V_l(\omega\,;\, x^l)\} e^{-i\tau H_\kappa(\omega)} f\, d\tau,\ g\right).\end{aligned} \tag{2.8}$$

With $A = \langle x^l \rangle^{-1-\varepsilon/4}$, (2.8) is bounded by

$$\begin{aligned}&(2\pi)^2 |\kappa| \sum_l \sup_{x^l \in R^\nu} |\langle x^l \rangle^{2+\varepsilon/2} \{V_l(\omega'\,;\, x^l) - V_l(\omega\,;\, x^l)\}|\\ &\qquad \times \|A\|_{H_\kappa(\omega)} \|A\|_{H_\kappa(\omega')^*} \|f\|\, \|g\|,\end{aligned} \tag{2.9}$$

where

$$\|A\|_T^2 = \sup_{\|f\|=1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|Ae^{-itT} f\|^2 dt. \tag{2.10}$$

Since $\|A\|_{H_\kappa(\omega)} \leqq (1 - N(\omega)|\kappa|)^{-1} \|A\|_{H_0}$ and similarly for $\|A\|_{H_\kappa(\omega')^*}$ by [6, (1.5)],

$\langle x^l\rangle^{2+\varepsilon/2}V_l(\omega;x^l)$ is continuous with respect to $\omega\in B$ in $\mathscr{B}^0(R^\nu)$ by Assumptions V and VII, Assumption VI-iii) follows from (2.8) and (2.9). This concludes the proof of the lemma.

Thus we have proved

THEOREM 2. *Let Assumptions V and VII be satisfied for the pair potentials* $V_{ij}(\omega;r)$ *in* (2.1). *Let* $|\kappa|<k(B)=\inf_{\omega\in B}\{N(\omega)^{-1}\}$, *where* $N(\omega)$ *is the norm of the matrix* $(a_{lk}(\omega))_{lk}$ *defined by* (2.7). *Then the conclusions of Theorem* 1 *hold for the Hamiltonian* $H_\kappa(\omega)$ *in* (2.4). *Further in this case* $\mathscr{F}_\pm(\omega;R^1)$ *is a bicontinuous bijection from* $\mathscr{H}$ *onto* $\hat{\mathscr{H}}$ *with the inverse* $\tilde{\mathscr{F}}_\pm(\omega;R^1)$, *and* $\phi_\pm(\omega;x,\xi)$ *and* $\tilde{\phi}_\pm(\omega;x,\xi)$ *give eigenfunction expansions on* $\mathscr{H}$.

2.2. *Real N-body plus complex 2-body short-range potentials* In this case $V(\omega;x)$ has the form

$$V(\omega;x)=V_N(x)+V_S(\omega;x), \tag{2.11}$$

$$V_N(x)=\sum_{1\leqq i\leqq j\leqq N}V_{ij}(x^{(ij)}), \tag{2.12}$$

where $V_{ij}(r)\in\mathscr{B}^\infty(R^\nu)(\nu\geqq 3)$ is real-valued and $V_S(\omega;x)\in\mathscr{B}^\infty(R^n)$ $(n=\nu(N-1))$ is complex-valued. We further assume

ASSUMPTION VIII. There exist constants ε_1 and $\varepsilon>0$ such that:

i) For $|\alpha|\leqq 1$ and $r\in R^\nu$

$$|\partial_r^\alpha V_{ij}(r)|\leqq C\langle r\rangle^{-|\alpha|-\varepsilon_1}. \tag{2.13}$$

ii) For any compact set K of B, $\omega\in K$, and $x\in R^n$

$$|V_S(\omega;x)|\leqq C_K\langle x\rangle^{-1-\varepsilon}, \tag{2.14}$$

and $V_S(\omega;x)$ is analytic in $\omega\in B$ for each $x\in R^n$.

We set

$$H_1=H_0+V_N(x),\qquad H(\omega)=H_1+V_S(\omega;x), \tag{2.15}$$

$$A=\langle x\rangle^{-(1+\varepsilon)/2},\qquad B(\omega)=\langle x\rangle^{(1+\varepsilon)/2}V_S(\omega;x), \tag{2.16}$$

$$Q(\omega;z)=B(\omega)R_1(z)A,\quad R_1(z)=(H_1-z)^{-1},\quad \mathrm{Im}\,z\neq 0. \tag{2.17}$$

Then by the results of Tamura [12] (cf. also Perry-Sigal-Simon [10]) and Assumption VIII, we have

$$\sup_{z\in\Pi_\pm(\Delta_0),\omega\in K}\{\|AR_1(z)A\|_{B(\mathscr{H})}+\|B(\omega)R_1(z)B(\omega)^*\|_{B(\mathscr{H})}\}<\infty \tag{2.18}$$

for any compact set K of B and a bounded open set Δ_0 of R^1 bounded away from the point spectra and threshold points of H_1, where

(2.19) $$\Pi_{\pm}(\Delta_0)=\{\mu\pm i\nu|\mu\in\Delta_0,\ \nu>0\}.$$

(Note that, if $\varepsilon_1>1$ in (2.13), we have only to assume (2.13) for $\alpha=0$. See [12].) Further, $Q(\omega; z)$ is an analytic family of compact operators on $\mathcal{H}$ in $(\omega, z)\in B\times\Pi_{\pm}(\Delta_0)$, and for each $\omega\in B$ there exists the boundary value

(2.20) $$Q(\omega; \mu\pm i0)\equiv\lim_{\nu\downarrow 0} Q(\omega; \mu\pm i\nu)$$

in $B(\mathcal{H})$ which is continuous in $\mu\in\Delta_0$ and analytic in $\omega\in B$. We here make the following

Assumption IX. For all $\omega\in B$ and $\mu\in\Delta_0$, -1 is not an eigenvalue of $Q(\omega; \mu\pm i0)$.

Then the inverse $(1+Q(\omega; \mu\pm i0))^{-1}$ exists, belongs to $B(\mathcal{H})$ locally uniformly in $\omega\in B$ and $\mu\in\Delta_0$, and is continuous in $B(\mathcal{H})$ with respect to $\omega\in B$ and $\mu\in\Delta_0$. Thus all assumptions (A-1), (B-2) and (A-3)′ of Kako-Yajima [5] are satisfied for $T_1=H_1$, A, $B(\omega)$, and Δ_0, hence all results in [5] hold in our case. In particular, Lemma 2.1 of [5] implies the unique existence of an operator $E(\omega; \Delta)$ (for a bounded Borel set Δ with $\bar{\Delta}\subset\Delta_0$) belonging to $B(\mathcal{H})$ locally uniformly in $\omega\in B$ such that

(2.21) $$(E(\omega; \Delta)f, g)$$
$$=\lim_{\nu\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}((R(\omega; \mu+i\nu)-R(\omega; \mu-i\nu))f, g)d\mu$$

for $f, g\in\mathcal{H}$, where for $\operatorname{Im} z\neq 0$

(2.22) $$R(\omega; z)=R_1(z)-R_1(z)A(1+Q(\omega; z))^{-1}B(\omega)R_1(z)$$
$$=(H(\omega)-z)^{-1}.$$

$E(\omega; \Delta)$ is a spectral measure for $H(\omega)$ in the sense of [5, Theorem 2.2]. We fix one bounded Borel set Δ of R^1 with $\bar{\Delta}\subset\Delta_0$ in the following.

With the same notation as in Sect. II.3.4 with $V(x)=V_N(x)$, we set in Sect. 1 for the channel $\alpha=\begin{pmatrix}C_1 & \cdots & C_k\\ \eta_1 & \cdots & \eta_k\end{pmatrix}$

(2.23) $$n_1=\nu(k-1),\qquad n_2=\nu(N-k),$$
$$\varphi(x_2)=\varphi_\alpha(x_2)\equiv\prod_{l=1}^{k}\eta_l(x^{(C_l)}),\quad \lambda=\sum_{l=1}^{k}\lambda_l^{(\alpha)},$$
$$T(\xi_1)=-\frac{1}{2}\sum_{j=1}^{k-1}\xi_j^2.$$

In these circumstances, we assume for H_1 the following

ASSUMPTION X. There exists a real-valued function $W(t, \xi_1) \in C^\infty(R^{n_1}-\{0\})$ such that the strong limit

$$(2.24) \qquad \Omega_\pm(\Delta)=\Omega_\pm^\alpha(\Delta)$$

$$=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH_1}(E_1(\Delta_\lambda)\otimes P_2)(e^{-iW(t)}\otimes e^{-it\lambda} I_2)$$

exists on $\mathcal{H}$, where $\Delta_\lambda=\Delta-\lambda\equiv\{\mu-\lambda|\mu\in\Delta\}$, and satisfies on $\mathcal{D}(T(D_{x_1})\otimes I_2+\lambda I_1\otimes I_2)$

$$(2.25) \qquad H_1\Omega_\pm(\Delta)=\Omega_\pm(\Delta)(T(D_{x_1})\otimes I_2+\lambda I_1\otimes I_2).$$

When ε_1 in (2.13) is greater than 1, it is known that (2.24) exists and satisfies (2.25) with $W(t, \xi_1)=tT(\xi_1)$ (see [II.12]) and that $\bigoplus_\alpha \mathcal{R}(\Omega_\pm^\alpha(\Delta))=E_{H_1}(\Delta)\mathcal{H}$ (see [I.19]) (note that the summation here is over all α including α with $k=1$). For the case $1>\varepsilon_1>0$, see [II.6], [II.2].

By [5, Theorem 4.1] and its proof, the strong limits on $\mathcal{H}$

$$(2.26) \qquad \tilde{W}_\pm(\omega\,;\,\Delta)=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH(\omega)}E(\omega\,;\,\Delta)e^{-itH_1}E_{H_1}(\Delta),$$

$$\tilde{Y}_\pm(\omega\,;\,\Delta)=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH(\omega)^*}E(\omega\,;\,\Delta)^*e^{-itH_1}E_{H_1}(\Delta)$$

exist and are bounded locally uniformly in $\omega\in B$, where $E_{H_1}(\Delta)$ is the spectral measure for the self-adjoint operator H_1. Further by [5, Theorem 3.2], the inverse $(\tilde{W}_\pm(\omega\,;\,\Delta)|_{E_{H_1}(\Delta)\mathcal{H}})^{-1}=\tilde{Y}_\pm(\omega\,;\,\Delta)^*$ exists on $E(\omega\,;\,\Delta)\mathcal{H}$ and is bounded locally uniformly in $\omega\in B$. We set

$$(2.27) \qquad P_\pm(\omega\,;\,\Delta)=\tilde{W}_\pm(\omega\,;\,\Delta)P_{\mathcal{R}(\Omega_\pm(\Delta))}\tilde{Y}_\pm(\omega\,;\,\Delta)^*,$$

where $P_{\mathcal{R}}$ is the orthogonal projection onto the closed subspace $\mathcal{R}$ of $\mathcal{H}$. By (2.25) and the intertwining property of $\tilde{W}_\pm(\omega\,;\,\Delta)$ and $\tilde{Y}_\pm(\omega\,;\,\Delta)^*$ ([5, (3.4), (3.5)]), $P_\pm(\omega\,;\,\Delta)$ commutes with $H(\omega)$.

LEMMA. *Under the above situation, Assumptions V and VI are satisfied with $E_1(\Delta)$ replaced by $E_1(\Delta_\lambda)$.*

Proof. Assumption V is clearly satisfied. By [5, (3.7)] (cf. also [6, (3.11)])

$$(2.28) \qquad e^{-itH(\omega)}E(\omega\,;\,\Delta)=\tilde{W}_\pm(\omega\,;\,\Delta)e^{-itH_1}\tilde{W}_\pm(\omega\,;\,\Delta)^{-1}E(\omega\,;\,\Delta).$$

This and the local boundedness in $\omega\in B$ of $\tilde{W}_\pm(\omega\,;\,\Delta)$ and $\tilde{W}_\pm(\omega\,;\,\Delta)^{-1}E(\omega\,;\,\Delta)=\tilde{Y}_\pm(\omega\,;\,\Delta)^*$ yield

$$(2.29) \qquad \sup_{\omega\in K,\, t}\|e^{-itH(\omega)}E(\omega\,;\,\Delta)\|_{B(\mathcal{H})}<\infty$$

for any compact set K of B. On the other hand, since $Q(\omega\,;\,\mu\pm i0)$ is analytic in $\omega\in B$ in $B(\mathcal{H})$, so is the inverse $(1+Q(\omega\,;\,\mu\pm i0))^{-1}$ for $\mu\in\Delta$ by Assumption IX. Then (2.18), (2.21), (2.22), Assumption VIII-ii), and the analyticity in $\omega\in B$ of $e^{itH(\omega)}$ imply that $e^{itH(\omega)}E(\omega\,;\,\Delta)$ is analytic in $\omega\in B$ in $B(\mathcal{H})$. Thus by (2.26), (2.29), and Morera's theorem, $\tilde{W}_{\pm}(\omega\,;\,\varDelta)$ and $\tilde{Y}_{\pm}(\omega\,;\,\varDelta)$ are analytic with respect to $\omega\in B$ in $B(\mathcal{H})$, hence $P_{\pm}(\omega\,;\,\varDelta)$ is also analytic in $\omega\in B$ by (2.27), which proves Assumption VI-i). Assumption VI-ii) follows from (2.29) and the relation

$$E(\omega\,;\,\varDelta)P_{\pm}(\omega\,;\,\varDelta)=P_{\pm}(\omega\,;\,\varDelta), \tag{2.30}$$

which is seen by (2.26) and (2.27).

To prove Assumption VI-iii), by the relation (2.30) and the analyticity of $P_{\pm}(\omega\,;\,\varDelta)$ in $\omega\in B$, we have only to show (1.6) with $P_{\pm}(\omega\,;\,\varDelta)$ replaced by $E(\omega\,;\,\varDelta)$. For this purpose, we use the representation (see [5, (4.2), (2.7)])

$$\begin{aligned}(2.31)\qquad &(e^{-itH(\omega)}E(\omega\,;\,\varDelta)f,\ g)\\ &=\lim_{\nu\downarrow 0}\frac{1}{2\pi i}\int_{\varDelta}e^{-it\mu}((R(\omega\,;\,\mu+i\nu)-R(\omega\,;\,\mu-i\nu))f,\ g)d\mu\\ &=\frac{1}{2\pi i}\lim_{\nu\downarrow 0}\Big[\int_{\varDelta}e^{-it\mu}((R_1(\mu+i\nu)-R_1(\mu-i\nu))f,\ g)d\mu\\ &-\int_{\varDelta}e^{-it\mu}((1+Q(\omega\,;\,\mu+i\nu))^{-1}B(\omega)R_1(\mu+i\nu)f,\ AR_1(\mu-i\nu)g)d\mu\\ &+\int_{\varDelta}e^{-it\mu}((1+Q(\omega\,;\,\mu-i\nu))^{-1}B(\omega)R_1(\mu-i\nu)f,\ AR_1(\mu+i\nu)g)d\mu\Big].\end{aligned}$$

It thus suffices to consider only the second and third terms to prove Assumption VI-iii). We write the second integral on the right hand side of (2.31) as $I_{\varDelta}^{\nu}(t,\,\omega,\,f,\,g)$. Then writing $z=\mu+iv$, $v>0$, we have

$$\begin{aligned}(2.32)\qquad &d_{\varDelta}^{\nu}(t\,;\,\omega',\,\omega\,;\,f,\,g)\\ &\equiv|I_{\varDelta}^{\nu}(t,\,\omega',\,f,\,g)-I_{\varDelta}^{\nu}(t,\,\omega,\,f,\,g)|^2\\ &\leqq\Big(\int_{\varDelta}\|AR_1(\bar{z})g\|^2d\mu\Big)\times\\ &\times\Big(\int_{\varDelta}\|(1+Q(\omega'\,;\,z))^{-1}-(1+Q(\omega\,;\,z))^{-1}\|^2\|B(\omega')R_1(z)f\|^2d\mu\\ &+\int_{\varDelta}\|(1+Q(\omega\,;\,z))^{-1}\|^2\|(B(\omega')-B(\omega))\langle x\rangle^{\delta}\|^2\|\langle x\rangle^{-\delta}R_1(z)f\|^2d\mu\Big),\end{aligned}$$

where $1/2<\delta<(1+\varepsilon)/2$. The operators $B(\omega')$, $\langle x\rangle^{-\delta}$, and A are H_1-smooth on $\varDelta_0\supset\bar{\varDelta}$ locally uniformly in $\omega'\in B$ by (2.18). Since $(1+Q(\omega\,;\,\mu\pm i0))^{-1}$ and $B(\omega)\langle x\rangle^{\delta}$ are continuous with respect to $\omega\in B$ in $B(\mathcal{H})$, and $\|(1+Q(\omega\,;\,\mu\pm i0))^{-1}\|_{B(\mathcal{H})}$ is uniformly bounded in $\mu\in\varDelta$ by the compactness of $\bar{\varDelta}\subset\varDelta_0$, we thus see that

$$\lim_{\omega'\to\omega}\lim_{\nu\downarrow 0}\sup_{t,\|f\|=\|g\|=1}d_{\varDelta}^{\nu}(t,\,\omega',\,\omega\,;\,f,\,g)=0. \tag{2.33}$$

The third term on the right hand side of (2.31) is treated similarly, and we obtain Assumption VI-iii).

From the chain rule and the existence and intertwining property of the limits (2.24) and (2.26) follow Assumption VI-iv)$_\pm$, v)$_\pm$ and

$$W_\pm(\omega;\, \Delta)=\widetilde{W}_\pm(\omega;\, \Delta)\Omega_\pm(\Delta), \tag{2.34}$$

$$Y_\pm(\omega;\, \Delta)=\widetilde{Y}_\pm(\omega;\, \Delta)\Omega_\pm(\Delta).$$

Then Assumption VI-vi)$_\pm$ follows from (2.34), (2.27), (2.25), and the relations $\widetilde{Y}_\pm(\omega;\, \Delta)^*\widetilde{W}_\pm(\omega;\, \Delta)=E_{H_1}(\Delta)$ ([5, (3.6)]) and $\Omega_\pm(\Delta)^*\Omega_\pm(\Delta)=E_1(\Delta_\lambda)\otimes P_2$, which concludes the proof of the lemma.

Thus we have proved

THEOREM 3. *Let* $V_{ij}(r)\in\mathcal{B}^\infty(R^\nu)$ $(\nu\geqq 3)$ *be real-valued, and let* $V_S(\omega;\, x)\in\mathcal{B}^\infty(R^n)$ $(n=\nu(N-1))$ *be complex-valued, and let Assumption VIII be satisfied for* V_{ij} *and* V_S. *Let* Δ_0 *be an open set of* R^1 *bounded away from the point spectra and thresholds of* H_1 *and satisfy Assumption IX, and let* Δ *be a bounded Borel set with* $\bar{\Delta}\subset\Delta_0$ *satisfying Assumption X. Then the conclusions of Theorem 1 hold for the Hamiltonian* $H(\omega)$, *with* $\hat{E}_1(\Delta)$ *and* $\Gamma_1(\Delta)$ *replaced by* $\hat{E}_1(\Delta_\lambda)$ *and* $\Gamma_1(\Delta_\lambda)$ *respectively. Further in the case that* ε_1 *in Assumption VIII is greater than* 1, $\{\phi_{\pm,\alpha}(\omega;\, x,\, \xi_{a(\alpha)})\otimes\bar{\varphi}_\alpha(\xi^{a(\alpha)})\}_\alpha$ *and* $\{\tilde{\phi}_{\pm,\alpha}(\omega;\, x,\, \xi_{a(\alpha)})\otimes\overline{\tilde{\varphi}_\alpha(\xi^{a(\alpha)})}\}_\alpha$ *give eigenfunction expansions on* $E(\omega;\, \Delta)\mathcal{H}=\widetilde{W}_\pm(\omega;\, \Delta)(\bigoplus_\alpha \mathcal{R}(\Omega_\pm^\alpha(\Delta)))$ *in the sense of* (1.13). (*For the notation here, see Sect.* II.3.4. *For* α *with* $k=1$, *we follow the convention that* $\phi_{\pm,\alpha}=\tilde{\phi}_{\pm,\alpha}=1$).

We remark that this theorem gives an extension of the results of Mochizuki [9], where he assumes that $V_S(x)=V_S(\omega;\, x)$ $(x\in R^3)$ is independent of ω and satisfies (2.14) with $\varepsilon>1$, allowing some local singularities.

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