A Topological Characterization of Hyperfunctions with Real Analytic Parameters

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Abstract

Let $f(x, t)$ be a hyperfunction whose support with respect to x is contained in a compact set K. We prove that a (necessary and) sufficient condition for $f(x, t)$ to contain t as real analytic parameters is that $\int f(x, t)\varphi(x)dx$ becomes real analytic in t for any $\varphi(x)\in\mathcal{A}(K)$.

1. The notion of hyperfunctions with real analytic parameters t was introduced by M. Sato in [4] as a section of the sheaf $\mathcal{B}\mathcal{A}:=\mathcal{H}^n_{\boldsymbol{R}^{n+m}}(\mathcal{O}_{z,\tau}|_{\boldsymbol{C}^n\times\boldsymbol{R}^m})$. Later, with the development of the theory of S.S. it was re-interpreted as a hyperfunction which satisfies

$$\text{S.S.}\, f(x, t)\cap iS^{m-1}dt=\emptyset. \tag{1}$$

Let T denote an open subset of the parameter space, and consider such a hyperfunction $f(x, t)\in\mathcal{B}\mathcal{A}(\boldsymbol{R}^n\times T)$. Let K be a compact subset of $\boldsymbol{R}^n$. A typical consequence of these definitions is that if supp $f\subset K\times T$, then for any $\varphi(x)\in\mathcal{A}(K)$ the integral

$$\int_{\boldsymbol{R}^n}\varphi(x)f(x, t)dt \tag{2}$$

becomes a real analytic function of $t\in T$. In this paper we shall show that conversely the analyticity of (2) for any φ implies that f contains t as real analytic parameters.

The author posed this problem about 20 years ago, asked to researchers in hyperfunction theory at that time, and was informed of a negative answer from the S-K-K group, but without a proof. The author then sought a concrete counter-example from time to time, and recently thus came to an opposite conclusion.

Note that this is rather a problem of pure curiosity. For, we know further that for any $\varphi(t, \xi)\in\mathcal{A}(T\times U)$, where $\xi\in U$ is another set of arbitrary parameters, the integral

$$\int_{\boldsymbol{R}^n}\varphi(x, \xi)f(x, t)dx \tag{3}$$

becomes real analytic in the joint variables (t, ξ), and in this form the converse is known to be true by S-K-K from long ago. This fact has been sufficient for the applications hitherto.

The proof of the converse of this latter may be easily proved, e.g. employing Kashiwara's twisted Radon decomposition of the delta function:

$$\delta(x) = \int_{S^{n-1}} W(x, \omega)d\omega.$$

We omit the concrete form of $W(x, \omega)$ or its defining function $W(z, \omega)$. (See e.g. S-K-K [4], Chapter III, Example 1.2.5, or [2], (2.3.25)). Let $\Delta_j \subset \boldsymbol{R}^n$, $j=1, \ldots, N$ denote convex polyhedral cones whose dual cones $\Delta_j{}^\circ$ give a partition of $\boldsymbol{S}^{n-1}$. Put

$$W(z, \Delta_j{}^\circ) = \int_{S^{n-1} \cap \Delta_j{}^\circ} W(z, \omega)d\omega. \tag{4}$$

Then, by the assumption the function

$$F_j(z, t) := \int_{\boldsymbol{R}^n} f(x, t) W(z-x, \Delta_j{}^\circ),$$

becomes real analytic in $(t, \operatorname{Re} z, \operatorname{Im} z)$, and complex holomorphic in z. Thus we can see that $F_j(z, \tau)$ becomes holomorphic on a 0-wedge of the form $(\boldsymbol{R}^n \times T) + i\Gamma_j 0$ such that $\Gamma_j \cap \{\theta=0\} = \Delta_j$, hence

$$f(x, t) = \sum_{j=1}^{N} F_j((x, t) + i\Gamma_j 0)$$

contains t as real analytic parameters. From this proof we can even see that in the assumption of (3) the number of arbitrary parameters ξ may be restricted to 2n.

2. Now we precisely formulate our result and prove it.

THEOREM. *Let $f(x, t)$ be a hyperfunction defined on $\boldsymbol{R}^n \times T$ with support contained in $K \times T$, where $K \subset \boldsymbol{R}^n$ is a compact subset. Assume that for any $\varphi(x) \in \mathcal{A}(K)$ the integral* (2) *becomes real analytic in $t \in T$. Then $f(x, t)$ contains t as real analytic parameters (i.e. satisfies* (1)).

Since the assertion is local in t, we assume that T is bounded in the sequel. Let $\tilde{f}(x, t)$ denote a hyperfunction extension of $f(x, t)$ with compact support $\subset K \times \overline{T}$. Let now Δ_j be cones in the t space $\boldsymbol{R}^m$ whose duals give a partition of $\boldsymbol{S}^{m-1}$, and let $W(t, \Delta_j{}^\circ)$ denote functions similar to (4) in these variables. Then

$$f(x, t) = \sum_{j=1}^{N} f_j(x, t) \qquad \text{on} \quad \boldsymbol{R}^n \times T,$$

$$\text{with} \qquad f_j(x, t) = \tilde{f}(x, t) *_t W(t, \Delta_j{}^\circ).$$

Since

$$\int_{R^n} \varphi(x) f_j(x,t)dx = \left(\int_{R^n} \varphi(x)\tilde{f}(x,t)dx\right) * W(t, \Delta_j^\circ) \qquad \text{on} \quad \boldsymbol{R}^n \times T,$$

$f_j(x, t)$ satisfies the same assumption as $f(x, t)$. Thus it suffices to prove the theorem for each $f_j(x, t)$. Hence we can assume from the beginning that

$$\text{S.S.}\, f(x,t) \subset \boldsymbol{R}^n \times T \times \{(\xi dx + \theta dt)\,;\ \theta \in \Delta^\circ\}$$

for some convex open cone $\Delta \subset \boldsymbol{R}^n$.

Let then $\bigcup_{j=1}^{N} \Gamma_j^\circ \supset \boldsymbol{S}^{n+m-1}$ be a partition by the duals of the convex polyhedral cones $\Gamma_j \subset \boldsymbol{R}^{n+m}$, and let $W((x, t), \Gamma_j^\circ)$ denote the components of Kashiwara's decomposition in dimension $n+m$. Assume that Γ_1° is a neighbohood of $\{0\} \times \Delta^\circ$ in $\boldsymbol{S}^{n+m-1}$. Then

$$\Gamma_j^\circ \cap \text{S.S.}\, f \cap \boldsymbol{S}^{m-1} dt = \emptyset, \qquad \text{for} \quad j \geqq 2,$$

and we can see by an elementary calculus of S.S. that

$$f(x,t) * W((x,t), \Gamma_j^\circ), \qquad j=2,\ldots,N$$

all contain t as real analytic parameters. Hence for any bounded domain $D \supset K$, for $\varphi \in \mathcal{A}(\bar{D})$ the integral

$$\int_D f_j(x,\ t)\varphi(x)dx, \qquad j=2,\ldots,N$$

becomes real analytic in t. Hence it suffices to show the following assertion which in itself may be interesting:

LEMMA 1. *Let $f(x, t) = F((x, t) + i\Gamma 0)$ be a hyperfunction which is the boundary value from a unique wedge such that $\Gamma^\circ \cap \boldsymbol{S}^{n-1}dx = \emptyset$, and which is real analytic on a neighborhood of $(\boldsymbol{R}^{n+m} \backslash K) \times T$. Assume that there exists a bounded region $D \supset K$, such that for any $\varphi \in \mathcal{A}(\bar{D})$ the integral*

$$\int_D f(x,t)\varphi(x)dx$$

becomes real analytic in t. Then $f(x, t)$ contains t as real analytic parameters.

In terms of the defining function, shrinking T a little if necessary, we can further specify the assumption of this lemma as follows:

LEMMA 2. *Let $\Gamma \subset \boldsymbol{R}^{n+m}$ be a convex open cone such that its projection to the t space is a proper convex cone Δ. Let $F(z, \tau)$ be a function satisfying the following conditions:*

1) *$F(z, \tau)$ is holomorphic in a wedge $(\bar{D} \times \bar{T}) + i(\Gamma \cap \{|\text{Im}\,\tau| < A\})$, and further can be analytically continued to a neighborhood of $(\bar{D} \backslash K) \times \bar{T}$.*

2) *For any $\varphi(x) \in \mathcal{A}(\bar{D})$, if we choose a path $\tilde{D}$ deforming D into the domain where $F(\cdot, \tau)$ and φ are both holomorphic, then the integral*

$$\int_{\tilde{D}} F(z, \tau)\varphi(z)dz,$$

which is originally holomorphic in the wedge $\bar{T}+i\{\tau\in\Delta;\ |\mathrm{Im}\,\tau|<A\}$, *can be analytically continued to a neighborhood of the edge* $\bar{T}$.
Then $F((x,t)+i\Gamma 0)$ *contains* t *as real analytic parameters in* $D\times T$.

Remark that this assumption does not necessarily imply that $F(z, \tau)$ can be continued to a neighborhood of T.

Proof of Lemma 2. Without loss of generality we can put

$$\Delta=\{s_1>B|s'|\},\qquad \Gamma=\{s_1>B|s'|+C|y|\},$$

where s, y denote the imaginary coordinates of τ, z respectively, and $s'=(s_2, \dots, s_m)$. Choose $a<A$, $b>B$. By the assumption, we have a well defined mapping

$$(5)\qquad \begin{array}{rccc} \Phi: & \mathcal{O}(\bar{D}) & \longrightarrow & \mathcal{O}(\bar{T}+i\{0\leqq b|s'|\leqq s_1\leqq a\}) \\ & \Psi & & \Psi \\ & \varphi(z) & \longmapsto & \int_{\tilde{D}} F(z, \tau)\varphi(z)dz. \end{array}$$

We shall first show that this becomes a continuous linear mapping between these DFS type spaces. We apply the closed graph theorem. Fix $\varepsilon>0$ arbitrarily and consider the diagram

$$\begin{array}{ccc} \mathcal{O}(\bar{D}) & \xrightarrow{\ \rho\circ\Phi\ } & \mathcal{O}(\bar{T}+i\{\varepsilon\leqq b|s'|\leqq a\}) \\ & \searrow^{\Phi} & \uparrow \rho \\ & & \mathcal{O}(\bar{T}+i\{0\leqq b|s'|\leqq a\}). \end{array}$$

The restriction ρ is obviously continuous. The horizontal arrow is easily seen to be continuous by means of the definition. Thus in view of the uniqueness of the analytic continuation we can see that the graph of Φ is closed.

Now let $\bigcup_{j=1}^{N} \Delta_j{}^\circ$ be a partition of S^{n-1} by the duals of the convex polyhedral cones $\Delta_j{}^\circ$ in the x space. Then

$$F_j(z, \tau)=\int_{\tilde{D}} F(\zeta, \tau)W(z-\zeta, \Delta_j{}^\circ)d\zeta$$

becomes an element of

$$(6)\qquad \mathcal{O}(\boldsymbol{R}^n+i\Delta_j{}', \mathcal{O}(\bar{T}+i\{0\leqq b|s'|\leqq s_1\leqq a\})),$$

where $\Delta_j{}'$ is a proper subcone of Δ_j. In fact, the correspondence

$$(7)\qquad \begin{array}{rccc} \Psi: & \boldsymbol{R}^n+i\Delta_j{}' & \longrightarrow & \mathcal{O}(\bar{D}) \\ & \Psi & & \Psi \\ & z & \longmapsto & W(z-\zeta, \Delta_j{}^\circ) \end{array}$$

is obviously holomorphic in the sense of the topology, hence it remains holomorphic after the composition with the continuous linear mapping (5). On the other hand, the space (6) is contained in

$$\mathcal{O}((\boldsymbol{R}^n+i\Delta_j')\times(\bar{T}+i\{0\leqq b|s'|\leqq s_1\leqq a\})).$$

This can be shown e.g. as follows: Choose a compact subset L of $\boldsymbol{R}^n+i\Delta_j'$. Then by the continuous mapping $\Phi\circ\Psi$, L is mapped to a compact subset of $\mathcal{O}(\bar{T}+i\{0\leqq b|s'|\leqq s_1\leqq a\})$. Recall that a compact set in the latter space comes from a bounded set in $\mathcal{O}(W)$ for some open neighborhood W of $\bar{T}+i\{0\leqq b|s'|\leqq s_1\leqq a\}$. Hence especially, the elements are holomorphic in τ in this fixed neighborhood W independent of $z\in L$. Thus for any point $t=(t_1,\ldots,t_m)\in\bar{T}$, we can find small constants $\delta>\varepsilon>0$ such that $F_j(z,\tau)$ is holomorphic in

$$\mathrm{Int}(L)\times\{|\tau_1-t_1-i\delta|<\varepsilon\}\times\{|\tau_j-t_j|<\varepsilon,\ j=2,\ldots,m\}$$

and for any fixed $z\in\mathrm{Int}(L)$ it is holomorphic in τ in

$$\{|\tau_1-t_1-i\delta|<\delta+\varepsilon\}\times\{|\tau_j-t_j|<\varepsilon,\ j=2,\ldots,m\}.$$

Thus by Hartogs' lemma (see e.g. [1], Chapter VII, Theorem 2), $F_j(z,\tau)$ becomes jointly holomorphic in

$$\mathrm{Int}(L)\times\{|\tau_1-t_1-i\delta|<\delta+\varepsilon\}\times\{|\tau_j-t_j|<\varepsilon,\ j=2,\ldots,m\}.$$

Since $t\in\bar{T}$ is arbitrary, we conclude that $F_j(z,\tau)$ is holomorphic in a neighborhood of $(\boldsymbol{R}^n+i\Delta_j')\times(\bar{T}+i\{0\leqq b|s'|\leqq s_1\leqq a\})$. By Kashiwara's lemma $F_j(z,\tau)$ is then continued analytically to a wedge domain whose opening becomes a cone Γ_j' containing $\Delta_j'\times\{0\leqq b|s'|\leqq s_1\}$, hence having a non-void intersection with $s=0$. Thus $F_j((x,t)+i\Gamma_j'0)$ contains t as real analytic parameters. Since their sum is equal to $F((x,t)+i\Gamma 0)$ on $D\times T$, this latter also contains t as real analytic parameters there.

The proof of our main theorem is thus completed.

3. Finally we apply our result to a characterization of differential operators. This improves results in [3] in some sense.

COROLLARY. *A sheaf endomorphism Φ of $\mathcal{A}$ which possesses a hyperfunction kernel function, is in fact a differential operator (of infinite order in hyperfunction sense).*

Proof. Let $K(x,y)$ be the kernel function. We have

$$\text{(8)}\qquad \operatorname{supp} K(x,y)\subset\{x=y\},$$

$$(\Phi\varphi)(x)=\int K(x,y)\varphi(y)dy\in\mathcal{A}(\Omega)\qquad\text{for every}\quad \varphi\in\mathcal{A}(\Omega).$$

Thus we can apply our theorem locally in Ω to conclude that $K(x,y)$ contains x as real analytic parameters. In view of (8) and the watermelon theorem, we then conclude that

$$\text{S.S.}\, K(x, y) \subset \{(x, x, \xi(dx-dy))\,;\ x\in\Omega,\ \xi\in S^{n-1}\}.$$

Thus we obtain another hyperfunction $L(x, y)$ with support in $y=0$ and containing x as real analytic parameters such that $K(x, y)=L(x, x-y)$. Such a hyperfunction $L(x, y)$ is obviously a $\mathcal{B}[\{0\}]$-valued real analytic function in the strong sense and hence can be written in the form $J(x, D_y)\delta(y)$, where J is an infinite order differential operator with real analytic parameters x. (See the proof of Corollary 2.3 in [3]). Thus we conclude that $\Phi\varphi=J(x, D_x)\varphi$.

It should be noted that the above Corollary does not assert that every sheaf endomorphism of $\mathcal{A}$ is a differential operator; the existence of a kernel hyperfunction amounts to assuming some continuity.

References

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