Continuous Dependence Problem in an Inverse Spectral Problem for Systems of Ordinary Differential Equations of First Order

Dedicated to Professor Kôtaro Oikawa on his 60th birthday

By Masahiro YAMAMOTO

Department of Mathematics, College of Arts and Sciences,
University of Tokyo

Abstract

We consider an eigenvalue problem (1)-(2):

$$(1)\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{du(x)}{dx}+P(x)u(x)=\lambda u(x) \quad (0\leq x\leq 1;\ u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}).$$

$(2)\quad u_2(0)+hu_1(0)=0,\ u_2(1)+Hu_1(1)=0.$

Here $P=\begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix}\in\{C^1[0,1]\}^4$: real-valued and h, H are real constants, and λ corresponds to an eigenvalue. We denote the set of eigenvalues of (1)-(2) by $\{\lambda_n(P,h,H)\}_{n\in\mathbf{Z}}$ under an appropriate numbering. For $Q=\begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}\in\{C^1[0,1]\}^4$ and h, H, H^*, J, $J^*\in\mathbf{R}\backslash\{-1,1\}$ $(H\neq H^*,\ J\neq J^*)$, we obtain the following result on continuous dependence of coefficients and boundary conditions upon eigenvalues: If $\delta_0\equiv\sum_{n=-\infty}^{\infty}(|\lambda_n(Q,h,J)-\lambda_n(P,h,H)|+|\lambda_n(Q,h,J^*)-\lambda_n(P,h,H^*)|)$ is sufficiently small, then $\|Q-P\|_{\{C^0[0,1]\}^4}+|J-H|+|J^*-H^*|\leq M\delta_0$ for some constant $M>0$. Moreover we get $\|Q-P\|_{\{C^1[0,1]\}^4}\leq M\sum_{n=-\infty}^{\infty}(|n|+1)(|\lambda_n(Q,h,J)-\lambda_n(P,h,H)|+|\lambda_n(Q,h,J^*)-\lambda_n(P,h,H^*)|)$. We show also that for given μ_n, $\mu_n^*\in\mathbf{C}$ $(n\in\mathbf{Z})$, there exists a unique $(Q,J,J^*)\in\{C^1[0,1]\}^4\times(\mathbf{R}\backslash\{-1,1\})^2$ satisfying $\lambda_n(Q,h,J)=\mu_n$ and $\lambda_n(Q,h,J^*)=\mu_n^*$ under appropriate assumptions on μ_n, μ_n^* $(n\in\mathbf{Z})$ (e.g. $\sum_{n=-\infty}^{\infty}(|\mu_n-\lambda_n(P,h,H)|+|\mu_n^*-\lambda_n(P,h,H^*)|)$ is sufficiently small.). We prove these results by the principle of contraction mappings and, in order to apply the principle, we establish a priori estimates of solutions to some hyperbolic systems and results on perturbation of Riesz bases.

§1. Introduction

We consider a system (1.1) of ordinary differential equations of first order in the interval (0, 1) with boundary conditions (1.2) and (1.3), or (1.2) and (1.4):

$$(1.1) \quad \begin{cases} \dfrac{du_2(x)}{dx}+p_{11}(x)u_1(x)+p_{12}(x)u_2(x)=\lambda u_1(x) \\[2mm] \dfrac{du_1(x)}{dx}+p_{21}(x)u_1(x)+p_{22}(x)u_2(x)=\lambda u_2(x) \quad (0\leqq x \leqq 1). \end{cases}$$

$(1.2) \quad u_2(0)+hu_1(0)=0.$

$(1.3) \quad u_2(1)+Hu_1(1)=0.$

$(1.4) \quad u_2(1)+H^*u_1(1)=0 \qquad (H\neq H^*).$

Here p_{ij} $(1\leqq i, j\leqq 2)$ are real-valued C^1-functions defined on the closed interval [0, 1], and h, H, H^* are real constants. A parameter λ corresponds to an eigenvalue.

For $P(\cdot)=(p_{ij}(\cdot))_{1\leqq i, j\leqq 2}\in\{C^1[0, 1]\}^4$, we define an operator $A_{P,h,H}$ in $\{L^2(0, 1)\}^2$ by the realization of a differential operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\dfrac{d\cdot}{dx}+P(x)\cdot$ with the boundary conditions (1.2) and (1.3). That is,

$$(1.5) \quad \begin{cases} (A_{P,h,H}u)(x)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\dfrac{du(x)}{dx}+P(x)u(x) \ (0<x<1), \qquad u\in\mathscr{D}(A_{P,h,H}) \\[3mm] \mathscr{D}(A_{P,h,H})=\left\{u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\in\{H^1(0, 1)\}^2 \, ; \ u_2(0)+hu_1(0)=u_2(1)+Hu_1(1)=0\right\}. \end{cases}$$

Here and henceforth, $L^2(0, 1)$ denotes the Hilbert space of complex-valued square integrable functions and $H^1(0, 1)$ is the ordinary Sobolev space, and $(\cdot, \cdot)\equiv$ $(\cdot, \cdot)_{\{L^2(0,1)\}^2}$ denotes the inner product in the product space $\{L^2(0, 1)\}^2$. Let $\sigma(A_{P,h,H})$ be the spectrum of $A_{P,h,H}$. Then $\sigma(A_{P,h,H})$ consists entirely of countable simple eigenvalues, if $|h|, |H|\neq 1$ (Russell [10], [11], for instance).

In Yamamoto [12], [13], we discuss an inverse problem of determining the coefficients p_{ij} $(1\leqq i, j\leqq 2)$, h, H, H^* from the one pair of the eigenvalues of (1.1)-(1.3) and (1.1), (1.2), (1.4). According to [12], we can see that at most two of the four coefficients p_{ij} $(1\leqq i, j\leqq 2)$ and H, H^* can be determined uniquely from such two sets of the eigenvalues. More precisely, we obtain

THEOREM 0 ([13]). *Let* h, H, H^*, J, $J^*\in R\backslash\{-1, 1\}$, $H\neq H^*$, *and let* P *and* Q *be of the form*

$$(1.6) \quad P=\begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix}, \qquad Q=\begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}, \ a, b, p_1, p_2, q_1, q_2 : \ real\text{-}valued \ C^1\text{-}functions.$$

If

(1.7) $\sigma(A_{Q,h,J}) = \sigma(A_{P,h,H})$ *and* $\sigma(A_{Q,h,J*}) = \sigma(A_{P,h,H*})$,

then we have

(1.8) $q_1(x) = p_1(x)$, $q_2(x) = p_2(x)$ $(0 \leqq x \leqq 1)$

and

(1.9) $J = H$, $J* = H*$.

In this paper we will consider in what sense coefficients and constants in boundary conditions continuously depend upon the eigenvalues. That is, the purpose of this paper is to discuss

PROBLEM. *Let* h, H, $H*$, J, $J* \in R \setminus \{-1, 1\}$ *and let* P *and* Q *be expressed in the form* (1.6). *Then, in order to assure that* $\|Q - P\|_{C^1[0,1]^4} + |J - H| + |J* - H*|$ *is small, in what sense should* $\{\sigma(A_{Q,h,J}),\ \sigma(A_{Q,h,J*})\}$ *be close to* $\{\sigma(A_{P,h,H}),\ \sigma(A_{P,h,H*})\}$?

In PROBLEM, we note that by Theorem 0, the equality $\{\sigma(A_{Q,h,J}),\ \sigma(A_{Q,h,J*})\}$ $= \{\sigma(A_{P,h,H}),\ \sigma(A_{P,h,H*})\}$ implies $Q = P$, $J = H$ and $J* = H*$.

Here and henceforth, we define

(1.10) $\begin{cases} \|P\|_{(C^0[0,1])^4} = \|P\|_{C^0} = \max\limits_{\substack{1 \leqq i,j \leqq 2 \\ 0 \leqq x \leqq 1}} |p_{ij}(x)| \\[2em] \|P\|_{(C^1[0,1])^4} = \|P\|_{C^1} \\[1em] = \max\left(\max\limits_{\substack{1 \leqq i,j \leqq 2 \\ 0 \leqq x \leqq 1}} |p_{ij}(x)|, \ \max\limits_{\substack{1 \leqq i,j \leqq 2 \\ 0 \leqq x \leqq 1}} \left| \dfrac{dp_{ij}(x)}{dx} \right| \right), \end{cases}$

for $P = (p_{ij})_{1 \leqq i,j \leqq 2} \in \{C^1[0,1]\}^4$. For $p \in \{C^1[0,1]\}^2$, etc., we adopt similar notation.

REMARK 1. For the Sturm-Liouville equation, a similar problem on continuous dependence is considered in Hochstadt [3] and Iwasaki [4]. Moreover we can refer to Mizutani [7].

In order to state our main result, we show Proposition 1 and introduce a class $A(a, b)$ of coefficients.

PROPOSITION 1. *For* $P \in \{C^1[0,1]\}^4$ *and* h, $H \in R \setminus \{-1, 1\}$, *the following facts hold.*

(I)   *The set* $\sigma(A_{P,h,H}) \cap R$ *is a finite set.*

(II) $\lambda \in \sigma(A_{P,h,H})$ *if and only if* $\bar{\lambda} \in \sigma(A_{P,h,H})$.

Here and henceforth, let $\bar{a}$ denote the complex conjugate of $a \in C$. In Appendix I, we prove this proposition.

Let us arbitrarily fix a, $b \in C^1[0, 1]$ and let us define a class $A(a, b)$ by

(1.11) $A(a, b) = \left\{ \begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix};\ p_1,\ p_2 \in C^1[0, 1],\ \textit{real-valued} \right\}.$

Throughout this paper, let us assume that all the coefficient matrices under consideration belong to $A(a, b)$.

Theorem 0 asserts that the two sets $\sigma(A_{P,h,H})$ and $\sigma(A_{P,h,H^*})$ determine P in the class $A(a, b)$.

Henceforth let $P \in A(a, b)$ and $h, H, H^* \in \boldsymbol{R} \backslash \{-1, 1\}$ be arbitrarily fixed.

By Proposition 1, we can number all the elements of $\sigma(A_{P,h,H})$ in the following manner:

Let us denote the number of elements of the set $\sigma(A_{P,h,H}) \cap \boldsymbol{R}$ by N_0. If N_0 is even, then we set

(1.12.1) $\sigma(A_{P,h,H}) \cap \boldsymbol{R} = \{\lambda_{-N_0/2}, \cdots, \lambda_{-1}, \lambda_1, \cdots, \lambda_{N_0/2}\}$

(1.12.2) $\sigma(A_{P,h,H}) \cap \{z;\ \mathrm{Im}\ z > 0\} = \{\lambda_n\}_{n \geq (N_0+2)/2}$

(1.12.3) $\lambda_n = \overline{\lambda_{-n}} \quad (n \leq -(N_0+2)/2)$

and

(1.12.4) $\mathrm{Im}\ \lambda_{n+1} \geq \mathrm{Im}\ \lambda_n \quad (n \geq N_0/2).$

Otherwise (i.e. N_0 is odd), we set

(1.13.1) $\sigma(A_{P,h,H}) \cap \boldsymbol{R} = \{\lambda_{-(N_0-1)/2}, \cdots, \lambda_{-1}, \lambda_0, \lambda_1, \cdots, \lambda_{(N_0-1)/2}\}$

(1.13.2) $\sigma(A_{P,h,H}) \cap \{z;\ \mathrm{Im}\ z > 0\} = \{\lambda_n\}_{n \geq (N_0+1)/2}$

(1.13.3) $\lambda_n = \overline{\lambda_{-n}} \quad (n \leq -(N_0+1)/2)$

and

(1.13.4) $\mathrm{Im}\ \lambda_{n+1} \geq \mathrm{Im}\ \lambda_n \quad (n \geq (N_0-1)/2).$

We put

(1.14) $\gamma = \dfrac{1}{2} \log \dfrac{(1+h)(1-H)}{(1-h)(1+H)} + \dfrac{1}{2} \int_0^1 (p_{11}(s) + p_{22}(s)) ds,$

where we take the principal value of the logarithm. For the asymptotic behavior of the eigenvalues, by Russell [10], [11], we know: if we renumber $\{\lambda_n\}_{n \in \boldsymbol{Z}}$ in an appropriate manner, then $\lambda_n = \gamma + n\pi\sqrt{-1} + O\left(\dfrac{1}{n}\right).$ Therefore $\mathrm{Im}\ \lambda_n$ $(n \geq N_1:$ a sufficiently large natural number) are mutually distinct. Hence the condition (1.12.4) or (1.13.4) gives a unique numbering to λ_n for sufficiently large n. Moreover, under the numbering (1.12) or (1.13), we see: for each $n \in \boldsymbol{Z}$, there exists $m(n) \in \boldsymbol{Z}$ such that

(1.15) $\lambda_n = \gamma + m(n)\pi\sqrt{-1} + O\left(\dfrac{1}{m(n)}\right)$ and $|m(n)| = O(|n|).$

For $\sigma(A_{P,h,H^*}) = \{\lambda_n^*\}_{n \in \boldsymbol{Z}}$, we can number all the elements in a similar manner. To sum up, for $\sigma(A_{P,h,H}) = \{\lambda_n\}_{n \in \boldsymbol{Z}}$ and $\sigma(A_{P,h,H^*}) = \{\lambda_n^*\}_{n \in \boldsymbol{Z}}$, there exist some $N_1, N_2 \in \boldsymbol{N} \cup \{0\}$ such that

(1.16.1) $\lambda_n \in R$ $(-N_1 \leqq n \leqq N_1)$
(1.16.2) $\lambda_n = \overline{\lambda_{-n}}$ $(n \leqq -N_1 - 1)$
(1.16.3)   Im $\lambda_{n+1} \geqq$ Im λ_n $(n \geqq N_1)$

and

(1.17.1) $\lambda_n^* \in R$ $(-N_2 \leqq n \leqq N_2)$
(1.17.2) $\lambda_n^* = \overline{\lambda^*_{-n}}$ $(n \leqq -N_2 - 1)$
(1.17.3)   Im $\lambda_{n+1}^* \geqq$ Im λ_n^* $(n \geqq N_2)$.

Here and henceforth, if N_1 and N_2 are even, then the subscript "0" of λ and λ^* is skipped, respectively. Similarly let us number $\sigma(A_{Q,h,J})$, $\sigma(A_{Q,h,J^*})$, etc.

Now we are ready to state our main result, which gives an answer to PROBLEM:

THEOREM 1.   *Let P, $Q \in A(a, b)$, h, H, H^*, J, $J^* \in R \setminus \{-1, 1\}$, $H \neq H^*$ and let us set*

(1.18) $\sigma(A_{P,h,H}) = \{\lambda_n\}_{n \in Z}$ *and* $\sigma(A_{P,h,H^*}) = \{\lambda_n^*\}_{n \in Z}$

and

(1.19) $\sigma(A_{Q,h,J}) = \{\mu_n\}_{n \in Z}$ *and* $\sigma(A_{Q,h,J^*}) = \{\mu_n^*\}_{n \in Z}$.

If $\sum\limits_{n=-\infty}^{\infty} (|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$ is sufficiently small for P, h, H and H^*, then we have the estimates

(1.20) $|J - H| + |J^* - H^*| \leqq C \times \sum\limits_{n=-\infty}^{\infty} (|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$

and

(1.21) $\|Q - P\|_{\{C^j[0,1]\}^4} \leqq C \times \sum\limits_{n=-\infty}^{\infty} (|n|^j + 1)(|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$ $(j = 0, 1)$,

where C is some positive constant depending on P, h, H and H^*.

We denote $P \in A(a, b)$ satisfying (1.18) by $\mathcal{F}((\{\lambda_n\}_{n \in Z}, \{\lambda_n^*\}_{n \in Z}))$. This mapping $\mathcal{F}: C^Z \times C^Z \longrightarrow A(a, b) \subset \{C^1[0, 1]\}^4$ is well-defined from Theorem 0. Theorem 1 means that $\mathcal{F}: C^Z \times C^Z \longrightarrow \{C^1[0, 1]\}^4$ is continuous when in $C^Z \times C^Z$ we introduce a distance function

(1.22) $\rho((\{\lambda_n\}_{n \in Z}, \{\lambda_n^*\}_{n \in Z}), (\{\mu_n\}_{n \in Z}, \{\mu_n^*\}_{n \in Z})) = \sum\limits_{n=-\infty}^{\infty} (|n| + 1)(|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$.

On the other hand, from $P \in \{C^1[0, 1]\}^4$, we can obtain *only* (1.15) as the asymptotic behavior of $\sigma(A_{P,h,H})$. Therefore, in general, the condition P, $Q \in \{C^1[0, 1]\}^4$ *does not* directly imply the convergence of the series at the right hand side of (1.22). In other words, Theorem 1 suggests that our inverse problem is *ill-posed* in the sense that the topology in $C^Z \times C^Z$ assuring the continuity of the mapping $\mathcal{F}$, is too strong in comparison with the asymptotic behavior (1.15).
We can observe the ill-posedness of this kind also for the inverse Sturm-Liouville problem (Hochstadt [3] and Iwasaki [4]).

We can derive Theorem 1 from Theorem 2, which assures the existence of $Q \in A(a, b)$, J, $J^* \in \mathbf{R} \setminus \{-1, 1\}$ satisfying $\sigma(A_{Q,h,J}) = \{\mu_n\}_{n \in \mathbf{Z}}$ and $\sigma(A_{Q,h,J^*}) = \{\mu_n^*\}_{n \in \mathbf{Z}}$ provided that $\sum_{n=-\infty}^{\infty} (|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$ is sufficiently small. That is,

THEOREM 2. *Let* $P \in A(a, b)$ *and* h, H, $H^* \in \mathbf{R} \setminus \{-1, 1\}$ $(H \neq H^*)$ *be fixed, and let* $\sigma(A_{P,h,H}) = \{\lambda_n\}_{n \in \mathbf{Z}}$ *and* $\sigma(A_{P,h,H^*}) = \{\lambda_n^*\}_{n \in \mathbf{Z}}$ *satisfy* (1.16) *and* (1.17), *respectively. If two sets of complex numbers* $\{\mu_n\}_{n \in \mathbf{Z}}$ *and* $\{\mu_n^*\}_{n \in \mathbf{Z}}$ *satisfy*

(1.23.1) $\mu_n \in \mathbf{R}$ $(-N_1 \leqq n \leqq N_1)$

(1.23.2) $\mu_n = \overline{\mu_{-n}}$ $(n \leqq -N_1 - 1)$

(1.23.3) $\operatorname{Im} \mu_{n+1} \geqq \operatorname{Im} \mu_n$ $(n \geqq N_1)$

and

(1.24.1) $\mu_n^* \in \mathbf{R}$ $(-N_2 \leqq n \leqq N_2)$

(1.24.2) $\mu_n^* = \overline{\mu_{-n}^*}$ $(n \leqq -N_2 - 1)$

(1.24.3) $\operatorname{Im} \mu_{n+1}^* \geqq \operatorname{Im} \mu_n^*$ $(n \geqq N_2)$,

respectively, and the inequality

(1.25) $\sum_{n=-\infty}^{\infty} (|n| + 1)(|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|) < \infty$

holds, then there exists a unique $(Q, J, J^*) \in A(a, b) \times (\mathbf{R} \setminus \{-1, 1\})^2$ *such that*

(1.26) $\sigma(A_{Q,h,J}) = \{\mu_n\}_{n \in \mathbf{Z}}$ *and* $\sigma(A_{Q,h,J^*}) = \{\mu_n^*\}_{n \in \mathbf{Z}}$,

provided that $\sum_{n=-\infty}^{\infty} (|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$ *is sufficiently small for* P, h, H *and* H^*.

This paper is composed of three sections and seven appendixes. In §2, we prove Theorem 2, while we postpone proofs of the technical lemmas required there to Appendixes II–VII. In §3, we prove Theorem 1 on the basis of Theorem 2. Appendix I is devoted to a proof of Proposition 1 in §1.

Acknowledgements

§2. Proof of Theorem 2

In this section, we prove Theorem 2 by constructing Q as a fixed point of a contraction mapping. To this end, in subsections §2.1–2.5, we define a contraction mapping G.

First let us consider a domain where our operator is defined.

2.1. *Definition of the domain*

By the conditions $\dfrac{H-1}{H+1}\neq 1$ and $\dfrac{H^*-1}{H^*+1}\neq 1$, we can choose a small constant M so that

(2.1)
$$\frac{H-1}{H+1},\ \frac{H^*-1}{H^*+1}\notin[e^{-M},\ e^{M}].$$

Henceforth let us fix M satisfying (2.1). As the domain, we define a set $\mathcal{A}_M$ by

(2.2)
$$\mathcal{A}_M=\{(q_1,\ q_2)\in\{C^0[0,\ 1]\}^2\ ;\ \|q_1-p_1\|_{C^0[0,1]},\ \|q_2-p_2\|_{C^0[0,1]}\leqq M\}.$$

Here in $\mathcal{A}_M$, we introduce the same norm as the one in $\{C^0[0,\ 1]\}^2$:
$\|(u_1,\ u_2)\|=\max\ \{\|u_1\|_{C^0[0,1]},\ \|u_2\|_{C^0[0,1]}\}$.

Next let us define an operator G on $\mathcal{A}_M$ by composing G_i $(1\leqq i\leqq 4)$ given in §§ 2.2-2.5:

(2.3)
$$G=G_4\circ G_3\circ G_2\circ G_1,\qquad \mathcal{D}(G)=\mathcal{A}_M.$$

NOTATION. Let $\sigma(A_{P,h,H})=\{\lambda_n\}_{n\in\mathbf{Z}}$ and $\sigma(A_{P,h,H^*})=\{\lambda_n^*\}_{n\in\mathbf{Z}}$, and let $\{\mu_n\}_{n\in\mathbf{Z}}$ and $\{\mu_n^*\}_{n\in\mathbf{Z}}$ satisfy (1.23)-(1.25). Then we set

(2.4)
$$\delta_0=\sum_{n=-\infty}^{\infty}(|\mu_n-\lambda_n|+|\mu_n^*-\lambda_n^*|),$$

(2.5)
$$\delta=\sum_{n=-\infty}^{\infty}(|n|+1)(|\mu_n-\lambda_n|+|\mu_n^*-\lambda_n^*|)$$

and

(2.6)
$$B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Moreover let us denote the solutions to (2.7) and to (2.8) by

$$\phi(\ \cdot\ ,\ \lambda)=\begin{pmatrix}\phi_1(\ \cdot\ ,\ \lambda)\\ \phi_2(\ \cdot\ ,\ \lambda)\end{pmatrix}\in\{C^1[0,\ 1]\}^2\ \text{ and }\ \phi^*(\ \cdot\ ,\ \lambda)=\begin{pmatrix}\phi_1^*(\ \cdot\ ,\ \lambda)\\ \phi_2^*(\ \cdot\ ,\ \lambda)\end{pmatrix}\in\{C^1[0,\ 1]\}^2,$$

respectively:

(2.7)
$$\begin{cases}B\dfrac{d\phi(x,\ \lambda)}{dx}+P(x)\phi(x,\ \lambda)=\lambda\phi(x,\ \lambda)\qquad (0\leqq x\leqq 1)\\[2mm] \phi(0,\ \lambda)=\begin{pmatrix}1\\ -h\end{pmatrix}.\end{cases}$$

(2.8)
$$\begin{cases}B\dfrac{d\phi^*(x,\ \lambda)}{dx}-{}^tP(x)\phi^*(x,\ \lambda)=\lambda\phi^*(x,\ \lambda)\qquad (0\leqq x\leqq 1)\\[2mm] \phi^*(0,\ \lambda)=\begin{pmatrix}1\\ h\end{pmatrix}.\end{cases}$$

Here ${}^t P(x)$ is the transpose matrix of $P(x)$. These notations are used throughout this paper.

Henceforth M_i $(1 \leq i \leq 32)$ denote positive constants depending on P, h, H, H^*, δ_0, δ, M, and each M_i is bounded as $\delta_0 \downarrow 0$. For simplicity, we adopt notation $\|P\|_{C^0}$ in place of $\|P\|_{(C^0[0,1])^4}$.

2.2. Definition of G_1

We define G_1 which transforms each element $q = (q_1, q_2)$ of $\mathcal{A}_M$ to $(\{a_n(q)\}_{n \in \mathbb{Z}}, \{b_n(q)\}_{n \in \mathbb{Z}} J, J^*, q) \in \mathbb{C}^{\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}} \times (\mathbb{R} \setminus \{-1, 1\})^2 \times \mathcal{A}_M$ in the following manner:

$$(2.9) \qquad a_n(q) = \frac{-2 \exp\left(\frac{1}{2} \int_0^1 (q_2(s) - p_2(s) + p_1(s) - q_1(s)) ds\right)}{(H+1) \exp\left(\int_0^1 (q_2(s) - p_2(s)) ds\right) + 1 - H}$$

$$\times (\phi_2(1, \mu_n) + H\phi_1(1, \mu_n)) \qquad (n \in \mathbb{Z}),$$

$$(2.10) \qquad b_n(q) = \frac{-2 \exp\left(\frac{1}{2} \int_0^1 (q_2(s) - p_2(s) + p_1(s) - q_1(s)) ds\right)}{(H^*+1) \exp\left(\int_0^1 (q_2(s) - p_2(s)) ds\right) + 1 - H^*}$$

$$\times (\phi_2(1, \mu_n^*) + H^*\phi_1(1, \mu_n^*)) \qquad (n \in \mathbb{Z}),$$

$$(2.11) \qquad J = J(q) = \left((H+1) \exp\left(\int_0^1 (q_2(s) - p_2(s)) ds\right) + H - 1\right)$$

$$\times \left((H+1) \exp\left(\int_0^1 (q_2(s) - p_2(s)) ds\right) + 1 - H\right)^{-1}$$

and

$$(2.12) \qquad J^* = J^*(q) = \left((H^*+1) \exp\left(\int_0^1 (q_2(s) - p_2(s)) ds\right) + H^* - 1\right)$$

$$\times \left((H^*+1) \exp\left(\int_0^1 (q_2(s) - p_2(s)) ds\right) + 1 - H^*\right)^{-1}.$$

Then we set

$$(2.13) \qquad G_1(q) = (\{a_n(q)\}_{n \in \mathbb{Z}}, \{b_n(q)\}_{n \in \mathbb{Z}}, J, J^*, q).$$

We see that G_1 is well-defined from

LEMMA 1. (I) *We have*

$$(2.14) \quad \begin{cases} \left| \left| (H+1) \exp\left(\int_0^1 (q_2(s)-p_2(s))\,ds \right) +1-H \right| \geqq M_1 \\ \left| \left| (H^*+1) \exp\left(\int_0^1 (q_2(s)-p_2(s))\,ds \right) +1-H^* \right| \geqq M_1 \end{cases}$$

for some positive constant M_1.

(II) *We have*

$$(2.15) \qquad J, J^* \in \mathbf{R}\backslash\{-1,\ 1\} \quad and \quad J \neq J^*.$$

Proof. (I) The first estimate in (2.14) is proved by:

$$\left| (H+1) \exp\left(\int_0^1 (q_2(s)-p_2(s))\,ds \right) +1-H \right|$$

$$= |H+1| \cdot \left| \exp\left(\int_0^1 (q_2(s)-p_2(s))\,ds \right) - \frac{H-1}{H+1} \right|$$

$$\geqq |H+1| \min\left\{ \left| e^M - \frac{H-1}{H+1} \right|,\ \left| e^{-M} - \frac{H-1}{H+1} \right| \right\}$$

$$\left(\text{by (2.1) and } e^{-M} \leqq \exp\left(\int_0^1 (q_2(s)-p_2(s))\,ds \right) \leqq e^M \right)$$

$$> 0.$$

The second one can be seen similarly.

(II) In view of (2.14), $|H| \neq 1$, $|H^*| \neq 1$ and $H \neq H^*$, we can prove (2.15) by direct computations.

Moreover we have

LEMMA 2. *Let $q \in \mathcal{A}_M$ and let $a_n(q)$ and $b_n(q)$ $(n \in \mathbf{Z})$ be defined by (2.9) and (2.10), respectively. Then we get*

$$(2.16) \qquad \sum_{n=-\infty}^{\infty} (|a_n(q)| + |b_n(q)|) \leqq M_2 \delta_0$$

and

$$(2.17) \qquad \sum_{n=-\infty}^{\infty} (|n|+1)(|a_n(q)| + |b_n(q)|) \leqq M_2 \delta$$

for some positive constant M_2.

Here we recall that δ_0 and δ are given by (2.4) and (2.5), respectively. In Appendix II, we prove this lemma.

2.3. *Definition of G_2*

For the definition, we show Lemma 3.

LEMMA 3. *Under all the assumptions of Theorem 2 except for (1.25), we have the following properties on* $\{\phi(\,\cdot\,,\mu_n)\}_{n\in\mathbf{Z}}$.

(I) *(the completeness of* $\{\phi(\,\cdot\,,\mu_n)\}_{n\in\mathbf{Z}}$) *The system* $\{\phi(\,\cdot\,,\mu_n)\}_{n\in\mathbf{Z}}$ *is a Riesz basis in* $\{L^2(0,1)\}^2$.

(II) *(the existence of a complete system biorthogonal to* $\{\phi(\,\cdot\,,\mu_n)\}_{n\in\mathbf{Z}}$) *There exists some system* $\{\psi_n^{(1)}\}_{n\in\mathbf{Z}}$ *satisfying (2.18)-(2.22).*

$$(2.18) \qquad \|\psi_n^{(1)}\|_{\{C^0[0,1]\}^2} \leqq M_3 \quad (n\in\mathbf{Z}) \text{ for some constant } M_3>0.$$

$$(2.19) \qquad \|\psi_n^{(1)}\|_{\{C^1[0,1]\}^2} \leqq M_3(|n|+1) \qquad (n\in\mathbf{Z}), \text{ if } (1.25) \text{ is assumed.}$$

$$(2.20) \qquad \psi_n^{(1)}: \text{ real-valued } (-N_1\leqq n\leqq N_1) \text{ and } \overline{\psi_n^{(1)}}=\psi_{-n}^{(1)} \ (n\geqq N_1+1).$$

$$(2.21) \qquad (\phi(\,\cdot\,,\mu_n),\,\psi_m^{(1)})=\delta_{nm}\equiv \begin{cases} 0, \text{ if } n\neq m \\ 1, \text{ if } n=m. \end{cases}$$

$$(2.22) \qquad u=\sum_{n=-\infty}^{\infty}(u,\,\overline{\phi(\,\cdot\,,\mu_n)})\cdot\overline{\psi_n^{(1)}} \text{ for each } u\in\{L^2(0,1)\}^2.$$

Here the series at the right hand side of (2.22) is convergent in $\{L^2(0,1)\}^2$.
Furthermore, for the system $\{\phi(\,\cdot\,,\mu_n^*)\}_{n\in\mathbf{Z}}$, *similar facts hold. That is,*

(I)′   *The system* $\{\phi(\,\cdot\,,\mu_n^*)\}_{n\in\mathbf{Z}}$ *is a Riesz basis.*

(II)′   *There exists some* $\{\psi_n^{(2)}\}_{n\in\mathbf{Z}}$ *satisfying*

$$(2.18)' \qquad \|\psi_n^{(2)}\|_{\{C^0[0,1]\}^2} \leqq M_3 \quad (n\in\mathbf{Z}) \text{ for some constant } M_3>0.$$

$$(2.19)' \qquad \|\psi_n^{(2)}\|_{\{C^1[0,1]\}^2} \leqq M_3(|n|+1) \qquad (n\in\mathbf{Z}), \text{ if } (1.25) \text{ is assumed.}$$

$$(2.20)' \qquad \psi_n^{(2)}: \text{ real-valued } (-N_2\leqq n\leqq N_2) \text{ and } \overline{\psi_n^{(2)}}=\psi_{-n}^{(2)} \ (n\geqq N_2+1).$$

$$(2.21)' \qquad (\phi(\,\cdot\,,\mu_n^*),\,\psi_m^{(2)})=\delta_{nm}.$$

$$(2.22)' \qquad u=\sum_{n=-\infty}^{\infty}(u,\,\overline{\phi(\,\cdot\,,\mu_n^*)})\cdot\overline{\psi_n^{(2)}} \text{ for each } u\in\{L^2(0,1)\}^2.$$

In Appendix III, we will prove Lemma 3.
As is seen by Lemma 4 stated below, for $\{a_n(q),\,b_n(q)\}_{n\in\mathbf{Z}}$ given by (2.9) and (2.10), we can set

$$(2.23) \qquad \begin{pmatrix} c_{11}(y,\,q) \\ c_{12}(y,\,q) \end{pmatrix} = \frac{1}{J-J^*}\sum_{n=-\infty}^{\infty}(a_n(q)\overline{\psi_n^{(1)}(y)}-b_n(q)\overline{\psi_n^{(2)}(y)})$$

and

$$(2.24) \qquad \begin{pmatrix} c_{21}(y,\,q) \\ c_{22}(y,\,q) \end{pmatrix} = \frac{1}{J^*-J}\sum_{n=-\infty}^{\infty}(J^*a_n(q)\overline{\psi_n^{(1)}(y)}-Jb_n(q)\overline{\psi_n^{(2)}(y)}),$$

where $J=J(q)$ and $J^*=J^*(q)$ are given by (2.11) and (2.12). Let us set $C(\cdot, q)$ $=(c_{ij}(\cdot, q))_{1\le i, j\le 2}$.

LEMMA 4. (I) *On all the assumptions of Theorem 2, we have*

(2.25) $C(\cdot, q)$: *real-valued.*

(2.26) $C(\cdot, q)\in\{C^1[0, 1]\}^4$.

(2.27) $\|C(\cdot, q)\|_{\{C^0[0,1]\}^4}\le M_4\delta_0$ *for some positive constant* M_4.

(2.28) $\|C(\cdot, q)\|_{\{C^1[0,1]\}^4}\le M_4\delta$.

   (II) *For* $q^{(i)}=(q_1^{(i)}, q_2^{(i)})\in\mathcal{A}_M$ $(i=1, 2)$, *we have the estimate*

(2.29) $\|C(\cdot, q^{(1)})-C(\cdot, q^{(2)})\|_{\{C^0[0,1]\}^4}\le M_4\delta_0\|q^{(1)}-q^{(2)}\|_{\{C^0[0,1]\}^2}$.

In Appendix IV, we will prove this lemma.
   Then we define an operator G_2 by

(2.30) $G_2((\{a_n(q)\}_{n\in\mathbb{Z}}, \{b_n(q)\}_{n\in\mathbb{Z}}, J, J^*, q))=(C(\cdot, q), q)$.

By Lemma 4, the operator G_2 sends $(\{a_n(q)\}_{n\in\mathbb{Z}}, \{b_n(q)\}_{n\in\mathbb{Z}}, J, J^*, q)$ to the four real-valued C^1-functions $c_{ij}(\cdot, q)$ $(1\le i, j\le 2)$ and two C^0-functions $q=(q_1, q_2)$.

2.4. *Definition of G_3*

Let us recall $B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and henceforth let us set

(2.31) $\Omega=\{(x, y); 0<y<x<1\}$.

For the definition of G_3, we show

LEMMA 5. *Let*

(2.32) $P\in A(a, b)\subset\{C^1[0, 1]\}^4$
and

(2.33) $Q=\begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}$, *where* $(q_1, q_2)\in\mathcal{A}_M$.

   (I) *For given* $D=(d_{ij})_{1\le i, j\le 2}\in\{C^1[0, 1]\}^4$, *there exists a unique solution* $K=$ $K(\cdot, \cdot, P, Q, D)\in\{C^1(\bar{\Omega})\}^4$ *to* (2.34)-(2.36):

(2.34) $$B\frac{\partial K(x, y)}{\partial x}+Q(x)K(x, y)-K(x, y)P(y)=-\frac{\partial K(x, y)}{\partial y}B$$
$$((x, y)\in\bar{\Omega}).$$

(2.35) $K_{12}(x, 0)=hK_{11}(x, 0)$ *and* $K_{22}(x, 0)=hK_{21}(x, 0)$ $(0\le x\le 1)$.

(2.36) $K(1, y)=D(y)$ $(0\leqq y \leqq 1)$.

Furthermore the estimates

(2.37) $\|K\|_{[C^0(\overline{\omega})]^4}\leqq M_5\|D\|_{[C^0[0,1]]^4}$

and

(2.38) $\|K\|_{[C^1(\overline{\omega})]^4}\leqq M_5\|D\|_{[C^1[0,1]]^4}$

hold for some positive constant M_5.

(II) *For given Q_1, Q_2 in the form (2.33) and $D_1, D_2 \in \{C^1[0, 1]\}^4$, we have the estimate*

(2.39) $\|K(\cdot, \cdot, P, Q_1, D_1)-K(\cdot, \cdot, P, Q_2, D_2)\|_{[C^0(\overline{\omega})]^4}$

$\leqq M_5(\|D_2\|_{[C^0[0,1]]^4}\times\|Q_1-Q_2\|_{[C^0[0,1]]^4}+\|D_1-D_2\|_{C^0})$.

In (I) of this lemma, we note that q_1, $q_2 \in C^0[0, 1]$ is sufficient for the existence of $\underline{C}^1$-solution K. In Appendix V, we will prove this lemma.

In (2.36), let us substitute $C(\cdot, q)=(c_{ij}(\cdot, q))_{1\leqq i,j\leqq 2}$ given by (2.23) and (2.24) into $D(\cdot)$. Then, by Lemma 5, we can set

(2.40) $G_3((C(\cdot, q), q))=K(\cdot, \cdot, P, Q, C)$.

Here we note that Q is given by (2.33).

2.5. Definition of G_4

Let us set

(2.41) $A(x)=\dfrac{1}{2}\begin{pmatrix} -a(x)-b(x)+p_1(x)+p_2(x) & a(x)+b(x)-p_1(x)-p_2(x) \\ -a(x)+b(x)-p_1(x)+p_2(x) & a(x)-b(x)+p_1(x)-p_2(x) \end{pmatrix}$

$(0\leqq x \leqq 1)$

and let us consider an initial value problem (2.42) and (2.43) for ordinary differential equations:

(2.42) $\dfrac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}=A(x)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}\cdot$

$\begin{pmatrix} K_{11}(x, x)-K_{22}(x, x)+K_{12}(x, x)-K_{21}(x, x) \\ K_{11}(x, x)-K_{22}(x, x)+K_{21}(x, x)-K_{12}(x, x) \end{pmatrix}$ $(0\leqq x \leqq 1)$.

(2.43) $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Here $K=(K_{ij})_{1\leqq i,j\leqq 2}=G_3(C(\cdot, q))$. Then we have

LEMMA 6. (I) *There exists some positive constant M_6 such that, if*

(2.44) $\delta_0 \leqq M_6$,

then the solution $\begin{pmatrix} u \\ v \end{pmatrix}$ *to* (2.42) *and* (2.43) *satisfies*

(2.45) $u(x),\ v(x) \geqq \dfrac{1}{2}$ $(0 \leqq x \leqq 1)$.

(II) *On the condition* (2.44), *we can define real-valued C^1-functions r_1, r_2 by*

(2.46) $\begin{cases} r_1(x) = p_1(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} - \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \\[3mm] r_2(x) = p_2(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} + \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \end{cases}$ $(0 \leqq x \leqq 1)$.

Furthermore the following estimates hold:

(2.47) $\|p_1 - r_1\|_{C^0} + \|p_2 - r_2\|_{C^0} \leqq M_7 \displaystyle\max_{\substack{1 \leqq i,\ j \leqq 2 \\ 0 \leqq x \leqq 1}} |K_{ij}(x,\ x)|$.

(2.48) $\|p_1 - r_1\|_{C^1} + \|p_2 - r_2\|_{C^1}$

$\leqq M_7 \max \left(\displaystyle\max_{\substack{1 \leqq i,\ j \leqq 2 \\ 0 \leqq x \leqq 1}} |K_{ij}(x,\ x)|,\ \max_{\substack{1 \leqq i,\ j \leqq 2 \\ 0 \leqq x \leqq 1}} \left| \dfrac{dK_{ij}(x,\ x)}{dx} \right| \right)$.

(III) *For* $q^{(i)} = (q_1^{(i)},\ q_2^{(i)}) \in \mathcal{A}_M$ $(i=1,\ 2)$, *let us put* $K^{(i)} = (G_3 \circ G_2 \circ G_1) q^{(i)}$ $(i=1,\ 2)$. *On the assumption* (2.44), *let* $(u^{(i)},\ v^{(i)})$ $(i=1,\ 2)$ *be the solution to*

(2.49) $\dfrac{d}{dx} \begin{pmatrix} u^{(i)}(x) \\ v^{(i)}(x) \end{pmatrix} = A(x) \begin{pmatrix} u^{(i)}(x) \\ v^{(i)}(x) \end{pmatrix} +$

$\begin{pmatrix} K_{11}^{(i)}(x,\ x) - K_{22}^{(i)}(x,\ x) + K_{12}^{(i)}(x,\ x) - K_{21}^{(i)}(x,\ x) \\ K_{11}^{(i)}(x,\ x) - K_{22}^{(i)}(x,\ x) + K_{21}^{(i)}(x,\ x) - K_{12}^{(i)}(x,\ x) \end{pmatrix}$

$(0 \leqq x \leqq 1,\ i=1,\ 2)$

and

(2.50) $\begin{pmatrix} u^{(i)}(0) \\ v^{(i)}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $(i=1,\ 2)$,

and let us set

$$(2.51) \quad \begin{cases} r_1^{(i)}(x) = p_1(x) - \dfrac{1}{u^{(i)}(x)} \dfrac{du^{(i)}(x)}{dx} - \dfrac{1}{v^{(i)}(x)} \dfrac{dv^{(i)}(x)}{dx} \\[3mm] r_2^{(i)}(x) = p_2(x) - \dfrac{1}{u^{(i)}(x)} \dfrac{du^{(i)}(x)}{dx} + \dfrac{1}{v^{(i)}(x)} \dfrac{dv^{(i)}(x)}{dx} \end{cases}$$

$$(0 \leq x \leq 1, \ i = 1, \ 2).$$

Then the estimate

$$(2.52) \qquad \|r_1^{(1)} - r_1^{(2)}\|_{C^0} + \|r_2^{(1)} - r_2^{(2)}\|_{C^0} \leq M_7 \max_{\substack{1 \leq i, \, j \leq 2 \\ 0 \leq x \leq 1}} |K_{ij}^{(1)}(x, \ x) - K_{ij}^{(2)}(x, \ x)|$$

holds.

In Appendix VI, we will prove Lemma 6.

Now we proceed to the definition of G_4. Under the assumption (2.44) of Lemma 6 (I), let us define G_4 by

$$(2.53) \qquad\qquad G_4(K(\cdot, \ \cdot, \ P, \ Q, \ C)) = (r_1, \ r_2),$$

where $K(\cdot, \ \cdot, \ P, \ Q, \ C)$ and $(r_1, \ r_2)$ are given by (2.40) and (2.46), respectively. Thus we complete the definitions of G_1, G_2, G_3 and G_4.

2.6. *Reduction to a fixed point*

In this subsection, we show Lemma 7 which asserts that a fixed point $(q_1, \ q_2)$ of the mapping G gives the functions satisfying all the conditions in Theorem 2. That is,

LEMMA 7. *On all the assumptions of Theorem 2, let* $q = (q_1, \ q_2) \in \{C^0[0, \ 1]\}^2$ *satisfy*

$$(2.54) \qquad\qquad q = Gq,$$

and let us put

$$(2.55) \qquad\qquad Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}.$$

Then we have

$$(2.56) \qquad\qquad q_i \in C^1[0, \ 1] \qquad (i = 1, \ 2)$$

and

$$(2.57) \qquad\qquad \sigma(A_{Q, h, J}) = \{\mu_n\}_{n \in Z} \quad and \quad \sigma(A_{Q, h, J^*}) = \{\mu_n^*\}_{n \in Z},$$

where J, J^* *are defined by* (2.11) *and* (2.12).

Proof of Lemma 7. Let us assume that $q = (q_1, \ q_2) \in \{C^0[0, \ 1]\}^2$ satisfies (2.54).

First we will prove (2.56). To this end, we have only to prove

(2.58) $$Gq \in \{C^1[0, 1]\}^2,$$

in view of (2.54). Since, by Lemma 5 (I), the relation (2.26) implies that

$K(\cdot, \cdot, P, Q, C) \in \{C^1(\bar{\Omega})\}^4$, where $P = \begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix}$ and $Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}$, we see (2.58) from

the definition of G and a, b, p_1, $p_2 \in C^1[0, 1]$.

Next we will prove (2.57). Firstly we have to show

(2.59) $$\sigma(A_{Q,h,J}) \supset \{\mu_n\}_{n \in Z} \quad \text{and} \quad \sigma(A_{Q,h,J^*}) \supset \{\mu_n^*\}_{n \in Z}.$$

For a unique solution $K = K(\cdot, \cdot, P, Q, C)$ to (2.34)–(2.36), the solution (u, v) to (2.42) and (2.43) satisfies

(2.60) $$\begin{cases} q_1(x) = p_1(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} - \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \\[3mm] q_2(x) = p_2(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} + \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \quad (0 \leqq x \leqq 1). \end{cases}$$

Noting (2.45) and (2.43), we integrate (2.60) with respect to x, so that we get
$\int_0^x (p_1(s) - q_1(s))ds = \log u(x) + \log v(x)$ and $\int_0^x (p_2(s) - q_2(s))ds = \log u(x) - \log v(x)$,
which imply

(2.61) $$\begin{cases} u(x) = \exp\left(\dfrac{1}{2} \int_0^x (p_1(s) + p_2(s) - q_1(s) - q_2(s))ds \right) \\[3mm] v(x) = \exp\left(\dfrac{1}{2} \int_0^x (p_1(s) - p_2(s) - q_1(s) + q_2(s))ds \right) \quad (0 \leqq x \leqq 1). \end{cases}$$

Solving (2.42) with respect to $K_{12} - K_{21}$ and $K_{11} - K_{22}$ by using (2.61), we obtain

(2.62) $$K_{12}(x, x) - K_{21}(x, x)$$

$$= \frac{1}{4} e^{-\theta_1(x) - \theta_2(x)} (2b(x) - q_1(x) - q_2(x) - p_1(x) + p_2(x))$$

$$+ \frac{1}{4} e^{-\theta_1(x) + \theta_2(x)} (-2b(x) + q_1(x) - q_2(x) + p_1(x) + p_2(x)) \quad (0 \leqq x \leqq 1)$$

and

(2.63) $$K_{11}(x, x) - K_{22}(x, x)$$

$$= \frac{1}{4} e^{-\theta_1(x) - \theta_2(x)} (2a(x) - q_1(x) - q_2(x) + p_1(x) - p_2(x))$$

$$+\frac{1}{4}e^{-\theta_1(x)+\theta_2(x)}(-2a(x)-q_1(x)+q_2(x)+p_1(x)+p_2(x)) \qquad (0\leqq x\leqq 1).$$

Here and henceforth, for $x\in[0, 1]$, we put

(2.64) $\qquad \theta_1(x)=\frac{1}{2}\int_0^x (q_1(s)-p_1(s))ds$ and $\theta_2(x)=\frac{1}{2}\int_0^x (q_2(s)-p_2(s))ds$

and

(2.65) $\qquad\qquad R(x)=e^{-\theta_1(x)}\begin{pmatrix} \cosh\theta_2(x) & -\sinh\theta_2(x) \\ -\sinh\theta_2(x) & \cosh\theta_2(x) \end{pmatrix}.$

Defining $\psi(\cdot, \lambda)=\begin{pmatrix} \phi_1(\cdot, \lambda) \\ \phi_2(\cdot, \lambda) \end{pmatrix}$ by

(2.66) $\qquad\qquad \psi(x, \lambda)=R(x)\phi(x, \lambda)+\int_0^x K(x, y)\phi(y, \lambda)dy \qquad (0\leqq x\leqq 1),$

in virtue of (2.34), (2.35), (2.62) and (2.63), we can apply Lemma 1 (II) in Yamamoto [12], so that we see that

(2.67) $\qquad \begin{cases} B\dfrac{d\phi(x, \lambda)}{dx}+Q(x)\phi(x, \lambda)=\lambda\phi(x, \lambda) \qquad (0\leqq x\leqq 1) \\ \phi(0, \lambda)=\begin{pmatrix} 1 \\ -h \end{pmatrix}. \end{cases}$

In order to prove (2.59), we have only to verify

(2.68) $\qquad\qquad \phi_2(1, \mu_n)+J\phi_1(1, \mu_n)=0 \qquad (n\in\boldsymbol{Z})$

and

(2.69) $\qquad\qquad \phi_2(1, \mu_n^*)+J^*\phi_1(1, \mu_n^*)=0 \qquad (n\in\boldsymbol{Z}).$

These verification is done as follows. We have

$$(\overline{\phi_m^{(1)}(\cdot)}, \overline{\phi(\cdot, \mu_n)})=\overline{(\phi(\cdot, \mu_n), \overline{\phi_m^{(1)}(\cdot)})}=(\phi(\cdot, \mu_n), \phi_m^{(1)}(\cdot))=\delta_{nm}$$

by (2.21), that is, we get

(2.70) $\qquad\qquad (\overline{\phi_m^{(1)}(\cdot)}, \overline{\phi(\cdot, \mu_n)})=\delta_{nm}.$

Similarly we get

(2.71) $\qquad\qquad (\overline{\phi_m^{(2)}(\cdot)}, \overline{\phi(\cdot, \mu_n^*)})=\delta_{nm}.$

Next by $K_{ij}(1, y)=c_{ij}(y, q)$ $(0\leqq y\leqq 1, 1\leqq i, j\leqq 2)$, we see

(2.72)
$$\begin{pmatrix} K_{11}(1,\ y) \\ K_{12}(1,\ y) \end{pmatrix} = \frac{1}{J-J^*}\ \sum_{n=-\infty}^{\infty}\ (a_n(q)\overline{\psi_n^{(1)}(y)} - b_n(q)\overline{\psi_n^{(2)}(y)})$$

and

(2.73)
$$\begin{pmatrix} K_{21}(1,\ y) \\ K_{22}(1,\ y) \end{pmatrix} = \frac{1}{J^*-J}\ \sum_{n=-\infty}^{\infty}\ (J^*a_n(q)\overline{\psi_n^{(1)}(y)} - Jb_n(q)\overline{\psi_n^{(2)}(y)}),$$

where the right hand sides of (2.72) and (2.73) are convergent in $\{L^2(0,\ 1)\}^2$. Thus, by using (2.70), the equalities (2.72) and (2.73) imply

(2.74)
$$\left\{ \begin{array}{l} \left(\begin{pmatrix} K_{11}(1,\ \cdot) \\ K_{12}(1,\ \cdot) \end{pmatrix},\ \overline{\phi(\cdot,\ \mu_n)} \right)_{\{L^2(0,1)\}^2} \\[2mm] = \dfrac{a_n(q)}{J-J^*} - \dfrac{1}{J-J^*}\ \sum_{m=-\infty}^{\infty}\ b_m(q)(\overline{\psi_m^{(2)}(\cdot)},\ \overline{\phi(\cdot,\ \mu_n)}) \qquad (n\in Z) \\[3mm] \left(\begin{pmatrix} K_{21}(1,\ \cdot) \\ K_{22}(1,\ \cdot) \end{pmatrix},\ \overline{\phi(\cdot,\ \mu_n)} \right)_{\{L^2(0,1)\}^2} \\[2mm] = \dfrac{J^*a_n(q)}{J^*-J} - \dfrac{J}{J^*-J}\ \sum_{m=-\infty}^{\infty}\ b_m(q)(\overline{\psi_m^{(2)}(\cdot)},\ \overline{\phi(\cdot,\ \mu_n)}) \qquad (n\in Z). \end{array} \right.$$

Therefore we have, for $n\in Z$,

$$\phi_2(1,\ \mu_n) + J\phi_1(1,\ \mu_n)$$

$$= e^{-\theta_1(1)}(J\cdot\cosh\theta_2(1) - \sinh\theta_2(1))\phi_1(1,\ \mu_n)$$

$$+ e^{-\theta_1(1)}(\cosh\theta_2(1) - J\cdot\sinh\theta_2(1))\phi_2(1,\ \mu_n)$$

$$+ \left(\begin{pmatrix} K_{21}(1,\ \cdot) \\ K_{22}(1,\ \cdot) \end{pmatrix},\ \overline{\phi(\cdot,\ \mu_n)} \right)_{L^2} + J \times \left(\begin{pmatrix} K_{11}(1,\ \cdot) \\ K_{12}(1,\ \cdot) \end{pmatrix},\ \overline{\phi(\cdot,\ \mu_n)} \right)_{L^2}$$

(by (2.66))

$$= \frac{2e^{\theta_2(1)-\theta_1(1)}}{(H+1)e^{2\theta_2(1)}+1-H} \times (\phi_2(1,\ \mu_n) + H\phi_1(1,\ \mu_n)) + a_n(q)$$

(by (2.11) and (2.74))

$$= 0 \qquad\qquad\qquad \text{(by the definition (2.9) of } a_n(q)\text{)},$$

which implies (2.68).

For (2.69), we can similarly proceed in view of (2.71), (2.72), (2.73), (2.66), (2.12) and (2.10). Thus we complete the proof of (2.68) and (2.69).

Finally we have to prove that

(2.75)
$$\sigma(A_{Q,h,J}) \subset \{\mu_n\}_{n\in Z}$$

and

(2.76) $\sigma(A_{Q,h,J^*}) \subset \{\mu_n^*\}_{n\in\mathbf{Z}}$.

To this end, we show

LEMMA 8. *Let us denote the solution to* (2.77) *by*

$$\psi^*(\cdot, \ \lambda) = \begin{pmatrix} \psi_1^*(\cdot, \ \lambda) \\ \psi_2^*(\cdot, \ \lambda) \end{pmatrix} :$$

(2.77)
$$\begin{cases} B \dfrac{d\psi^*(x, \ \lambda)}{dx} - {}^t Q(x)\psi^*(x, \ \lambda) = \lambda\psi^*(x, \ \lambda) \qquad (0 \leq x \leq 1) \\ \psi^*(0, \ \lambda) = \begin{pmatrix} 1 \\ h \end{pmatrix}. \end{cases}$$

Then

(I) *The equalities*

(2.78) $\psi_2^*(1, \ \overline{-\mu_n}) - J\psi_1^*(1, \ \overline{-\mu_n}) = 0 \qquad (n \in \mathbf{Z})$

hold.

(II) *For each* $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \{L^2(0, \ 1)\}^2,$ *we get*

(2.79) $u = \displaystyle\sum_{n=-\infty}^{\infty} \dfrac{(u, \ \psi^*(\cdot, \ \overline{-\mu_n}))}{\alpha_n} \ \phi(\cdot, \ \mu_n),$

where the right hand side is convergent in $\{L^2(0, \ 1)\}^2$ *and we set*
$\alpha_n = (\phi(\cdot, \ \mu_n), \ \psi^*(\cdot, \ \overline{-\mu_n})).$

Moreover, for $\{\phi(\cdot, \ \mu_n^*)\}_{n\in\mathbf{Z}},$ *we obtain similar results.*

In Appendix VII, a proof of this lemma is given.

Now we return to the proof of (2.75) and (2.76). Assume that there exists $\nu \in \sigma(A_{Q,h,J})$ such that

(2.80) $\nu \neq \mu_n \qquad (n \in \mathbf{Z}).$

Let $u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq 0$ be an eigenvector of $A_{Q,h,J}$ associated with ν. Then, since $\psi^*(\cdot, \ \overline{-\mu_n})$ satisfies (2.77) with $\lambda = \overline{-\mu_n}$ and (2.78), and u satisfies $u_2(0) + hu_1(0) = u_2(1) + Ju_1(1) = 0$, we see by integration by parts that

(2.81) $(A_{Q,h,J}u, \ \psi^*(\cdot, \ \overline{-\mu_n})) = \left(u, \ -B \dfrac{d}{dx} \psi^*(\cdot, \ \overline{-\mu_n}) + {}^t Q(\cdot)\psi^*(\cdot, \ \overline{-\mu_n})\right)$

$$= (u, \ \overline{\mu_n} \psi^*(\cdot, \ \overline{-\mu_n})).$$

Therefore, for $n \in \mathbf{Z}$, we have

$$\nu(u, \ \psi^*(\cdot, \ \overline{-\mu_n})) = (A_{Q,h,J}u, \ \psi^*(\cdot, \ \overline{-\mu_n})) \qquad \text{(by } A_{Q,h,J}u = \nu u)$$

$$= (u, \ \overline{\mu_n} \psi^*(\cdot, \ \overline{-\mu_n})) \qquad\qquad\qquad \text{(by (2.81))}$$

$$= \mu_n(u, \ \psi^*(\cdot, \ \overline{-\mu_n})),$$

which implies $(u, \ \psi^*(\cdot, \ \overline{-\mu_n})) = 0$ for each $n \in \mathbf{Z}$ by (2.80). Therefore it follows from (2.79) that $u = 0$. This contradicts that $u \neq 0$. Thus we see (2.75). Similarly we can prove (2.76). This completes the proof of Lemma 7.

2.7. Completion of the proof of Theorem 2

Applying the principle of contraction mappings (Kolmogorov and Fomin [6, p. 66], for instance), we complete the proof of Theorem 2. To this end, provided that δ_0 is sufficiently small, we have only to verify

(I) *G is a contraction mapping, that is, there exists some constant* $0 \leq \kappa < 1$ *such that*

$$\|Gq^{(1)} - Gq^{(2)}\|_{C^0} \leq \kappa \|q^{(1)} - q^{(2)}\|_{C^0} \qquad (q^{(1)}, \ q^{(2)} \in \mathcal{A}_M).$$

(II) $G\mathcal{A}_M \subset \mathcal{A}_M$.

In fact, if (I) and (II) are proved, then since $\mathcal{A}_M$ is a closed set in $\{C^0[0, 1]\}^2$ by the definition (2.2), the principle of contraction mappings implies the unique existence of fixed point $q = (q_1, \ q_2)$ of G. For this $(q_1, \ q_2)$, we define J and J^* by (2.11) and (2.12), respectively. Then, in view of Lemma 7, $Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}$ and $J, \ J^*$ satisfy (1.26).

Now we proceed to

Proof of (I). For $q^{(i)} = (q_1^{(i)}, \ q_2^{(i)}) \in \mathcal{A}_M \ (i = 1, \ 2)$, let us set $Q_i = \begin{pmatrix} a & b \\ q_1^{(i)} & q_2^{(i)} \end{pmatrix}$ and $r^{(i)} = G_i(K(\cdot, \ \cdot, \ P, \ Q_i, \ C(\cdot, \ q^{(i)}))) = Gq^{(i)}$, where $r^{(i)} = (r_1^{(i)}, \ r_2^{(i)})$. Henceforth, for brevity, we put $K^{(i)}(x, \ y) = K(x, \ y, \ P, \ Q_i, \ C(\cdot, \ q^{(i)}))$ $((x, \ y) \in \bar{\Omega}, \ i = 1, \ 2)$. Then, by the estimate (2.39) of Lemma 5, we have

$$\|K^{(1)} - K^{(2)}\|_{C^0} \leq M_5(\|C(\cdot, \ q^{(2)})\|_{C^0}\|Q_1 - Q_2\|_{C^0} + \|C(\cdot, \ q^{(1)}) - C(\cdot, \ q^{(2)})\|_{C^0}).$$

Thus, by the estimates (2.27) and (2.29) in Lemma 4, we get

$$(2.82) \qquad \|K^{(1)} - K^{(2)}\|_{\{C^0(\bar{\omega})\}^4} \leq 2M_4 M_5 \delta_0 \|q^{(1)} - q^{(2)}\|_{\{C^0[0,1]\}^2}.$$

On the other hand, by Lemma 6 (III), we have the estimate

$$\|r_1^{(1)} - r_1^{(2)}\|_{C^0} + \|r_2^{(1)} - r_2^{(2)}\|_{C^0} \leq M_7 \|K^{(1)} - K^{(2)}\|_{C^0},$$

with which we combine (2.82), so that we reach

(2.83) $\|r^{(1)}-r^{(2)}\|_{C^0}\leqq 2\,M_4 M_5 M_7\delta_0\|q^{(1)}-q^{(2)}\|_{C^0}$.

Therefore, if δ_0 is sufficiently small so that

(2.84) $\kappa\equiv 2\,M_4 M_5 M_7\delta_0<1$,

then we see that G is a contraction mapping. This shows the assertion (I).

   Proof of (II). Let $q=(q_1,\ q_2)\in\mathcal{A}_M$. Since c_{ij} ($1\leqq i,\ j\leqq 2$) are real-valued functions by (2.25), also the solution K to (2.34)-(2.36) with $D=C$, is real-valued. Therefore we see that Gq is real-valued. Next we have to prove that for $(r_1,\ r_2)=Gq$,

(2.85) $\|r_1-p_1\|_{C^0}\leqq M$ and $\|r_2-p_2\|_{C^0}\leqq M$.

Firstly, by the inequality (2.47) of Lemma 6, we see

(2.86) $\max\{\|r_1-p_1\|_{C^0},\ \|r_2-p_2\|_{C^0}\}\leqq M_7\max_{\substack{1\leqq i,\ j\leqq 2\\0\leqq x\leqq 1}}|K_{ij}(x,\ x,\ P,\ Q,\ C)|$.

Here we put $P=\begin{pmatrix}a&b\\p_1&p_2\end{pmatrix}$ and $Q=\begin{pmatrix}a&b\\q_1&q_2\end{pmatrix}$.

Secondly, by the inequality (2.37) of Lemma 5, we have

(2.87) $\|K(\cdot,\ \cdot,\ P,\ Q,\ C)\|_{(C^0(\overline{D}))^4}\leqq M_5\|C(\cdot,\ q)\|_{(C^0[0,1])^4}$.

Finally, by the estimate (2.27) of Lemma 4, we have

(2.88) $\|C(\cdot,\ q)\|_{(C^0[0,1])^4}\leqq M_4\delta_0$.

Therefore, combining (2.86), (2.87) and (2.88), we reach

(2.89) $\max\{\|r_1-p_1\|_{C^0[0,1]},\ \|r_2-p_2\|_{C^0[0,1]}\}\leqq M_4 M_5 M_7\delta_0$.

Consequently we take a sufficiently small δ_0 so that

(2.90) $M_4 M_5 M_7\delta_0\leqq M$,

we see (2.85).

   Thus we complete the proof of the assertions (I) and (II), so that Theorem 2 is proved.

§3. Proof of Theorem 1

   We prove Theorem 1 separately in the two cases:

   CASE 1. $\delta\equiv\sum_{n=-\infty}^{\infty}(|n|+1)(|\mu_n-\lambda_n|+|\mu_n^*-\lambda_n^*|)<\infty$.

CASE 2. $\delta = \infty$.

Proof in Case 1. Let $Q \in A(a, b)$, J, $J^* \in \mathbf{R} \setminus \{-1, 1\}$ and let

(3.1) $\sigma(A_{Q,h,J}) = \{\mu_n\}_{n \in \mathbf{Z}}$ and $\sigma(A_{Q,h,J^*}) = \{\mu_n^*\}_{n \in \mathbf{Z}}$.

If

(3.2) $$\delta_0 \equiv \sum_{n=-\infty}^{\infty} (|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$$

is sufficiently small for P, h, H and H^*, then by Theorem 2, there exist
$\widetilde{Q} = \begin{pmatrix} a & b \\ \tilde{q}_1 & \tilde{q}_2 \end{pmatrix} \in A(a, b)$ and $\widetilde{J}$, $\widetilde{J}^* \in \mathbf{R} \setminus \{-1, 1\}$ such that

(3.1)′ $\sigma(A_{\widetilde{Q},h,\widetilde{J}}) = \{\mu_n\}_{n \in \mathbf{Z}}$ and $\sigma(A_{\widetilde{Q},h,\widetilde{J}^*}) = \{\mu_n^*\}_{n \in \mathbf{Z}}$.

In view of the estimate (2.89), we get

(3.3) $$\|\tilde{q}_1 - p_1\|_{C^0} + \|\tilde{q}_2 - p_2\|_{C^0} \leqq M_8 \delta_0,$$

and, combining the estimates (2.28), (2.38) and (2.48), we can obtain

(3.4) $$\|\tilde{q}_1 - p_1\|_{C^1} + \|\tilde{q}_2 - p_2\|_{C^1} \leqq M_8 \delta.$$

Moreover, since $\widetilde{J}$ and $\widetilde{J}^*$ are given by (2.11) and (2.12), it follows from (3.3) that

(3.5) $|\widetilde{J} - H| + |\widetilde{J}^* - H^*|$

$\leqq M_1^{-1}(|1 - H^2| + |1 - H^{*2}|) \left| \exp\left(\int_0^1 (\tilde{q}_2(s) - p_2(s)) ds \right) - 1 \right|$ (by (2.14))

$\leqq M_8 \delta_0$ (by (3.3) and the mean value theorem for e^x).

On the other hand, by Corollary 1 in Yamamoto [13], the relations (3.1) and (3.1)′ imply

(3.6) $\begin{cases} \tilde{q}_1(x) = q_1(x), \ \tilde{q}_2(x) = q_2(x) & (0 \leqq x \leqq 1) \\ \widetilde{J} = J, \ \widetilde{J}^* = J^*. \end{cases}$

Thus we can obtain (1.20) and (1.21) by (3.3)–(3.6). This completes the proof of Theorem 1 in Case 1.

Proof in Case 2. In this case, we have to prove

(3.7) $|H - J| + |H^* - J^*| \leqq M_8 \delta_0$

and

(3.8) $\|q_1 - p_1\|_{C^0[0,1]} + \|q_2 - p_2\|_{C^0[0,1]} \leqq M_8 \delta_0.$

Without loss of generality, we may assume that $\delta_0 < \infty$. Along the line of the proof of Theorem 2, we divide the proof into three steps.

First Step. In this step, we derive

(3.9.1)
$$J = \left((H+1) \exp \left(\int_0^1 (q_2(s) - p_2(s)) ds \right) + H - 1 \right)$$
$$\times \left((H+1) \exp \left(\int_0^1 (q_2(s) - p_2(s)) ds \right) + 1 - H \right)^{-1}$$

and

(3.9.2)
$$J^* = \left((H^*+1) \exp \left(\int_0^1 (q_2(s) - p_2(s)) ds \right) + H^* - 1 \right)$$
$$\times \left((H^*+1) \exp \left(\int_0^1 (q_2(s) - p_2(s)) ds \right) + 1 - H^* \right)^{-1}.$$

Derivation of (3.9). Since $\delta_0 < \infty$, we have $\lim_{|n| \to \infty} (\mu_n - \lambda_n) = 0$ and $\lim_{|n| \to \infty} (\mu_n^* - \lambda_n^*) = 0$, which imply

(3.10)
$$\lim_{|n| \to \infty} (\theta + \gamma_1 + (m_1(n) - m_2(n)) \pi \sqrt{-1}) = 0$$

and

(3.11)
$$\lim_{|n| \to \infty} (\theta + \gamma_2 + (m_3(n) - m_4(n)) \pi \sqrt{-1}) = 0$$

by Russell [10], [11] (cf. (1.15)). Here and henceforth, we set

(3.12)
$$\theta = \frac{1}{2} \int_0^1 (q_2(s) - p_2(s)) ds$$

and

(3.13)
$$\begin{cases} \gamma_1 = \frac{1}{2} \left\{ \log \frac{(1+h)(1-J)}{(1-h)(1+J)} - \log \frac{(1+h)(1-H)}{(1-h)(1+H)} \right\} \\ \gamma_2 = \frac{1}{2} \left\{ \log \frac{(1+h)(1-J^*)}{(1-h)(1+J^*)} - \log \frac{(1+h)(1-H^*)}{(1-h)(1+H^*)} \right\} \end{cases}$$

and $m_i(n)$ $(1 \leq i \leq 4)$ denote integers depending upon n such that $\lim_{n \to \infty} |m_i(n)| = \infty$ and $|m_i(n)| = O(|n|)$, and in (3.13), we take the principal values of the logarithms. Since h, H, H^*, J, $J^* \in R \setminus \{-1, 1\}$, and $\theta \in R$, we have $\mathrm{Im}(\theta + \gamma_i) = 0$ or $\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$ $(i=1, 2)$. Therefore (3.10) and (3.11) imply $\mathrm{Im}(\theta + \gamma_i) = 0$ and $\mathrm{Re}(\theta + \gamma_i) = 0$ $(i=1, 2)$, which are seen to be equivalent to (3.9) by direct computations.

Second Step. In this step, we prove Lemma 9, which is a converse of Lemma 7:

LEMMA 9. *Let* $\delta_0 \equiv \sum_{n=-\infty}^{\infty} (|\mu_n - \lambda_n| + |\mu_n^* - \lambda_n^*|)$ *be sufficiently small. If* $q = (q_1, q_2)$ $\in \{C^1[0, 1]\}^2$ *satisfies*

$$(3.14) \qquad \sigma(A_{Q,h,J}) = \{\mu_n\}_{n \in \mathbf{Z}} \quad and \quad \sigma(A_{Q,h,J^*}) = \{\mu_n^*\}_{n \in \mathbf{Z}},$$

then q *is a fixed point of the mapping* G *defined in* §2.

Proof of Lemma 9. By Lemma 1 in [12], there exists a unique solution $K = K(x, y) \in \{C^1(\bar{\Omega})\}^4$ to (2.34), (2.35), (2.62) and (2.63), and $\psi(\cdot, \lambda) = \begin{pmatrix} \phi_1(\cdot, \lambda) \\ \phi_2(\cdot, \lambda) \end{pmatrix}$ defined by (2.66) satisfies $B \dfrac{d\psi(x, \lambda)}{dx} + Q(x)\psi(x, \lambda) = \lambda\psi(x, \lambda)$ $(0 \leq x \leq 1)$ and $\psi(0, \lambda) = \begin{pmatrix} 1 \\ -h \end{pmatrix}$. Therefore, since $\sigma(A_{Q,h,J}) = \{\mu_n\}_{n \in \mathbf{Z}}$ and μ_n is a simple eigenvalue, we see that $\psi(\cdot, \mu_n)$ is an eigenvector of $A_{Q,h,J}$ associated with μ_n, so that we get

$$(3.15) \qquad \phi_2(1, \mu_n) + J\phi_1(1, \mu_n) = 0 \qquad (n \in \mathbf{Z}).$$

Similarly we can get

$$(3.16) \qquad \phi_2(1, \mu_n^*) + J^*\phi_1(1, \mu_n^*) = 0 \qquad (n \in \mathbf{Z}).$$

Substituting (2.66) into (3.15) and (3.16), we obtain for $n \in \mathbf{Z}$,

$$\left(\begin{pmatrix} (K_{21} + JK_{11})(1, \cdot) \\ (K_{22} + JK_{12})(1, \cdot) \end{pmatrix}, \overline{\phi(\cdot, \mu_n)} \right)_{\{L^2(0,1)\}^2} = a_n(q)$$

and

$$\left(\begin{pmatrix} (K_{21} + J^*K_{11})(1, \cdot) \\ (K_{22} + J^*K_{12})(1, \cdot) \end{pmatrix}, \overline{\phi(\cdot, \mu_n^*)} \right)_{\{L^2(0,1)\}^2} = b_n(q),$$

where $a_n(q)$, $b_n(q)$ are defined by (2.9) and (2.10). From the assumption that δ_0 is sufficiently small, we can apply (2.22) and (2.22)′ in Lemma 3, so that we see that

$$(3.17) \qquad K_{ij}(1, y) = c_{ij}(y, q) \qquad (0 \leq y \leq 1, \ 1 \leq i, \ j \leq 2).$$

Here $c_{ij}(\cdot, q)$ $(1 \leq i, \ j \leq 2)$ are nothing but the functions defined by (2.23) and (2.24). Considering the hyperbolic equation (2.34) with (2.35) and (3.17), in view of uniqueness of solutions to the problem (Lemma 5), we see that

(3.18) $K = (G_3 \circ G_2 \circ G_1)(q)$,

G_1, G_2 and G_3 being defined by (2.13), (2.30) and (2.40), respectively.

As is seen by direct computations, the equalities (2.62) and (2.63) imply the following: $\begin{pmatrix} u \\ v \end{pmatrix}$ given by

(3.19)
$$\begin{cases} u(x) = \exp\left(\dfrac{1}{2}\int_0^x (p_1(s) + p_2(s) - q_1(s) - q_2(s))ds\right) \\ v(x) = \exp\left(\dfrac{1}{2}\int_0^x (p_1(s) - p_2(s) - q_1(s) + q_2(s))ds\right) \quad (0 \le x \le 1) \end{cases}$$

is the solution to

(3.20) $$\frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = A(x)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} + $$

$$\begin{pmatrix} K_{11}(x,\ x) - K_{22}(x,\ x) + K_{12}(x,\ x) - K_{21}(x,\ x) \\ K_{11}(x,\ x) - K_{22}(x,\ x) + K_{21}(x,\ x) - K_{12}(x,\ x) \end{pmatrix} \quad (0 \le x \le 1)$$

and

(3.21) $$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Here let us recall that $A(x)$ is defined by (2.41).

By (3.19), we get $q_1(x) = p_1(x) - \dfrac{1}{u(x)}\dfrac{du(x)}{dx} - \dfrac{1}{v(x)}\dfrac{dv(x)}{dx}$ and $q_2(x) = p_2(x) - \dfrac{1}{u(x)}\dfrac{du(x)}{dx} + \dfrac{1}{v(x)}\dfrac{dv(x)}{dx}$ $(0 \le x \le 1)$, which imply

(3.22) $q = (q_1, q_2) = G_4 K$.

Here we recall that G_4 is defined by (2.53). The relations (3.18) and (3.22) imply that $q = (q_1, q_2)$ is a fixed point of $G = G_4 \circ G_3 \circ G_2 \circ G_1$. Thus we complete the proof of Lemma 9.

Third Step. By the final stage of the proof of Theorem 2 (§ 2.7), we see that if δ_0 is sufficiently small, then G possesses a unique fixed point $(\tilde{q}_1, \tilde{q}_2)$, and the estimate

(3.23) $\|\tilde{q}_1 - p_1\|_{C^0} + \|\tilde{q}_2 - p_2\|_{C^0} \le M_8 \delta_0$

holds.

On the other hand, by Lemma 9, $q = (q_1, q_2)$ is a fixed point of G, so that the

uniqueness of fixed points imply $\tilde{q}_1 = q_1$ and $\tilde{q}_2 = q_2$, that is, we obtain (3.8), namely, (1.21) for $j=0$.

For $j=1$, the estimate (1.21) is trivial by $\delta = \infty$. In a way similar to Case 1, we can prove (3.7) by using (3.8). Thus we complete the proof of Theorem 1.

Appendix I. Proof of Proposition 1

In view of the results of Russell [10], [11] (cf. (1.15)), we see that for $\lambda_n \in \sigma(A_{P,h,H})$, we have $\lambda_n = \gamma + n\pi \sqrt{-1} + O\left(\dfrac{1}{n}\right)$ (as $|n| \to \infty$) under an appropriate renumbering. Here γ is a constant depending on P, h, H (cf. (1.14)). Therefore we see that $\lambda_n \notin \boldsymbol{R}$ for sufficiently large $|n|$, which means the part (I) of this proposition.

Now we proceed to a proof of the part (II). Let us assume that $\lambda \in \sigma(A_{P,h,H})$, and let $\phi(\cdot, \lambda) = \begin{pmatrix} \phi_1(\cdot, \lambda) \\ \phi_2(\cdot, \lambda) \end{pmatrix}$ be an eigenvector of $A_{P,h,H}$ associated with λ. Since P is real-valued and h, $H \in \boldsymbol{R}$, the complex conjugate $\overline{\phi(\cdot, \lambda)}$ satisfies

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d\overline{\phi(x, \lambda)}}{dx} + P(x)\overline{\phi(x, \lambda)} = \bar{\lambda} \cdot \overline{\phi(x, \lambda)} \qquad (0 \le x \le 1),$$

and

$$\overline{\phi_2(0, \lambda)} + h\overline{\phi_1(0, \lambda)} = 0$$

$$\overline{\phi_2(1, \lambda)} + H\overline{\phi_1(1, \lambda)} = 0,$$

by which we see that $\overline{\phi(\cdot, \lambda)}$ is an eigenvector of $A_{P,h,H}$ associated with $\bar{\lambda}$. That is, we see that $\lambda \in \sigma(A_{P,h,H})$ implies $\bar{\lambda} \in \sigma(A_{P,h,H})$. Similarly we can show that $\bar{\lambda} \in \sigma(A_{P,h,H})$ implies $\lambda \in \sigma(A_{P,h,H})$. Thus we complete the proof of Proposition 1.

Appendix II. Proof of Lemma 2

First, we show the following Lemmas II.1 and II.1, which are useful also in Appendix III. Let us recall that $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Omega = \{(x, y); 0 < y < x < 1\}$.

LEMMA II.1. (I) *For given* $P = \begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix} \in \{C^1[0, 1]\}^4$ *and* $h \in \boldsymbol{R} \setminus \{-1, 1\}$, *there exists a unique* $U = U(x, y) = (U_{ij}(x, y))_{1 \le i, j \le 2} \in \{C^1(\bar{\Omega})\}^4$ *satisfying* (II.1)–(II.4):

(II.1) $B \dfrac{\partial U(x, y)}{\partial x} + P(x)U(x, y) = -\dfrac{\partial U(x, y)}{\partial y} B \qquad ((x, y) \in \bar{\Omega}).$

(II.2) $U_{12}(x,\ 0)=hU_{11}(x,\ 0)\ \ and\ \ U_{22}(x,\ 0)=hU_{21}(x,\ 0)$ $(0\leqq x\leqq 1)$.

(II.3) $U_{12}(x,\ x)-U_{21}(x,\ x)=\dfrac{1}{4}e^{-\eta_1(x)-\eta_2(x)}(a(x)+b(x)-p_1(x)-p_2(x))$

$$+\dfrac{1}{4}e^{-\eta_1(x)+\eta_2(x)}(a(x)-b(x)+p_1(x)-p_2(x))\qquad (0\leqq x\leqq 1).$$

(II.4) $U_{11}(x,\ x)-U_{22}(x,\ x)=\dfrac{1}{4}e^{-\eta_1(x)-\eta_2(x)}(a(x)+b(x)-p_1(x)-p_2(x))$

$$+\dfrac{1}{4}e^{-\eta_1(x)+\eta_2(x)}(-a(x)+b(x)-p_1(x)+p_2(x))\qquad (0\leqq x\leqq 1).$$

Here and henceforth, for $x\in[0,\ 1]$, *we put*

(II.5) $\eta_1(x)=\dfrac{1}{2}\displaystyle\int_0^x(b(s)+p_1(s))ds\ \ and\ \ \eta_2(x)=\dfrac{1}{2}\int_0^x(a(s)+p_2(s))ds$.

   (II) *For the solution* $U=(U_{ij})_{1\leq i,\ j\leq 2}$, *we have the estimates*

(II.6) $\|U\|_{\{C^0(\bar{\Omega})\}^4}\leqq M_9(\|P\|_{\{C^0[0,1]\}^4},\ h)$

and

(II.7) $\|U\|_{\{C^1(\bar{\Omega})\}^4}\leqq M_{10}(\|P\|_{\{C^1[0,1]\}^4},\ h)$.

   (III) *For* $\lambda\in\boldsymbol{C}$, *let us set*

(II.8) $f(x,\ \lambda)=\begin{pmatrix}f_1(x,\ \lambda)\\f_2(x,\ \lambda)\end{pmatrix}=\begin{pmatrix}\cosh\lambda x-h\sinh\lambda x\\\sinh\lambda x-h\cosh\lambda x\end{pmatrix}$

and

(II.9) $S(x)=e^{-\eta_1(x)}\begin{pmatrix}\cosh\eta_2(x)&-\sinh\eta_2(x)\\-\sinh\eta_2(x)&\cosh\eta_2(x)\end{pmatrix}$ $(0\leqq x\leqq 1)$.

Then $\phi(\cdot,\ \lambda)$ *defined by*

(II.10) $\phi(x,\ \lambda)=S(x)f(x,\ \lambda)+\displaystyle\int_0^x U(x,\ y)f(y,\ \lambda)dy$ $(0\leqq x\leqq 1)$

satisfies (2.7).

   Similarly the following facts hold:

   (I)′ *There exists a unique* $V=V(x,\ y)=(V_{ij}(x,\ y))_{1\leq i,j\leq 2}\in\{C^1(\bar{\Omega})\}^4$ *satisfying*
(II.1)′–(II.4)′ :

(II.1)′ $B\dfrac{\partial V(x,\ y)}{\partial x}-{}^tP(x)V(x,\ y)=-\dfrac{\partial V(x,\ y)}{\partial y}B$ $((x,\ y)\in\bar{\Omega})$.

(II.2)′ $V_{12}(x,\ 0)=-hV_{11}(x,\ 0)\ \ and\ \ V_{22}(x,\ 0)=-hV_{21}(x,\ 0)$ $(0\leqq x\leqq 1)$.

(II. 3)′ $\qquad V_{12}(x, \; x)-V_{21}(x, \; x)=\dfrac{1}{4}e^{\eta_1(x)+\eta_2(x)}\left(-a(x)+b(x)-p_1(x)+p_2(x)\right)$

$\qquad\qquad +\dfrac{1}{4}e^{\eta_1(x)-\eta_2(x)}\left(-a(x)-b(x)+p_1(x)+p_2(x)\right) \qquad (0\leqq x\leqq 1).$

(II. 4)′ $\qquad V_{11}(x, \; x)-V_{22}(x, \; x)=\dfrac{1}{4}e^{\eta_1(x)+\eta_2(x)}\left(-a(x)+b(x)-p_1(x)+p_2(x)\right)$

$\qquad\qquad +\dfrac{1}{4}e^{\eta_1(x)-\eta_2(x)}\left(a(x)+b(x)-p_1(x)-p_2(x)\right) \qquad (0\leqq x\leqq 1).$

(II)′ *The estimates*

(II. 6)′ $\qquad\qquad\qquad \|V\|_{[C^0(\overline{D})]^4}\leqq M_9(\|P\|_{[C^0[0,1]]^4}, \; h)$

and

(II. 7)′ $\qquad\qquad\qquad \|V\|_{[C^1(\overline{D})]^4}\leqq M_{10}(\|P\|_{[C^1[0,1]]^4}, \; h)$

hold.

(III)′ *Let us set*

(II. 8)′ $\qquad f^*(x, \; \lambda)=\begin{pmatrix} f_1^*(x, \; \lambda) \\ f_2^*(x, \; \lambda) \end{pmatrix}=\begin{pmatrix} \cosh\lambda x+h\sinh\lambda x \\ \sinh\lambda x+h\cosh\lambda x \end{pmatrix}$

and

(II. 9)′ $\qquad T(x)=e^{\eta_1(x)}\begin{pmatrix} \cosh\eta_2(x) & \sinh\eta_2(x) \\ \sinh\eta_2(x) & \cosh\eta_2(x) \end{pmatrix} \qquad (0\leqq x\leqq 1).$

Then $\phi^*(\cdot, \; \lambda)$ *defined by*

(II. 10)′ $\qquad \phi^*(x, \; \lambda)=T(x)f^*(x, \; \lambda)+\displaystyle\int_0^x V(x, \; y)f^*(y, \; \lambda)dy \qquad (0\leqq x\leqq 1)$

satisfies (2.8).

$\qquad$ *Proof of Lemma* II. 1. The parts (I) and (III) follow directly from the parts (I) and (II) of Lemma 1 in [12], respectively. On the other hand, we can show the estimate (II. 6) by means of the inequalities for the iterative approximate solutions for (II. 1)-(II. 4). Those inequalities are derived in the course of the proof of Proposition 1 in [12], and so we omit those derivation. (See Appendix I in [12].) Similarly the estimate (II. 7) can be obtained and so we omit the detail.

$\qquad$ Now we recall that $\sigma(A_{P,h,H})=\{\lambda_n\}_{n\in Z}$ and $\sigma(A_{P,h,H^*})=\{\lambda_n^*\}_{n\in Z}$.

$\qquad$ LEMMA II. 2. *For* $n\in Z$, *we have*

(II. 11) $\qquad\qquad \|\phi(\cdot, \; \mu_n)-\phi(\cdot, \; \lambda_n)\|_{[C^0[0,1]]^2}\leqq M_{11}|\mu_n-\lambda_n|$

and

(II. 12) $\|\phi(\cdot,\ \mu_n^*)-\phi(\cdot,\ \lambda_n^*)\|_{[C^0[0,1]]^2} \leq M_{11}|\mu_n^*-\lambda_n^*|.$

Here M_{11} is a positive constant depending on $\|P\|_{[C^0[0,1]]^4}$, h, H, H^*, δ_0, and M_{11} remains bounded as δ_0 is bounded.

Proof of Lemma II. 2.   We have only to prove

(II. 13) $|\exp(\mu_n x)-\exp(\lambda_n x)| \leq M_{11}'|\mu_n-\lambda_n|$ $(0\leq x\leq 1,\ n\in\mathbf{Z})$

for a positive constant M_{11}' with a property similar to M_{11}.   In fact, assume that (II. 13) is proved.   Then, from (II. 13) we see

(II. 14) $\|f(\cdot,\ \mu_n)-f(\cdot,\ \lambda_n)\|_{[C^0[0,1]]^2} \leq M_{11}'|\mu_n-\lambda_n|$ $(n\in\mathbf{Z}).$

Applying the estimates (II. 14) and (II. 6) in (II. 10), we reach (II. 11).   Similarly we can prove (II. 12).

Now we return to the proof of (II. 13).   By the mean value theorem, we get

(II. 15) $|\exp(\mu_n x)-\exp(\lambda_n x)| \leq 2\max_{0\leq t\leq 1}|\exp(t\mu_n x+(1-t)\lambda_n x)||\mu_n-\lambda_n|.$

On the other hand, by (1. 15) (cf. Russell [10]), for each $n\in\mathbf{Z}$, there exists $m(n)\in\mathbf{Z}$ such that $\lim_{|n|\to\infty}|m(n)|=\infty$, $|m(n)|=O(|n|)$ and

(II. 16) $\lambda_n=\gamma+m(n)\pi\sqrt{-1}+\dfrac{\alpha_{m(n)}}{m(n)},$

where γ is a constant given by (1. 14) and $\alpha_n\in\mathbf{C}$ $(n\in\mathbf{Z})$ satisfy

(II. 17) $\alpha\equiv\sup_{n\in\mathbf{Z}}|\alpha_n|<\infty.$

Furthermore, by $\delta_0=\sum\limits_{n=-\infty}^{\infty}(|\mu_n-\lambda_n|+|\mu_n^*-\lambda_n^*|)$, we have

(II. 18) $|\mathrm{Re}\mu_n| \leq |\mathrm{Re}(\mu_n-\lambda)|+|\mathrm{Re}\lambda_n| \leq |\mu_n-\lambda_n|+|\mathrm{Re}\lambda_n| \leq \delta_0+|\mathrm{Re}\lambda_n|$
$(n\in\mathbf{Z}).$

From (II. 16)–(II. 18), we can see

        $|\exp(t\mu_n x+(1-t)\lambda_n x)| \leq \exp(2|\mathrm{Re}\ \gamma|+2\alpha+\delta_0)$ $(0\leq t,\ x\leq 1)$

and therefore, we see (II. 13) by (II. 15).

Now we proceed to the proof of Lemma 2.   Since $\lambda_n\in\sigma(A_{P,h,H})$, we have $\phi_2(1,\ \lambda_n)+H\phi_1(1,\ \lambda_n)=0$ $(n\in\mathbf{Z})$.   Therefore we get

    $|\phi_2(1,\ \mu_n)+H\phi_1(1,\ \mu_n)| = |\phi_2(1,\ \mu_n)-\phi_2(1,\ \lambda_n)+H(\phi_1(1,\ \mu_n)-\phi_1(1,\ \lambda_n))|$

    $\leq |\phi_2(1,\ \mu_n)-\phi_2(1,\ \lambda_n)|+|H||\phi_1(1,\ \mu_n)-\phi_1(1,\ \lambda_n)|$

$$\leqq (1+|H|)\|\phi(\cdot,\ \mu_n)-\phi(\cdot,\ \lambda_n)\|_{\{C^0[0,1]\}^2}\qquad(n\in\mathbf{Z})\,.$$

Thus Lemma II.2 implies

(II.19) $$\qquad\qquad|\phi_2(1,\ \mu_n)+H\phi_1(1,\ \mu_n)|\leqq M_{11}|\mu_n-\lambda_n|\qquad(n\in\mathbf{Z})\,.$$

Similarly we can get

(II.20) $$\qquad\qquad|\phi_2(1,\ \mu_n^*)+H^*\phi_1(1,\ \mu_n^*)|\leqq M_{11}|\mu_n^*-\lambda_n^*|\qquad(n\in\mathbf{Z})\,.$$

Noting the inequalities (2.14) and the definitions (2.4) and (2.5) of δ_0 and δ, we see that the inequalities (II.19) and (II.20) imply the estimates (2.16) and (2.17), the conclusion of Lemma 2.

Appendix III. Proof of Lemma 3

Proof of the part (I). A theorem on perturbation of Riesz bases by K. Barĭ (Gohberg and Kreĭn [2]) is a key. That is, in order to prove that $\{\phi(\cdot,\ \mu_n)\}_{n\in\mathbf{Z}}$ is a Riesz basis in $\{L^2(0,\ 1)\}^2$, we have to show the following two facts:

(III.1) $$\sum_{n=-\infty}^{\infty}\|\phi(\cdot,\ \mu_n)-\phi(\cdot,\ \lambda_n)\|_{\{L^2(0,1)\}^2}^2<\infty\,.$$

(III.2) $\qquad$ If $\displaystyle\sum_{n=-\infty}^{\infty}c_n\phi(x,\ \mu_n)=0$ *almost everywhere in* $(0,\ 1)$, *then* $c_n=0$ $(n\in\mathbf{Z})$.

Proof of (III.1). In view of (II.11) of Lemma II.2 in Appendix II, we get

$$\sum_{n=-\infty}^{\infty}\|\phi(\cdot,\ \mu_n)-\phi(\cdot,\ \lambda_n)\|_{\{L^2(0,1)\}^2}^2\leqq\sum_{n=-\infty}^{\infty}\|\phi(\cdot,\ \mu_n)-\phi(\cdot,\ \lambda_n)\|_{\{C^0[0,1]\}^2}^2$$

$$\leqq M_{11}^2\sum_{n=-\infty}^{\infty}|\mu_n-\lambda_n|^2\ \leqq M_{11}^2\Big(\sum_{n=-\infty}^{\infty}|\mu_n-\lambda_n|\Big)^2\ \leqq M_{11}^2\delta_0^2\,.$$

This proves (III.1).

Proof of (III.2). Let us assume that $\displaystyle\sum_{n=-\infty}^{\infty}c_n\phi(x,\ \mu_n)=0$ almost everywhere in $(0,\ 1)$. Then we have for almost all $x\in(0,\ 1)$,

(III.3) $$\qquad\qquad-\sum_{n=-\infty}^{\infty}c_n\phi(x,\ \lambda_n)=\sum_{n=-\infty}^{\infty}c_n(\phi(x,\ \mu_n)-\phi(x,\ \lambda_n))\,.$$

On the other hand, since $\{\phi(\cdot,\ \lambda_n)\}_{n\in\mathbf{Z}}$ is a Riesz basis in $\{L^2(0,\ 1)\}^2$ (e.g. Russell [10], [11]), there exists some positive constant M_{12} such that

(III.4) $$\qquad\qquad M_{12}\|\sum_{n=-\infty}^{\infty}c_n\phi(\cdot,\ \lambda_n)\|_{\{L^2(0,1)\}^2}\geqq\Big(\sum_{n=-\infty}^{\infty}|c_n|^2\Big)^{1/2}\,.$$

Applying (III.4) in (III.3), we have

$$M_{12}^{-1}\Big(\sum_{n=-\infty}^{\infty}|c_n|^2\Big)^{1/2}\leqq\|\sum_{n=-\infty}^{\infty}c_n\phi(\cdot,\ \lambda_n)\|_{[L^2(0,1)]^2}$$

$$=\|\sum_{n=-\infty}^{\infty}c_n(\phi(\cdot,\ \mu_n)-\phi(\cdot,\ \lambda_n))\|_{[L^2(0,1)]^2}$$

$$\leqq\sum_{n=-\infty}^{\infty}|c_n|\ \|\phi(\cdot,\ \mu_n)-\phi(\cdot,\ \lambda_n)\|_{[C^c[0,1]]^2}$$

$$\leqq M_{11}\sum_{n=-\infty}^{\infty}|c_n|\,|\mu_n-\lambda_n| \qquad\qquad\qquad\text{(by (II.11))}$$

$$\leqq M_{11}\Big(\sum_{n=-\infty}^{\infty}|c_n|^2\Big)^{1/2}\Big(\sum_{n=-\infty}^{\infty}|\mu_n-\lambda_n|^2\Big)^{1/2} \quad\text{(by Schwarz's inequality)}$$

$$\leqq M_{11}\delta_0\Big(\sum_{n=-\infty}^{\infty}|c_n|^2\Big)^{1/2}.$$

Therefore if δ_0 is so small that $M_{11}M_{12}\delta_0<1$, then we see $\Big(\sum_{n=-\infty}^{\infty}|c_n|^2\Big)^{1/2}=0$, which implies (III.2).

Proof of the part (II). We divide the proof into the following five steps.

First Step. For the operator $A_{P,h,H}$ defined by (1.5), the adjoint operator $A_{P,h,H}^*$ is given by

(III.5)
$$\begin{cases} (A_{P,h,H}^*v)(x)=-\begin{pmatrix}0&1\\1&0\end{pmatrix}\dfrac{dv(x)}{dx}+{}^tP(x)v(x)\ (0<x<1),\quad u\in\mathscr{D}(A_{P,h,H}^*)\\[2mm] \mathscr{D}(A_{P,h,H}^*)=\Big\{v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in\{H^1(0,\ 1)\}^2\ ;\ v_2(0)-hv_1(0)=v_2(1)-Hv_1(1)=0\Big\}.\end{cases}$$

Here ${}^tP(x)$ denotes the transpose of the matrix $P(x)$. Then, by integration by parts, we obtain

(III.6) $\qquad\qquad (A_{P,h,H}u,\ v)_{[L^2(0,1)]^2}=(u,\ A_{P,h,H}^*v)_{[L^2(0,1)]^2}$

for each $u\in\mathscr{D}(A_{P,h,H})$ and $v\in\mathscr{D}(A_{P,h,H}^*)$.

Let us recall that $\phi(\cdot,\ \lambda_n)$ and $\phi^*(\cdot,\ \overline{-\lambda_n})$ is given by (II.10) and (II.10)' in Appendix II, respectively. In this step, we show Lemmas III.1 and III.2.

LEMMA III.1. $\phi(\cdot,\ \lambda_n)$ *and* $\phi^*(\cdot,\ \overline{-\lambda_n})$ *are eigenvectors of* $A_{P,h,H}$ *and* $A_{P,h,H}^*$ *associated with the eigenvalues* λ_n *and* $\overline{\lambda_n}$, *respectively.*

Proof. Since $A_{P,h,H}$ is a differential operator of the first order, all the eigenvalues are simple, as is easily proved.
Therefore, noting that $\phi(\cdot,\ \lambda_n)$ satisfies (2.7) with $\lambda=\lambda_n$, we see that $\phi(\cdot,\ \lambda_n)$ is an eigenvector of $A_{P,h,H}$. For $\phi^*(\cdot,\ \overline{-\lambda_n})$, we can proceed similarly.

LEMMA III. 2. *Let us set*

(III. 7) $\rho_n = (\phi(\cdot, \lambda_n), \phi^*(\cdot, \overline{-\lambda_n}))_{[L^2(0,1)]^2}$ $(n \in \mathbb{Z})$.

Then, for each $n \in \mathbb{Z}$, there exists $m(n) \in \mathbb{Z}$ such that $\lim\limits_{|n| \to \infty} |m(n)| = \infty$, $|m(n)| = O(|n|)$
and

(III. 8) $\rho_n = 1 - h^2 + O\left(\dfrac{1}{m(n)}\right)$ (as $|n| \to \infty$)

and

(III. 9) $|\rho_n| \geqq M_{13}$

for some positive constant M_{13}.

   Proof. For the proof, the equalities (II. 10) and (II. 10)′ are essential. In (II. 10), by integration by parts, we can get

(III. 10) $\displaystyle\int_0^x \left[U_{i1}(x, y) f_1(y, \lambda_n) + U_{i2}(x, y) f_2(y, \lambda_n) \right] dy$

$\displaystyle = \frac{1}{2} \int_0^x \left[(1-h)(U_{i1}(x, y) + U_{i2}(x, y)) e^{\lambda_n y} \right.$

$\displaystyle \left. \qquad + (1+h)(U_{i1}(x, y) - U_{i2}(x, y)) e^{-\lambda_n y} \right] dy$

$\displaystyle = \frac{1}{\lambda_n} d_n^{(i)}(x)$ $(i=1, 2, \ n \in \mathbb{Z})$.

Here and henceforth we set

(III. 11) $d_n^{(i)}(x)$

$\displaystyle = \frac{1}{2} \Big((1-h)(U_{i1}(x, x) + U_{i2}(x, x)) e^{\lambda_n x}$

$\displaystyle \qquad - (1+h)(U_{i1}(x, x) - U_{i2}(x, x)) e^{-\lambda_n x}$

$\displaystyle \qquad - (1-h)(U_{i1}(x, 0) + U_{i2}(x, 0)) + (1+h)(U_{i1}(x, 0) - U_{i2}(x, 0))$

$\displaystyle \qquad + \int_0^x \left[(1+h)\left(\frac{\partial U_{i1}(x, y)}{\partial y} - \frac{\partial U_{i2}(x, y)}{\partial y} \right) e^{-\lambda_n y} \right.$

$\displaystyle \qquad \left. - (1-h)\left(\frac{\partial U_{i1}(x, y)}{\partial y} + \frac{\partial U_{i2}(x, y)}{\partial y} \right) e^{\lambda_n y} \right] dy \Big).$

Similarly we can get

(III. 12) $\displaystyle\int_0^x (V_{i1}(x, y) f_1^*(y, \overline{-\lambda_n}) + V_{i2}(x, y) f_2^*(y, \overline{-\lambda_n})) dy$

$$= \frac{1}{\lambda_n} e_n^{(i)}(x) \qquad (i=1, 2, \quad n \in \mathbf{Z}),$$

where

(III. 13) $e_n^{(i)}(x)$

$$= \frac{1}{2} \Big((1-h)(V_{i1}(x, x) - V_{i2}(x, x)) e^{\overline{\lambda_n} x}$$

$$-(1+h)(V_{i1}(x, x) + V_{i2}(x, x)) e^{-\overline{\lambda_n} x}$$

$$-(1-h)(V_{i1}(x, 0) - V_{i2}(x, 0)) + (1+h)(V_{i1}(x, 0) + V_{i2}(x, 0))$$

$$+ \int_0^x \Big[(1+h)\Big(\frac{\partial V_{i1}(x, y)}{\partial y} + \frac{\partial V_{i2}(x, y)}{\partial y} \Big) e^{-\overline{\lambda_n} y}$$

$$-(1-h)\Big(\frac{\partial V_{i1}(x, y)}{\partial y} - \frac{\partial V_{i2}(x, y)}{\partial y} \Big) e^{\overline{\lambda_n} y} \Big] dy \Big).$$

Substituting (III.10) and (III.12) into (II.10) and (II.10)′, respectively, we have

(III. 14) $\displaystyle\int_0^1 {}^t\phi(x, \lambda_n) \overline{\phi^*(x, -\lambda_n)} dx$

$$= 1 - h^2 + \frac{1}{\lambda_n} \int_0^1 c_{1,n}(x) dx + \frac{1}{\lambda_n^2} \int_0^1 c_{2,n}(x) dx,$$

where

(III. 15) $c_{1,n}(x)$

$$= e^{-\eta_1(x)}(f_1(x, \lambda_n) \cosh\eta_2(x) - f_2(x, \lambda_n) \sinh\eta_2(x)) \overline{e_n^{(1)}(x)}$$

$$+ e^{-\eta_1(x)}(-f_1(x, \lambda_n) \sinh\eta_2(x) + f_2(x, \lambda_n) \cosh\eta_2(x)) \overline{e_n^{(2)}(x)}$$

$$+ e^{\eta_1(x)}(f_1^*(x, -\lambda_n) \cosh\eta_2(x) + f_2^*(x, -\lambda_n) \sinh\eta_2(x)) d_n^{(1)}(x)$$

$$+ e^{\eta_1(x)}(f_1^*(x, -\lambda_n) \sinh\eta_2(x) + f_2^*(x, -\lambda_n) \cosh\eta_2(x)) d_n^{(2)}(x)$$

and

(III. 16) $c_{2,n}(x) = d_n^{(1)}(x) \overline{e_n^{(1)}(x)} + d_n^{(2)}(x) \overline{e_n^{(2)}(x)} \qquad (0 \leqq x \leqq 1).$

On the other hand, for $n \in \mathbf{Z}$, there exists $m(n) \in \mathbf{Z}$ such that $\displaystyle\lim_{|n| \to \infty} |m(n)| = \infty$, $|m(n)| = O(|n|)$ and

(III. 17) $\displaystyle\lambda_n = \gamma + m(n)\pi\sqrt{-1} + O\Big(\frac{1}{m(n)} \Big),$

where γ is given by (1.14) (e.g. [10]. See (1.15).). Therefore we have

$$(\text{III}.18)' \quad \begin{cases} \sup_{n \in \mathbf{Z}} \|f(\cdot,\ \lambda_n)\|_{[C^0[0,1]]^2} < \infty \\[2mm] \sup_{n \in \mathbf{Z}} \|f^*(\cdot,\ \overline{-\lambda_n})\|_{[C^0[0,1]]^2} < \infty. \end{cases}$$

Thus, by means of the estimates (II.6), (II.7), (II.6)′ and (II.7)′ in Lemma II.1, and (III.17), we see

$$\sup_{n \in \mathbf{Z}} \|d_n^{(i)}(\cdot)\|_{C^0[0,1]} < \infty \quad \text{and} \quad \sup_{n \in \mathbf{Z}} \|e_n^{(i)}(\cdot)\|_{C^0[0,1]} < \infty \qquad (i=1,\ 2),$$

that is,

$$(\text{III}.19) \qquad \sup_{n \in \mathbf{Z}} \|c_{i,n}(\cdot)\|_{C^0[0,1]} < \infty \qquad (i=1,\ 2)$$

Again, by (III.17), we have

$$(\text{III}.20) \qquad \frac{1}{\lambda_n} = O\!\left(\frac{1}{m(n)}\right) \qquad (as\ |n| \to \infty).$$

Applying (III.19) and (III.20) in (III.14), we reach

$$\int_0^1 {}^t\phi(x,\ \lambda_n)\overline{\phi^*(\cdot,\ \overline{-\lambda_n})}dx = 1 - h^2 + O\!\left(\frac{1}{m(n)}\right),$$

which proves (III.8).

Next we proceed to a proof of (III.9). First we will show

$$(\text{III}.21) \qquad (\phi(\cdot,\ \lambda_n),\ \phi^*(\cdot,\ \overline{-\lambda_n})) \neq 0 \qquad (n \in \mathbf{Z}).$$

In fact, assume that

$$(\text{III}.22) \qquad (\phi(\cdot,\ \lambda_{n_0}),\ \phi^*(\cdot,\ \overline{-\lambda_{n_0}})) = 0 \qquad for\ some\ n_0 \in \mathbf{Z}.$$

Then, by Lemma III.1, we see

$$\lambda_m(\phi(\cdot,\ \lambda_m),\ \phi^*(\cdot,\ \overline{-\lambda_{n_0}})) = (A_{P,h,H}\phi(\cdot,\ \lambda_m),\ \phi^*(\cdot,\ \overline{-\lambda_{n_0}}))$$

$$= (\phi(\cdot,\ \lambda_m),\ A_{P,h,H}^* \phi^*(\cdot,\ \overline{-\lambda_{n_0}})) \qquad (\text{by (III.6)})$$

$$= (\phi(\cdot,\ \lambda_m),\ \overline{\lambda_{n_0}}\, \phi^*(\cdot,\ \overline{-\lambda_{n_0}})),$$

so that we get

$$(\text{III}.23) \qquad (\phi(\cdot,\ \lambda_m),\ \phi^*(\cdot,\ \overline{-\lambda_{n_0}})) = 0 \qquad for\ each\ m \neq n_0.$$

Combining (III.23) with (III.22), we have $(\phi(\cdot,\ \lambda_m),\ \phi^*(\cdot,\ \overline{-\lambda_{n_0}})) = 0$ for each $m \in \mathbf{Z}$. Therefore, noting that $\{\phi(\cdot,\ \lambda_m)\}_{m \in \mathbf{Z}}$ is a Riesz basis in $\{L^2(0,\ 1)\}^2$, we see $\phi^*(\cdot,\ \overline{-\lambda_{n_0}}) = 0$, which is a contradiction. Thus we see (III.21).

Now we complete the proof of (III.9). The asymptotic behavior (III.8) and

$|h| \neq 1$ imply that $|\rho_n| \geq M'_{13} > 0$ for each $|n| \geq N_0$, where N_0 is sufficiently large. On the other hand, $M''_{13} = \min\limits_{|n| < N_0} |\rho_n| > 0$ by (III.21). Therefore, setting $M_{13} = \min\{M'_{13}, M''_{13}\}$, we see (III.9). This completes the proof of Lemma III.2.

Second Step. In this step, we show

LEMMA III.3. *For $n \in \mathbf{Z}$, we have*

(III.24) $$\|\phi(\cdot, \lambda_n)\|_{(C^0[0,1])^2} \leq M_{14}$$

(II.11) *bis* $$\|\phi(\cdot, \mu_n) - \phi(\cdot, \lambda_n)\|_{(C^0[0,1])^2} \leq M_{14} |\mu_n - \lambda_n|$$

(III.25) $$\|\phi(\cdot, \mu_n) - \phi(\cdot, \lambda_n)\|_{(C^1[0,1])^2} \leq M_{14}(|n|+1)|\mu_n - \lambda_n|$$

(III.26) $$\|\phi^*(\cdot, -\lambda_n)\|_{(C^0[0,1])^2} \leq M_{14}$$

and

(III.27) $$\|\phi^*(\cdot, -\lambda_n)\|_{(C^1[0,1])^2} \leq M_{14}(|n|+1).$$

Proof of (III.24). By (II.10), we immediately get for $n \in \mathbf{Z}$,
$$\|\phi(\cdot, \lambda_n)\|_{(C^0[0,1])^2} \leq 2(\|S\|_{C^0} + \|U\|_{C^0})\|f(\cdot, \lambda_n)\|_{(C^0[0,1])^2}.$$
Therefore, by (II.9) and (II.6), we obtain

$$\|\phi(\cdot, \lambda_n)\|_{(C^0[0,1])^2} \leq M'_{14} \|f(\cdot, \lambda_n)\|_{(C^0[0,1])^2} \qquad (n \in \mathbf{Z}),$$

which means (III.24), in view of (III.18).

Proof of (III.25). Since $\phi(\cdot, \lambda)$ satisfies the differential equations in (2.7), we get

(III.28)
$$\begin{cases}
\dfrac{d}{dx}(\phi_1(x, \mu_n) - \phi_1(x, \lambda_n)) \\[2mm]
= -p_1(x)(\phi_1(x, \mu_n) - \phi_1(x, \lambda_n)) - p_2(x)(\phi_2(x, \mu_n) - \phi_2(x, \lambda_n)) \\[2mm]
\quad + \mu_n(\phi_2(x, \mu_n) - \phi_2(x, \lambda_n)) + (\mu_n - \lambda_n)\phi_2(x, \lambda_n) \\[2mm]
\dfrac{d}{dx}(\phi_2(x, \mu_n) - \phi_2(x, \lambda_n)) \\[2mm]
= -a(x)(\phi_1(x, \mu_n) - \phi_1(x, \lambda_n)) - b(x)(\phi_2(x, \mu_n) - \phi_2(x, \lambda_n)) \\[2mm]
\quad + \mu_n(\phi_1(x, \mu_n) - \phi_1(x, \lambda_n)) + (\mu_n - \lambda_n)\phi_1(x, \lambda_n) \qquad (n \in \mathbf{Z}).
\end{cases}$$

On the other hand, since $|\mu_n| \leq |\mu_n - \lambda_n| + |\lambda_n| \leq \delta_0 + |\lambda_n|$ by $|\mu_n - \lambda_n| \leq \delta_0 \equiv \sum\limits_{m=-\infty}^{\infty} (|\mu_m - \lambda_m| + |\mu_m^* - \lambda_m^*|)$, we have

(III.29) $$|\mu_n| \leq \delta_0 + |\gamma| + O(|n|) \qquad (n \in \mathbf{Z})$$

by (III.17).

Applying (III. 24), (II. 11) and (III. 29) at the right hand side of (III. 28), we get

$$\|d(\phi_1(\cdot,\ \mu_n)-\phi_1(\cdot,\ \lambda_n))/dx\|_{C^0}+\|d(\phi_2(\cdot,\ \mu_n)-\phi_2(\cdot,\ \lambda_n))/dx\|_{C^0}$$

$$\leqq M_{14}(|n|+1)|\mu_n-\lambda_n|\qquad(n\in\mathbf{Z}),$$

from which we see (III. 25).

Proof of (III. 26) *and* (III. 27). We can prove (III. 26) by a way analogous with the one in the proof of (III. 24), noting (II. 10)' and (II. 6)'. Next we proceed to a proof of (III. 27). Since $\phi^*(\cdot,\ -\lambda_n)$ satisfies (2.8), we have for each $n\in\mathbf{Z}$,

(III. 30)
$$\begin{cases}\dfrac{d\phi_1^*(x,\ -\lambda_n)}{dx}=b(x)\phi_1^*(x,\ -\lambda_n)+p_2(x)\phi_2^*(x,\ -\lambda_n)-\lambda_n\phi_2^*(x,\ -\lambda_n)\\[2mm]\dfrac{d\phi_2^*(x,\ -\lambda_n)}{dx}=a(x)\phi_1^*(x,\ -\lambda_n)+p_1(x)\phi_2^*(x,\ -\lambda_n)-\lambda_n\phi_1^*(x,\ -\lambda_n).\end{cases}$$

In (III. 30), we apply (III. 26) and the asymptotic behavior (III. 17) of λ_n, so that we have

$$\|d\phi_1^*(\cdot,\ -\lambda_n)/dx\|_{C^0}+\|d\phi_2^*(\cdot,\ -\lambda_n)/dx\|_{C^0}\leqq M_{14}(|n|+1).$$

This completes the proof of (III. 27), and so Lemma III. 3 is proved.

Third Step. We set

(III. 31)
$$Z(x,\ y)=(Z_{ij}(x,\ y))_{1\leqq i,j\leqq 2}$$

$$=\sum_{n=-\infty}^{\infty}\frac{\phi(x,\ \mu_n)-\phi(x,\ \lambda_n)}{\rho_n}{}^t\phi^*(y,\ -\lambda_n)\qquad((x,\ y)\in[0,\ 1]^2).$$

Then $Z(x,\ y)$ is well-defined and has the following properties:

(III. 32) $Z(\cdot,\ \cdot)\in\{C^0([0,\ 1]^2)\}^4.$

(III. 33) $Z(\cdot,\ \cdot)\in\{C^1([0,\ 1]^2)\}^4,\ if\ \delta<\infty.$

(III. 34) $Z(x,\ y):\ real\text{-}valued.$

(III. 35) $\phi(x,\ \mu_n)=\phi(x,\ \lambda_n)+\displaystyle\int_0^1 Z(x,\ y)\phi(y,\ \lambda_n)dy\qquad(0\leqq x\leqq 1,\ n\in\mathbf{Z}).$

The purpose of this step is to verify (III. 32)–(III. 35).

Verification of (III. 32) *and* (III. 33). First by (III. 9) of Lemma III. 2, we see

(III. 36) $|\rho_n|^{-1}\leqq M_{13}^{-1}.$

By (II. 11), (III. 26) and (III. 36), we have

$$\|(\phi(\cdot,\ \mu_n)-\phi(\cdot,\ \lambda_n))\cdot{}^t\phi^*(\cdot,\ -\lambda_n)/\rho_n\|_{\{C^0([0,1]^2)\}^4}$$

$$= |\rho_n|^{-1} \max_{\substack{1 \le i,j \le 2 \\ 0 \le x,y \le 1}} |(\phi_i(x, \mu_n) - \phi_i(x, \lambda_n))\phi_j^*(y, -\lambda_n)| \le M_{14}^2 M_{13}^{-1} |\mu_n - \lambda_n| \qquad (n \in \mathbf{Z}) .$$

Since $\sum_{n=-\infty}^{\infty} |\mu_n - \lambda_n| \le \delta_0 < \infty$, the majorant series for the right hand side of (III.31) is convergent. This proves (III.32). By (III.25), (III.27) and (III.36), we can similarly see (III.33).

Verification of (III.34). We show

LEMMA III.4. *The equalities*

(III.37) $$\overline{\phi(x, \lambda)} = \phi(x, \bar{\lambda})$$

and

(III.38) $$\overline{\phi^*(x, \lambda)} = \phi^*(x, \bar{\lambda})$$

hold for $\lambda \in \mathbf{C}$.

Proof of Lemma III.4. Since, by the definition, $\phi(x, \lambda)$ satisfies (2.7), noting that $P(x)$ is real-valued and $h \in \mathbf{R}$, we have

(III.39)
$$\begin{cases} B \dfrac{d\overline{\phi(x, \lambda)}}{dx} + P(x)\overline{\phi(x, \lambda)} = \bar{\lambda}\overline{\phi(x, \lambda)} \qquad (0 \le x \le 1) \\[2mm] \overline{\phi(0, \lambda)} = \begin{pmatrix} 1 \\ -h \end{pmatrix}. \end{cases}$$

On the other hand, by (2.7), also $\phi(x, \bar{\lambda})$ satisfies (III.39). Therefore the uniqueness of solutions to the initial value problem (III.39) means (III.37). Similarly we can prove (III.38).

Now we return to the verification of (III.34). Let us recall that the integer N_1 is given in (1.16). By Lemma III.4, we have

(III.40)′ $\overline{\phi(\cdot, \mu_n)} = \phi(\cdot, \overline{\mu_n})$, $\overline{\phi(\cdot, \lambda_n)} = \phi(\cdot, \overline{\lambda_n})$ and $\overline{\phi^*(\cdot, -\lambda_n)} = \phi^*(\cdot, \overline{-\lambda_n})$

$$(n \ge N_1 + 1) .$$

By the conditions (1.16.2) and (1.23.2), the equalities (III.40)′ imply

(III.40) $\overline{\phi(\cdot, \mu_n)} = \phi(\cdot, \mu_{-n})$, $\overline{\phi(\cdot, \lambda_n)} = \phi(\cdot, \lambda_{-n})$ and $\overline{\phi^*(\cdot, -\lambda_n)} = \phi^*(\cdot, -\lambda_{-n})$

$$(n \ge N_1 + 1) .$$

Furthermore since ρ_n is given by (III.7), we see by (III.40) that

(III.41) $$\overline{\rho_n} = \rho_{-n} \qquad\qquad (n \ge N_1 + 1) .$$

On the other hand, by (1.16.1) and (1.23.1), the equalities (III.37) and (III.38) imply that

(III. 42) $\phi(\cdot, \mu_n),\ \phi(\cdot, \lambda_n),\ \phi^*(\cdot, -\lambda_n)\qquad (-N_1 \leqq n \leqq N_1):\ \text{real-valued}$

and hence

(III. 43) $\rho_n \in \boldsymbol{R}\qquad (-N_1 \leqq n \leqq N_1).$

Since the series at the right hand side of (III. 31) is absolutely convergent as is seen in the verification of (III. 32), we have

$$Z(x,\ y) = \sum_{n=-N_1}^{N_1} \frac{(\phi(x,\ \mu_n) - \phi(x,\ \lambda_n))^t \phi^*(y,\ -\lambda_n)}{\rho_n}$$

$$+ \lim_{N \to \infty} \sum_{n=N_1+1}^{N} \left(\frac{(\phi(x,\ \mu_n) - \phi(x,\ \lambda_n))^t \phi^*(y,\ -\lambda_n)}{\rho_n} \right.$$

$$\left. + \frac{\overline{(\phi(x,\ \mu_n) - \phi(x,\ \lambda_n))^t \phi^*(y,\ \lambda_n)}}{\overline{\rho_n}} \right)$$

by (III. 40) and (III. 41).

Therefore, noting also (III. 42) and (III. 43), we can see that Z is real-valued. Thus (III. 34) is verified.

Verification of (III. 35)   Since the right hand side of (III. 31) is uniformly convergent with respect to $(x,\ y) \in [0,\ 1]^2$, we have

$$\int_0^1 Z(x,\ y)\phi(y,\ \lambda_n)dy$$

$$= \sum_{m=-\infty}^{\infty} \frac{(\phi(x,\ \mu_m) - \phi(x,\ \lambda_m))}{\rho_m} \int_0^1 {}^t\phi^*(y,\ -\lambda_m)\phi(y,\ \lambda_n)dy$$

$$= (\phi(x,\ \mu_n) - \phi(x,\ \lambda_n)) \left(\frac{1}{\rho_n} \int_0^1 {}^t\phi^*(y,\ -\lambda_n)\phi(y,\ \lambda_n)dy \right) \qquad (n \in \boldsymbol{Z}).$$

Here we use the equalities

(III. 44) $\int_0^1 {}^t\phi^*(y,\ -\lambda_m)\phi(y,\ \lambda_n)dy = (\phi(\cdot,\ \lambda_n),\ \phi^*(x,\ \overline{-\lambda_m})) = 0 \qquad (n \neq m),$

which are derived in the same way as (III. 23).
Noting the definition (III. 7) of ρ_n, we obtain (III. 35).

Fourth Step.   Let us define an operator F from $\{C^0[0,\ 1]\}^2$ into itself by

(III. 45) $(Fu)(x) = \int_0^1 {}^t Z(y,\ x)u(y)dy \qquad (0 \leqq x \leqq 1).$

Henceforth $\mathcal{L}(X)$ denotes the set of bounded linear operators defined on a Banach space X to itself.

The purpose of this step is to show that

(III. 46) $(1+F)^{-1} \in \mathcal{L}(\{C^0[0, 1]\}^2)$

and

(III. 47) $(1+F)^{-1} \in \mathcal{L}(\{C^1[0, 1]\}^2)$, if $\delta < \infty$.

To this end, we have only to prove the facts:

(III. 48)   *F is a compact operator on* $\{C^0[0, 1]\}^2$.

(III. 49)   *F is a compact operator on* $\{C^1[0, 1]\}^2$, *if* $\delta < \infty$.

(III. 50) -1 *is not an eigenvalue of F*.

In fact, by the Riesz-Schauder theorem (Yosida [15, p. 283], for example), by (III. 50), the facts (III. 48) and (III. 49) imply (III. 46) and (III. 47), respectively.

Verification of (III. 48) *and* (III. 49). Let us consider any sequence $\{u_n\}_{n \geq 1}$ $\subset \{C^0[0, 1]\}^2$ such that $\{\|u_n\|_{C^0}\}_{n \geq 1}$ is bounded. Since $Z(\cdot, \cdot)$ is bounded and uniformly continuous on $[0, 1]^2$ by (III. 32), we see that $\{\|Fu_n\|_{C^0}\}_{n \geq 1}$ is bounded and that Fu_n is equi-continuous with respect to n, that is,

$$\lim_{\substack{\epsilon \to 0}} \sup_{\substack{n \geq 1 \\ |x-x'|<\epsilon}} |(Fu_n)(x) - (Fu_n)(x')| = 0.$$

Therefore, by the Ascoli-Arzelà theorem, $\{Fu_n\}_{n \geq 1}$ contains a subsequence convergent in $\{C^0[0, 1]\}^2$, which means (III. 48). We can similarly prove (III. 49) in view of (III. 33), provided that $\delta < \infty$.

Verification of (III. 50). Regarding F defined by (III. 45) as an operator on $\{L^2(0, 1)\}^2$, we can easily see

(III. 51) $(F^*u)(x) = \displaystyle\int_0^1 Z(x, y)u(y)dy$.

Here F^* is the adjoint operator of $F \in \mathcal{L}(\{L^2(0, 1)\}^2)$. Therefore we have $\overline{\sigma(F^*)}$ $= \sigma(F)$. Thus, in order to verify (III. 50), it is sufficient to show that

(III. 52) $u + F^*u = 0$ *implies* $u = 0$.

Since $\{\phi(\cdot, \lambda_n)\}_{n \in \mathbf{Z}}$ is a Riesz basis in $\{L^2(0, 1)\}^2$ (Russell [10], [11]), we can expand u: $u = \sum\limits_{n=-\infty}^{\infty} c_n \phi(\cdot, \lambda_n)$ for some $c_n \in \mathbf{C}$ ($n \in \mathbf{Z}$), where this series is convergent in $\{L^2(0, 1)\}^2$. Hence we have

(III. 53) $0 = (1+F^*)u = \sum\limits_{n=-\infty}^{\infty} c_n(1+F^*)\phi(\cdot, \lambda_n)$

                                                    (by the boundedness of $1+F^*$)

$$= \sum_{n=-\infty}^{\infty} c_n \phi(\cdot, \mu_n) \qquad\qquad \text{(by (III.35))}.$$

Since as is proved in the part (I) of this lemma, also $\{\phi(\cdot, \mu_n)\}_{n \in \mathbf{Z}}$ is a Riesz basis in $\{L^2(0, 1)\}^2$, it follows from (III.53) that $c_n = 0$ $(n \in \mathbf{Z})$, namely, $u = 0$. This proves (III.52).

Fifth Step. Let us set

(III.54) $$\phi_n^{(1)} = \frac{1}{\rho_n}(1+F)^{-1}\overline{\phi^*(\cdot, -\lambda_n)} \qquad (n \in \mathbf{Z}).$$

Then we can verify that $\phi_n^{(1)}$ satisfies (2.18)–(2.22).

Verification of (2.18) *and* (2.19). We see (2.18) by combining (III.36), (III.26) and (III.46). If $\delta < \infty$, then (III.36), (III.27) and (III.47) imply (2.19).

Verification of (2.20). Let us consider $\{C^0[0, 1]\}^2$ as a real Banach space of all *real-valued* C^0-functions. Then, by (III.34), we can regard F as an operator from the real Banach space $\{C^0[0, 1]\}^2$ to itself. Therefore $(1+F)^{-1}u$ is real-valued for real-valued $u \in \{C^0[0, 1]\}^2$. Since $\rho_n \in \mathbf{R}$ and $\phi^*(\cdot, -\lambda_n)$ is real-valued $(-N_1 \leq n \leq N_1)$ by (III.42) and (III.43), also $\phi_n^{(1)}$ is real-valued for $-N_1 \leq n \leq N_1$.

On the other hand, since $\dfrac{1}{\rho_n}\overline{\phi^*(\cdot, -\lambda_n)} + \dfrac{1}{\rho_{-n}}\overline{\phi^*(\cdot, -\lambda_{-n})}$ is real-valued for $n \geq N_1 + 1$ from (III.40) and (III.41), we see that

also $\phi_n^{(1)} + \phi_{-n}^{(1)} = (1+F)^{-1}\left(\dfrac{1}{\rho_n}\overline{\phi^*(\cdot, -\lambda_n)} + \dfrac{1}{\rho_{-n}}\overline{\phi^*(\cdot, -\lambda_{-n})}\right)$ $(n \geq N_1+1)$ is real-valued

and so, we get (2.20).

Verification of (2.21). First we have

(III.55) $$((1+F)^{-1})^* = (1+F^*)^{-1},$$

where the operators are considered in the Hilbert space $\{L^2(0, 1)\}^2$ (Kato [5, p.169], for example). Now we get

$$(\phi(\cdot, \mu_n), \phi_m^{(1)})_{\{L^2(0,1)\}^2} = (\phi(\cdot, \mu_n), \frac{1}{\rho_m}(1+F)^{-1}\overline{\phi^*(\cdot, -\lambda_m)})$$

$$= \frac{1}{\rho_m}((1+F^*)^{-1}\phi(\cdot, \mu_n), \phi^*(\cdot, \overline{-\lambda_m})) \qquad \text{(by (III.38) and (III.55))}$$

$$= \frac{1}{\rho_m}(\phi(\cdot, \lambda_n), \phi^*(\cdot, \overline{-\lambda_m})) \qquad \text{(by (III.51) and (III.35))}.$$

Therefore, in view of (III.7), for the verification of (2.21), we have only to use

(III.56) $$(\phi(\cdot, \lambda_n), \phi^*(\cdot, \overline{-\lambda_m})) = 0 \qquad (n \neq m),$$

which is nothing but (III. 23).

Verification of (2.22). Since $\{\phi(\cdot, \mu_n)\}_{n \in Z}$ is a Riesz basis in $\{L^2(0, 1)\}^2$ from the part (I) of Lemma 3, the biorthogonality (2.21) implies that also $\{\psi_n^{(1)}\}_{n \in Z}$ is a Riesz basis in $\{L^2(0, 1)\}^2$ (Gohberg and Kreĭn [2, p. 310]). Therefore, for each $u \in \{L^2(0, 1)\}^2$, we have

$$\text{(III. 57)} \qquad\qquad u = \sum_{m=-\infty}^{\infty} c_m \overline{\psi_m^{(1)}} \quad in \ \{L^2(0, 1)\}^2,$$

for appropriate $c_m \in C$ $(m \in Z)$. Then we get

$$(u, \overline{\phi(\cdot, \mu_n)}) = \sum_{m=-\infty}^{\infty} c_m(\overline{\psi_m^{(1)}}, \overline{\phi(\cdot, \mu_n)}) = \sum_{m=-\infty}^{\infty} c_m \delta_{mn} \qquad\qquad \text{(by (2.21))}$$

$$= c_n,$$

which implies (2.22).

For (I)′ and (II)′ of Lemma 3, we can proceed similarly. Thus we complete the proof of Lemma 3.

Appendix IV. Proof of Lemma 4

(I)   Proof of (2.25)-(2.28). We have for $0 \leqq y \leqq 1$,

$$|a_n(q)\overline{\psi_n^{(1)}(y)} - b_n(q)\overline{\psi_n^{(2)}(y)}|$$

$$\leqq (|a_n(q)| + |b_n(q)|) \cdot \max \{||\psi_n^{(1)}||_{\{C^0[0,1]\}^2}, \ ||\psi_n^{(2)}||_{\{C^0[0,1]\}^2}\}$$

$$\leqq M_3(|a_n(q)| + |b_n(q)|)$$

by (2.18) and (2.18)′ in Lemma 3. Therefore, by (2.16) in Lemma 2, we get

$$\text{(IV. 1)} \qquad\qquad \sum_{n=-\infty}^{\infty} |a_n(q)\overline{\psi_n^{(1)}(y)} - b_n(q)\overline{\psi_n^{(2)}(y)}|$$

$$\leqq M_3 \sum_{n=-\infty}^{\infty} (|a_n(q)| + |b_n(q)|) \leqq M_2 M_3 \delta_0 \qquad (0 \leqq y \leqq 1).$$

On the other hand, since the equality

$$\frac{1}{J - J^*} = \frac{1}{4(H - H^*)} \left((H+1) \exp \left(\int_0^1 (q_2(s) - p_2(s)) \, ds \right) + 1 - H \right)$$

$$\times \left((H^*+1) \exp \left(\int_0^1 (q_2(s) - p_2(s)) \, ds \right) + 1 - H^* \right) \times \exp \left(-\int_0^1 (q_2(s) - p_2(s)) \, ds \right)$$

holds, noting that $\|q_2 - p_2\|_{C^0} \leqq M$, we have

$$\text{(IV. 2)} \qquad \left| \frac{1}{J - J^*} \right| \leqq \frac{e^M}{4} (|H+1|e^M + |1 - H|)(|H^*+1|e^M + |1 - H^*|) \left| \frac{1}{H - H^*} \right| \equiv M_{16}.$$

Combining (IV.1) with (IV.2), we see $||c_{11}||_{C^0}$, $||c_{12}||_{C^0} \leqq M_4 \delta_0$. For c_{21} and c_{22}, we can proceed similarly, so that (2.27) is proved.

In a manner analogous with the one in (IV.1), we get

$$
\text{(IV.3)} \qquad || \sum_{n=-\infty}^{\infty} (a_n(q)\overline{\psi_n^{(1)}}(\cdot) - b_n(q)\overline{\psi_n^{(2)}}(\cdot)) ||_{[C^1[0,1]]^2} \leqq M_2 M_3 \delta ,
$$

in virtue of (2.19), (2.19)′ and (2.17).

The inequalities (IV.2) and (IV.3) imply $||c_{11}||_{C^1}$, $||c_{12}||_{C^1} \leqq M_4 \delta$. For c_{21} and c_{22}, we can proceed similarly. Thus we complete the proof of (2.26)–(2.28).

Next we have to prove (2.25). From (IV.1), we see that the series at the right hand side of (2.23) is absolutely convergent. Therefore we can rewrite (2.23) as

$$
\text{(IV.4)} \qquad \begin{pmatrix} c_{11}(y, q) \\ c_{12}(y, q) \end{pmatrix} = \frac{1}{J - J^*} \sum_{n=-N_1}^{N_1} a_n(q)\overline{\psi_n^{(1)}(y)} - \frac{1}{J - J^*} \sum_{n=-N_2}^{N_2} b_n(q)\overline{\psi_n^{(2)}(y)}
$$

$$
+ \frac{1}{J - J^*} \lim_{N \to \infty} \sum_{n=N_1+1}^{N} (a_n(q)\overline{\psi_n^{(1)}(y)} + a_{-n}(q)\overline{\psi_{-n}^{(1)}(y)})
$$

$$
- \frac{1}{J - J^*} \lim_{N \to \infty} \sum_{n=N_2+1}^{N} (b_n(q)\overline{\psi_n^{(2)}(y)} + b_{-n}(q)\overline{\psi_{-n}^{(2)}(y)}).
$$

Applying (III.40) and (III.42) in the definition (2.9) of $a_n(q)$, we have

$$
\text{(IV.5)} \qquad \overline{a_n(q)} = a_{-n}(q) \ (n \geqq N_1 + 1) \text{ and } a_n(q) \in \boldsymbol{R} \ (-N_1 \leqq n \leqq N_1).
$$

Similarly we can get

$$
\text{(IV.6)} \qquad \overline{b_n(q)} = b_{-n}(q) \ (n \geqq N_2 + 1) \text{ and } b_n(q) \in \boldsymbol{R} \ (-N_2 \leqq n \leqq N_2).
$$

By using (IV.5), (IV.6) and (2.20), (2.20)′ of Lemma 3 in (IV.4), we conclude that c_{11} and c_{12} are real-valued functions. Similarly we can prove that c_{21} and c_{22} are real-valued. Thus the part (I) of Lemma 4 is proved.

(II) *Proof of* (2.29). In view of (2.9), (2.10), (2.16), (2.18), (2.18)′, (2.23), (2.24) and (II.19), (II.20), (IV.2), we have only to show that, for each $q^{(i)} = (q_1^{(i)}, q_2^{(i)}) \in \mathcal{A}_M$ $(i = 1, 2)$,

$$
\text{(IV.7)} \qquad \left| 2 \exp\left(\frac{1}{2} \int_0^1 (q_2^{(1)}(s) - p_2(s) + p_1(s) - q_1^{(1)}(s)) ds \right) \right.
$$

$$
\times \left((H+1) \exp\left(\int_0^1 (q_2^{(1)}(s) - p_2(s)) ds \right) + 1 - H \right)^{-1}
$$

$$
\left. - 2 \exp\left(\frac{1}{2} \int_0^1 (q_2^{(2)}(s) - p_2(s) + p_1(s) - q_1^{(2)}(s)) ds \right) \right.
$$

$$\times \left((H+1)\exp\left(\int_0^1 (q_2^{(2)}(s)-p_2(s))ds\right)+1-H \right)^{-1} \Bigg|$$

$$\leqq M_{17}\|q^{(1)}-q^{(2)}\|_{(C^0[0,1])^2},$$

(IV.8)
$$\left| \frac{1}{J(q^{(1)})-J^*(q^{(1)})} - \frac{1}{J(q^{(2)})-J^*(q^{(2)})} \right|$$

$$\leqq M_{17}\|q^{(1)}-q^{(2)}\|_{(C^0[0,1])^2},$$

(IV.9)
$$|J(q^{(1)})-J(q^{(2)})|,\ \ |J^*(q^{(1)})-J^*(q^{(2)})|$$

$$\leqq M_{17}\|q^{(1)}-q^{(2)}\|_{(C^0[0,1])^2}$$

and

(IV.10)
$$|J(q)|,\ \ |J^*(q)| \leqq M_{17}.$$

Proof of (IV.7). We have

$$|[\text{the left hand side of (IV.7)}]|$$

$$=\left| \left((H+1)\exp\left(\int_0^1 (q_2^{(1)}(s)-p_2(s))ds\right)+1-H \right)^{-1} \right.$$

$$\times \left((H+1)\exp\left(\int_0^1 (q_2^{(2)}(s)-p_2(s))ds\right)+1-H \right)^{-1}$$

$$\times \left[2(H+1)\exp\left(\frac{1}{2}\int_0^1 (q_2^{(1)}(s)+q_2^{(2)}(s)+p_1(s)-3p_2(s))ds\right) \right.$$

$$\times \left\{ \exp\left(\frac{1}{2}\int_0^1 (q_2^{(2)}(s)-q_1^{(1)}(s))ds\right) - \exp\left(\frac{1}{2}\int_0^1 (q_2^{(1)}(s)-q_1^{(2)}(s))ds\right) \right\}$$

$$+2(1-H)\exp\left(\frac{1}{2}\int_0^1 (p_1(s)-p_2(s))ds\right)$$

$$\left. \left. \times \left\{ \exp\left(\frac{1}{2}\int_0^1 (q_2^{(1)}(s)-q_1^{(1)}(s))ds\right) - \exp\left(\frac{1}{2}\int_0^1 (q_2^{(2)}(s)-q_1^{(2)}(s))ds\right) \right\} \right] \right|$$

$$\leqq M_1^{-2}(2|H+1|\exp(2M+\|p_1\|_{C^0}+\|p_2\|_{C^0})+2|1-H|\exp(M+\|p_1\|_{C^0}+\|p_2\|_{C^0}))$$

$$\times \|q^{(1)}-q^{(2)}\|_{(C^0[0,1])^2},$$

by (2.14) and the mean value theorem for e^x. This shows (IV.7).

Proof of (IV.8). We have

$$\left| \frac{1}{J(q^{(1)})-J^*(q^{(1)})} - \frac{1}{J(q^{(2)})-J^*(q^{(2)})} \right|$$

$$= \frac{1}{4|H-H^*|} \Bigg| (H+1)(H^*+1) \exp\left(-\int_0^1 p_2(s)ds\right)$$

$$\times \left(\exp\left(\int_0^1 q_2^{(1)}(s)ds\right) - \exp\left(\int_0^1 q_2^{(2)}(s)ds\right) \right)$$

$$+ (H-1)(H^*-1) \exp\left(\int_0^1 p_2(s)ds\right)$$

$$\times \left(\exp\left(-\int_0^1 q_2^{(1)}(s)ds\right) - \exp\left(-\int_0^1 q_2^{(2)}(s)ds\right) \right) \Bigg|$$

<div align="right">(by (2.11) and (2.12))</div>

$$\leq \frac{1}{4|H-H^*|} \Bigg(|(H+1)(H^*+1)| \exp(2\|p_2\|_{C^0}+M) \|q_2^{(1)}-q_2^{(2)}\|_{C^0}$$

$$+ |(H-1)(H^*-1)| \exp(2\|p_2\|_{C^0}+M)\|q_2^{(1)}-q_2^{(2)}\|_{C^0} \Bigg),$$

<div align="right">(by the mean value theorem).</div>

This shows (IV.8).

Proof of (IV.9). We have

$$|J(q^{(1)}) - J(q^{(2)})|$$

$$= \left| (H+1)\exp\left(\int_0^1 (q_2^{(1)}(s)-p_2(s))ds\right) + 1 - H \right|^{-1}$$

$$\times \left| (H+1)\exp\left(\int_0^1 (q_2^{(2)}(s)-p_2(s))ds\right) + 1 - H \right|^{-1}$$

$$\times 2 \left| (1-H^2)\exp\left(-\int_0^1 p_2(s)ds\right)\left(\exp\left(\int_0^1 q_2^{(1)}(s)ds\right) - \exp\left(\int_0^1 q_2^{(2)}(s)ds\right)\right) \right|$$

$$\leq 2M_1^{-2}|1-H^2|\exp(2\|p_2\|_{C^0}+M)\|q_2^{(1)}-q_2^{(2)}\|_{C^0},$$

by (2.14) and the mean value theorem. For $|J^*(q^{(1)})-J^*(q^{(2)})|$, we can similarly carry out a proof. This completes the proof of (IV.9).

Proof of (IV.10). By using (2.14) in the definition (2.11) of $J(q)$, we get

$$|J(q)| \leq M_1^{-1}\left| (H+1)\exp\left(\int_0^1 (q_2(s)-p_2(s))ds\right) + H - 1 \right| \leq M_1^{-1}(|H+1|e^M + |H-1|),$$

which implies (IV.10) for $|J(q)|$. For $|J^*(q)|$, we can proceed similarly.
    Thus we complete the proof of Lemma 4.

Appendix V. Proof of Lemma 5

Let us set

(V.1) $$\Omega_1 = \{(x, y) \; ; \; 1-x < y < x, \frac{1}{2} < x < 1\}$$

and

(V.2) $$\Omega_2 = \Omega \backslash \Omega_1 \backslash \{(x, y) \; ; \; 1-x=y\} .$$

As in Appendix I of Yamamoto [12], putting

(V.3)
$$\begin{cases} L_1(x, y) = L_1(x, y, q_1, q_2, D) = K_{12}(x, y) - K_{21}(x, y) \\ L_2(x, y) = L_2(x, y, q_1, q_2, D) = K_{11}(x, y) - K_{22}(x, y) \\ L_3(x, y) = L_3(x, y, q_1, q_2, D) = K_{11}(x, y) + K_{22}(x, y) \\ L_4(x, y) = L_4(x, y, q_1, q_2, D) = K_{12}(x, y) + K_{21}(x, y) \end{cases}$$

$$((x, y) \in \bar{\Omega}),$$

we can rewrite (2.34)-(2.36), so that we obtain (V.4)-(V.6):

(V.4) $$\frac{\partial L_i(x, y)}{\partial x} + \delta_i \frac{\partial L_i(x, y)}{\partial y} = f_i(x, y, L_1, L_2, L_3, L_4)$$

$$((x, y) \in \bar{\Omega}, \; 1 \leq i \leq 4) .$$

(V.5)
$$\begin{cases} L_3(x, 0) = k L_1(x, 0) + l L_2(x, 0) \\ L_4(x, 0) = -l L_1(x, 0) - k L_2(x, 0) \end{cases} \qquad (0 \leq x \leq 1) .$$

(V.6) $$L_i(1, y) = r_i(y) \qquad (0 \leq y \leq 1, \; 1 \leq i \leq 4) .$$

Here we set (V.7)-(V.11):

(V.7) $$\delta_i = \begin{cases} -1, & \text{if } i=1, 2 \\ 1, & \text{if } i=3, 4 . \end{cases}$$

(V.8) $$f_i(x, y, L_1, L_2, L_3, L_4) = f_i(x, y, L_1, L_2, L_3, L_4, q_1, q_2)$$

$$= \sum_{j=1}^{4} a_{ij}(x, y) L_j(x, y) \qquad ((x, y) \in \bar{\Omega}) .$$

$$\begin{cases} a_{11}(x, y) \equiv a_{11}(x, y, q_1, q_2) = \frac{1}{2}(-b(y)-p_1(y)-b(x)-q_1(x)) \\[2mm] a_{12}(x, y) \equiv a_{12}(x, y, q_1, q_2) = \frac{1}{2}(-a(y)-p_2(y)+a(x)+q_2(x)) \\[2mm] a_{13}(x, y) \equiv a_{13}(x, y, q_1, q_2) = \frac{1}{2}(-a(y)+p_2(y)+a(x)-q_2(x)) \end{cases}$$

$$
\text{(V.9)}
\begin{cases}
a_{14}(x,\ y)\equiv a_{14}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(b(y)-p_1(y)+b(x)-q_1(x)) \\[6pt]
a_{21}(x,\ y)\equiv a_{21}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(-a(y)-p_2(y)+a(x)+q_2(x)) \\[6pt]
a_{22}(x,\ y)\equiv a_{22}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(-b(y)-p_1(y)-b(x)-q_1(x)) \\[6pt]
a_{23}(x,\ y)\equiv a_{23}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(-b(y)+p_1(y)+b(x)-q_1(x)) \\[6pt]
a_{24}(x,\ y)\equiv a_{24}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(a(y)-p_2(y)+a(x)-q_2(x)) \\[6pt]
a_{31}(x,\ y)\equiv a_{31}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(-a(y)+p_2(y)-a(x)+q_2(x)) \\[6pt]
a_{32}(x,\ y)\equiv a_{32}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(b(y)-p_1(y)+b(x)-q_1(x)) \\[6pt]
a_{33}(x,\ y)\equiv a_{33}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(b(y)+p_1(y)-b(x)-q_1(x)) \\[6pt]
a_{34}(x,\ y)\equiv a_{34}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(a(y)+p_2(y)-a(x)-q_2(x)) \\[6pt]
a_{41}(x,\ y)\equiv a_{41}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(-b(y)+p_1(y)+b(x)-q_1(x)) \\[6pt]
a_{42}(x,\ y)\equiv a_{42}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(a(y)-p_2(y)-a(x)+q_2(x)) \\[6pt]
a_{43}(x,\ y)\equiv a_{43}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(a(y)+p_2(y)-a(x)-q_2(x)) \\[6pt]
a_{44}(x,\ y)\equiv a_{44}(x,\ y,\ q_1,\ q_2)=\frac{1}{2}\,(b(y)+p_1(y)-b(x)-q_1(x))\ ((x,y)\in\bar{\Omega}).
\end{cases}
$$

$$
\text{(V.10)} \qquad k=\frac{-2h}{1-h^2} \quad \text{and} \quad l=\frac{1+h^2}{1-h^2}.
$$

$$
\text{(V.11)}
\begin{cases}
r_1(y)\equiv r_1(y,\ D)=d_{12}(y)-d_{21}(y) \\[4pt]
r_2(y)\equiv r_2(y,\ D)=d_{11}(y)-d_{22}(y) \\[4pt]
r_3(y)\equiv r_3(y,\ D)=d_{11}(y)+d_{22}(y) \\[4pt]
r_4(y)\equiv r_4(y,\ D)=d_{12}(y)+d_{21}(y)
\end{cases}
\qquad (0\leqq y\leqq 1).
$$

We will prove Lemma 5 separately in each of $\overline{\Omega_1}$ and $\overline{\Omega_2}$. In $\overline{\Omega_1}$, our problem (V.4) and (V.6) is a Cauchy problem and, for the unique existence of solutions, we can refer to Nagumo [8] and Petrovsky [9, pp. 67-73], for instance. Moreover, in this lemma, we have to prove the estimates, which depend upon the C^1-*norms* of p_i and the C^0-*norms* of q_i $(i=1,\ 2)$.

Proof of Lemma 5 in Ω_1. First we show

LEMMA V.1. *Let $f(x, y)$ and $\dfrac{\partial f(x, y)}{\partial y}$ be continuous functions on $\overline{\Omega_1}$ and satisfy*

(V.12) $\qquad |f(x, y)| \leqq g(x) \quad and \quad \left| \dfrac{\partial f(x, y)}{\partial y} \right| \leqq h(x) \qquad ((x, y) \in \overline{\Omega_1})$

for some g, $h \in C^0 \left[\dfrac{1}{2}, 1 \right]$. Then, for each $a \in C^1[0, 1]$, there exists a unique solution $u \in C^1(\overline{\Omega_1})$ to each of (V.13) and (V.13)′:

(V.13) $\qquad \dfrac{\partial u(x, y)}{\partial x} = \dfrac{\partial u(x, y)}{\partial y} + f(x, y) \ ((x, y) \in \overline{\Omega_1}), \ u(1, y) = a(y) \ (0 \leqq y \leqq 1).$

(V.13)′ $\qquad \dfrac{\partial u(x, y)}{\partial x} = -\dfrac{\partial u(x, y)}{\partial y} + f(x, y) \ ((x, y) \in \overline{\Omega_1}), \ u(1, y) = a(y) \ (0 \leqq y \leqq 1).$

Moreover the solution to each of (VI.13) and (VI.13)′ satisfies

(V.14) $\qquad\qquad\qquad |u(x, y)| \leqq ||a||_{C^0} + \displaystyle\int_x^1 g(s)ds \qquad\qquad ((x, y) \in \overline{\Omega_1})$

and

(V.15) $\qquad\qquad\qquad \left| \dfrac{\partial u(x, y)}{\partial y} \right| \leqq ||a||_{C^1} + \displaystyle\int_x^1 h(s)ds \qquad\qquad ((x, y) \in \overline{\Omega_1}).$

Proof of Lemma V.1. Since the solutions u to (V.13) and v to (V.13)′ are represented in the forms

$$u(x, y) = a(x+y-1) + \int_1^x f(s, -s+x+y)ds \qquad ((x, y) \in \overline{\Omega_1})$$

and

$$v(x, y) = a(1-x+y) + \int_1^x f(s, s-x+y)ds \qquad ((x, y) \in \overline{\Omega_1}),$$

respectively, we can immediately see this lemma.

In $\overline{\Omega_1}$, as is proved below, the solution L_i $(1 \leqq i \leqq 4)$ to (V.4) and (V.6) is given as the limit of uniformly convergent sequence $\{L_i^{(n)}\}_{n \geqq 0}$ $(1 \leqq i \leqq 4)$ defined inductively by (V.16) and (V.17):

(V.16) $\qquad\qquad L_i^{(0)}(x, y) = 0 \qquad ((x, y) \in \overline{\Omega_1}, \ 1 \leqq i \leqq 4)$

(V.17) $\quad \begin{cases} L_i^{(n)} \ (1 \leqq i \leqq 4) \ is \ the \ solution \ to \\[2mm] \dfrac{\partial L_i^{(n)}(x, y)}{\partial x} + \delta_i \dfrac{\partial L_i^{(n)}(x, y)}{\partial y} \\[4mm] = f_i(x, y, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)}) \qquad ((x, y) \in \overline{\Omega_1}) \\[3mm] L_i^{(n)}(1, y) = r_i(y) \qquad (0 \leqq y \leqq 1). \end{cases}$

In view of Lemma V.1, the sequences $\{L_i^{(n)}\}_{n\geq 0}\subset C^1(\overline{\Omega_1})$ $(1\leq i\leq 4)$ are well-defined. Furthermore we will prove the estimates

(V.18)
$$\left| L_i^{(n+1)}(x,\ y)-L_i^{(n)}(x,\ y)\right|\leq\frac{M_{19}M_{18}^n(1-x)^n}{n!}$$

and

(V.19)
$$\left|\frac{\partial L_i^{(n+1)}(x,\ y)}{\partial y}-\frac{\partial L_i^{(n)}(x,\ y)}{\partial y}\right|\leq\frac{M_{20}M_{18}^n(1-x)^n}{n!}$$

$$((x,\ y)\in\overline{\Omega_1},\ n\geq 0,\ 1\leq i\leq 4),$$

where

(V.20)
$$\begin{cases} M_{18}=\max_{1\leq i\leq 4}\sum_{j=1}^4\ (\|a_{ij}\|_{C^0(\overline{\Omega_1})}+\|\partial a_{ij}/\partial y\|_{C^0(\overline{\Omega_1})}) \\[2mm] M_{19}=\max_{1\leq j\leq 4}\ \|r_j\|_{C^0[0,1]} \\[2mm] M_{20}=\max_{1\leq j\leq 4}\ \|r_j\|_{C^1[0,1]}. \end{cases}$$

Here we see that M_{18} is independent of $\|dq_i/dx\|_{C^0[0,1]}$ $(i=1,\ 2)$, as is see from the forms (V.9) of a_{ij} $(1\leq i,\ j\leq 4)$. That is, for $(q_1,\ q_2)\in\mathcal{A}_M$, we have

(V.21)
$$M_{18}=M_{18}(M,\ \|P\|_{(C^1[0,1])^4}).$$

Now the proof of (V.18) and (V.19) is done as follows. For $n=0$, we immediately see (V.18) and (V.19). Assume that (V.18) and (V.19) hold true for $n=m$. Then since

$$\frac{\partial}{\partial x}(L_i^{(m+2)}-L_i^{(m+1)})(x,\ y)+\partial_i\frac{\partial}{\partial y}(L_i^{(m+2)}-L_i^{(m+1)})(x,\ y)$$

$$=\sum_{j=1}^4 a_{ij}(x,\ y)(L_j^{(m+1)}(x,\ y)-L_j^{(m)}(x,\ y))\qquad((x,\ y)\in\overline{\Omega_1},\ 1\leq i\leq 4)$$

and $(L_i^{(m+2)}-L_i^{(m+1)})(1,\ y)=0$ $(0\leq y\leq 1,\ 1\leq i\leq 4)$, by (V.18) and (V.19) for $n=m$, we can put

$$g(x)=\frac{M_{19}M_{18}^{m+1}(1-x)^m}{m!}\ \ \text{and}\ \ h(x)=\frac{M_{20}M_{18}^{m+1}(1-x)^m}{m!}$$

in (V.12), so that we get

$$\left|L_i^{(m+2)}(x,\ y)-L_i^{(m+1)}(x,\ y)\right|\leq\int_x^1\frac{M_{19}M_{18}^{m+1}(1-s)^m}{m!}\ ds=\frac{M_{19}M_{18}^{m+1}(1-x)^{m+1}}{(m+1)!}$$

and

$$\left|\frac{\partial L_i^{(m+2)}(x,\ y)}{\partial y}-\frac{\partial L_i^{(m+1)}(x,\ y)}{\partial y}\right|\leq\int_x^1\frac{M_{20}M_{18}^{m+1}(1-s)^m}{m!}\ ds=\frac{M_{20}M_{18}^{m+1}(1-x)^{m+1}}{(m+1)!}$$

in view of Lemma V.1. Thus, by induction, we obtain (V.18) and (V.19) for each $n \geq 0$.

By (V.18) and (V.19), for $1 \leq i \leq 4$, the series

$$\sum_{n=0}^{\infty} (L_i^{(n+1)}(x, y) - L_i^{(n)}(x, y)) \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{\partial L_i^{(n+1)}(x, y)}{\partial y} - \frac{\partial L_i^{(n)}(x, y)}{\partial y} \right)$$

are absolutely convergent to $L_i(x, y)$ and $\dfrac{\partial L_i(x, y)}{\partial y}$, respectively, and the convergences are uniform with respect to $(x, y) \in \overline{\Omega_1}$. Therefore we see that

$L_i, \dfrac{\partial L_i}{\partial y} \in C^0(\overline{\Omega_1})$. By (V.17), as $n \to \infty$, also $\dfrac{\partial L_i^{(n)}(x, y)}{\partial x}$ $(1 \leq i \leq 4)$ are convergent uniformly with respect to $(x, y) \in \overline{\Omega_1}$, so that $L_i \in C^1(\overline{\Omega_1})$ $(1 \leq i \leq 4)$ and L_i $(1 \leq i \leq 4)$ satisfy (V.4) and (V.6). Furthermore, by (V.18), (V.19), for $1 \leq i \leq 4$, we get the estimates

(V.22) $$\|L_i\|_{C^0(\overline{\Omega_1})} \leq e^{M_{18}} \max_{1 \leq j \leq 4} \|r_j\|_{C^0[0,1]},$$

(V.23) $$\|\partial L_i/\partial y\|_{C^0(\overline{\Omega_1})} \leq e^{M_{18}} \max_{1 \leq j \leq 4} \|r_j\|_{C^1[0,1]}$$

and, by (V.4),

(V.24) $$\|\partial L_i/\partial x\|_{C^0(\overline{\Omega_1})} \leq \|\partial L_i/\partial y\|_{C^0(\overline{\Omega_1})} + \|f_i(\cdot, \cdot, L_1, L_2, L_3, L_4)\|_{C^0(\overline{\Omega_1})}$$

$$\leq e^{M_{18}} \max_{1 \leq j \leq 4} \|r_j\|_{C^1[0,1]} + M_{18} \, e^{M_{18}} \max_{1 \leq j \leq 4} \|r_j\|_{C^0[0,1]}.$$

The estimates (V.22)-(V.24) show the inequalities on $\overline{\Omega_1}$ corresponding to (2.37) and (2.38).

Next we proceed to the proof of the estimate corresponding to (2.39) in Ω_1. To this end, we have only to prove

(V.25) $$\|L_i(\cdot, \cdot, q_1^{(1)}, q_2^{(1)}, D_1) - L_i(\cdot, \cdot, q_1^{(2)}, q_2^{(2)}, D_2)\|_{C^0(\overline{\Omega_1})}$$

$$\leq M_6 \left(\|D_2\|_{C^0}(\|q_1^{(1)} - q_1^{(2)}\|_{C^0} + \|q_2^{(1)} - q_2^{(2)}\|_{C^0}) + \|D_1 - D_2\|_{C^0} \right)$$

for $(q_1^{(m)}, q_2^{(m)}) \in \mathcal{A}_M$ $(m=1, 2)$. Here and henceforth, for brevity, we set

(V.26) $$\begin{cases} D_m = \begin{pmatrix} d_{11}^{(m)} & d_{12}^{(m)} \\ d_{21}^{(m)} & d_{22}^{(m)} \end{pmatrix} \quad (m=1, 2) \\ L_{m,i}(x, y) = L_i(x, y, q_1^{(m)}, q_2^{(m)}, D_m) \quad (x \in \overline{\Omega_1}, \ 1 \leq i \leq 4, \ m=1, 2). \end{cases}$$

Since for $m=1, 2$, the functions $L_{m,i}$ $(1 \leq i \leq 4)$ are the solutions to hyperbolic problems similar to (V.4) and (V.6), we have equations with respect to $L_{1,i} - L_{2,i}$ $(1 \leq i \leq 4)$:

(V. 27) $$\frac{\partial}{\partial x}(L_{1,i}-L_{2,i})(x,\ y)+\delta_i\frac{\partial}{\partial y}(L_{1,i}-L_{2,i})(x,\ y)$$

$$=\sum_{j=1}^{4}a_{ij}(x,\ y,\ q_1^{(1)},\ q_2^{(1)})(L_{1,j}-L_{2,j})(x,\ y)$$

$$+\sum_{j=1}^{4}(a_{ij}(x,\ y,\ q_1^{(1)},\ q_2^{(1)})-a_{ij}(x,\ y,\ q_1^{(2)},\ q_2^{(2)}))L_{2,j}(x,\ y)$$

$$((x,\ y)\in\overline{\Omega_1},\ 1\leqq i\leqq 4)$$

and

(V. 28) $(L_{1,i}-L_{2,i})(1,\ y)=r_i(y,\ D_1)-r_i(y,\ D_2)\qquad(0\leqq y\leqq 1,\ 1\leqq i\leqq 4).$

Then, since $(x,\ y)\in\overline{\Omega_1}$ implies $1-x\leqq y\leqq x$, we have

$$|[\text{the right hand side of (V. 27)}]|$$

$$\leqq M_{18}\max_{1\leqq j\leqq 4}\max_{1-x\leqq y\leqq x}|(L_{1,j}-L_{2,j})(x,\ y)|$$

$$+\sum_{j=1}^{4}||a_{ij}(\cdot,\ \cdot,\ q_1^{(1)},\ q_2^{(1)})-a_{ij}(\cdot,\ \cdot,\ q_1^{(2)},\ q_2^{(2)})||_{C^0}\times\max_{1\leqq j\leqq 4}||L_{2,j}||_{C^0}$$

$$\leqq M_{18}\max_{1\leqq j\leqq 4}\max_{1-x\leqq y\leqq x}|(L_{1,j}-L_{2,j})(x,\ y)|$$

$$+4e^{M_{18}}||D_2||_{(C^0[0,1])^4}(||q_1^{(1)}-q_1^{(2)}||_{C^0[0,1]}+||q_2^{(1)}-q_2^{(2)}||_{C^0[0,1]})$$

$$\text{(by (V. 9) and (V. 22))}$$

$$\equiv g(x).$$

Applying Lemma V.1, we get

(V. 29) $$\max_{1-x\leqq y\leqq x}|(L_{1,i}-L_{2,i})(x,\ y)|$$

$$\leqq||r_i(\cdot,\ D_1)-r_i(\cdot,\ D_2)||_{C^0[0,1]}+\int_x^1 g(s)ds\qquad(1\leqq i\leqq 4).$$

Let us set $\theta(x)=\max\limits_{1\leqq i\leqq 4}\max\limits_{1-x\leqq y\leqq x}|(L_{1,i}-L_{2,i})(x,\ y)|.$
Then (V. 29) is rewritten as

$$\theta(x)\leqq\Big(\max_{1\leqq i\leqq 4}||r_i(\cdot,\ D_1)-r_i(\cdot,\ D_2)||_{C^0[0,1]}$$

$$+4e^{M_{18}}||D_2||_{(C^0[0,1])^4}(||q_1^{(1)}-q_1^{(2)}||_{C^0[0,1]}+||q_2^{(1)}-q_2^{(2)}||_{C^0})\Big)$$

$$+M_{18}\int_x^1\theta(s)ds\qquad\qquad\Big(\frac{1}{2}\leqq x\leqq 1\Big),$$

which implies

(V.30) $\theta(x) \leqq e^{M_{18}}\Big(\max_{1 \leqq i \leqq 4} \|r_i(\cdot,\ D_1) - r_i(\cdot,\ D_2)\|_{C^0[0,1]}$

$$+ 4e^{M_{18}}\|D_2\|_{\{C^0[0,1]\}^4}(\|q_1^{(1)} - q_1^{(2)}\|_{C^0[0,1]} + \|q_2^{(1)} - q_2^{(2)}\|_{C^0})\Big)\Big(\frac{1}{2} \leqq x \leqq 1\Big),$$

by Gronwall's inequality. The inequality (V.30) is equivalent to (V.25), the conclusion. Thus in Ω_1, we complete the proof of Lemma 5.

Proof of Lemma 5 in Ω_2. In Ω_2, we have to consider a hyperbolic problem (V.4), (V.5) and

(V.31) $L_j(x,\ 1-x) = b_j(x)$ $\Big(\frac{1}{2} \leqq x \leqq 1,\ j=3,\ 4\Big).$

Here and henceforth we set

(V.32) $b_j(x) = L_j(x,\ 1-x) = \lim_{\substack{x' \to x,\ y' \to 1-x \\ (x',y') \in \Omega_1}} L_j(x',\ y')$ $\Big(\frac{1}{2} \leqq x \leqq 1,\ j=3,\ 4\Big),$

where $L_j \in C^1(\overline{\Omega_1})$ $(j=3,\ 4)$ is the solution to (V.4) and (V.6).

As the approximate sequences for the solution in $\overline{\Omega_2}$, let us inductively define $\{L_i^{(n)}\}_{n \geqq 0}$ $(1 \leqq i \leqq 4)$ by (V.33)-(V.35):

(V.33) $L_i^{(0)}(x,\ y) = 0$ $((x,\ y) \in \overline{\Omega_2},\ 1 \leqq i \leqq 4).$

(V.34)
$$
\begin{cases}
L_1^{(n+1)}(x,\ y) \\[4pt]
= \displaystyle\int_{\frac{1+x+y}{2}}^{x+y} (-kf_3 - lf_4)(s,\ s-x-y,\ L_1^{(n)},\ L_2^{(n)},\ L_3^{(n)},\ L_4^{(n)})ds \\[10pt]
+ \displaystyle\int_{x+y}^{x} f_1(s,\ -s+x+y,\ L_1^{(n)},\ L_2^{(n)},\ L_3^{(n)},\ L_4^{(n)})ds \\[10pt]
- kb_3\Big(\dfrac{1+x+y}{2}\Big) - lb_4\Big(\dfrac{1+x+y}{2}\Big) \\[10pt]
L_2^{(n+1)}(x,\ y) \\[4pt]
= \displaystyle\int_{\frac{1+x+y}{2}}^{x+y} (lf_3 + kf_4)(s,\ s-x-y,\ L_1^{(n)},\ L_2^{(n)},\ L_3^{(n)},\ L_4^{(n)})ds \\[10pt]
+ \displaystyle\int_{x+y}^{x} f_2(s,\ -s+x+y,\ L_1^{(n)},\ L_2^{(n)},\ L_3^{(n)},\ L_4^{(n)})ds \\[10pt]
+ lb_3\Big(\dfrac{1+x+y}{2}\Big) + kb_4\Big(\dfrac{1+x+y}{2}\Big) \qquad ((x,\ y) \in \overline{\Omega_2},\ n \geqq 0).
\end{cases}
$$

(V.35) $L_i^{(n+1)}(x,\ y) = b_i\Big(\dfrac{1+x-y}{2}\Big)$

$$+\int_{\frac{1+x-y}{2}}^{x} f_i(s,\ s-x+y,\ L_1^{(n)},\ L_2^{(n)},\ L_3^{(n)},\ L_4^{(n)})ds$$

$$((x,\ y)\in\overline{\Omega_2},\ n\geqq0,\ i=3,\ 4).$$

Obviously we see that $\{L_i^{(n)}\}_{n\geqq0}$ $(1\leqq i\leqq4)$ are well-defined and $L_i^{(n)}\in C^1(\overline{\Omega_2})$. Moreover, by induction, in a way similar to (V.18) and (V.19), we can obtain the estimates:

$$(V.36) \qquad \left|L_i^{(n+1)}(x,\ y)-L_i^{(n)}(x,\ y)\right|\leqq\frac{M_{22}M_{21}^n(1-x)^n}{n!}$$

$$((x,\ y)\in\overline{\Omega_2},\ n\geqq0,\ 1\leqq i\leqq4)$$

and

$$(V.37) \qquad \left|\frac{\partial L_i^{(n+1)}(x,\ y)}{\partial y}-\frac{\partial L_i^{(n)}(x,\ y)}{\partial y}\right|\leqq\frac{M_{24}M_{23}^{n-1}(1-x)^{n-1}}{(n-1)!}$$

$$((x,\ y)\in\overline{\Omega_2},\ n\geqq1,\ 1\leqq i\leqq4).$$

Here and henceforth, we set

$$(V.38) \qquad \begin{cases} M_{21}=(|k|+|l|+1)\ \displaystyle\max_{1\leqq i\leqq4}\ \sum_{j=1}^{4}\ \|a_{ij}\|_{C^0(\overline{\Omega_2})} \\[2mm] M_{22}=(|k|+|l|+1)\ \displaystyle\max_{j=3,4}\ \|b_j\|_{C^0[\frac{1}{2},1]} \\[2mm] M_{23}=2M_{21}(|k|+|l|+1) \\[2mm] \qquad \times(\displaystyle\max_{1\leqq i\leqq4}\ \sum_{j=1}^{4}\ \|a_{ij}\|_{C^0}+\max_{1\leqq i\leqq4}\ \sum_{j=1}^{4}\ \|\partial a_{ij}/\partial y\|_{C^0}+1) \\[2mm] \qquad +M_{21} \\[2mm] M_{24}=3(|k|+|l|+1)^2\ \displaystyle\max_{j=3,4}\ \|b_j\|_{C^1[\frac{1}{2},1]} \\[2mm] \qquad \times(\displaystyle\max_{1\leqq i\leqq4}\ \sum_{j=1}^{4}\ \|a_{ij}\|_{C^0}+\max_{1\leqq i\leqq4}\ \sum_{j=1}^{4}\ \|\partial a_{ij}/\partial y\|_{C^0}+1). \end{cases}$$

From the forms (V.9) of a_{ij} $(1\leqq i,\ j\leqq4)$, the constants M_{23} and M_{24} are independent of $\|dq_i/dx\|_{C^0}$ $(i=1,\ 2)$. Hence we can rewrite M_{21}, M_{22}, M_{23} and M_{24} as

$$(V.39) \qquad \begin{cases} M_{21}=M_{21}(M,\ \|P\|_{C^0},\ h) \\[2mm] M_{22}=(|k|+|l|+1)\ \displaystyle\max_{j=3,4}\ \|b_j\|_{C^0[\frac{1}{2},1]} \\[2mm] M_{23}=M_{23}(M,\ \|P\|_{C^1},\ h) \\[2mm] M_{24}=M_{24}(M,\ \|P\|_{C^1},\ h)\ \displaystyle\max_{j=3,4}\ \|b_j\|_{C^1[\frac{1}{2},1]}. \end{cases}$$

By (V.36) and (V.37), the series $\sum_{n=0}^{\infty}\ (L_i^{(n+1)}(x,\ y)-L_i^{(n)}(x,\ y))$ and

$\sum_{n=0}^{\infty}\left(\dfrac{\partial L_i^{(n+1)}(x,\ y)}{\partial y}-\dfrac{\partial L_i^{(n)}(x,\ y)}{\partial y}\right)$ are absolutely convergent to $L_i(x,\ y)$ and $\dfrac{\partial L_i(x,\ y)}{\partial y}$,

respectively, and the convergences are uniform with respect to $(x,\ y)\in\overline{\Omega_2}$, that $L_i,\ \dfrac{\partial L_i}{\partial y}\in C^0(\overline{\Omega_2})$ $(1\leqq i\leqq 4)$. Moreover, letting $n\to\infty$ in (V.34) and (V.35), we get

(V.40)

$$
\begin{cases}
\begin{aligned}
L_1(x,\ y)=&\int_{\frac{1+x+y}{2}}^{x+y}(-kf_3-lf_4)(s,\ s-x-y,\ L_1,\ L_2,\ L_3,\ L_4)ds\\[2mm]
&+\int_{x+y}^{x}f_1(s,\ -s+x+y,\ L_1,\ L_2,\ L_3,\ L_4)ds\\[2mm]
&-kb_3\left(\frac{1+x+y}{2}\right)-lb_4\left(\frac{1+x+y}{2}\right)\qquad((x,\ y)\in\overline{\Omega_2})\\[3mm]
L_2(x,\ y)=&\int_{\frac{1+x+y}{2}}^{x+y}(lf_3+kf_4)(s,\ s-x-y,\ L_1,\ L_2,\ L_3,\ L_4)ds\\[2mm]
&+\int_{x+y}^{x}f_2(s,\ -s+x+y,\ L_1,\ L_2,\ L_3,\ L_4)ds\\[2mm]
&+lb_3\left(\frac{1+x+y}{2}\right)+kb_4\left(\frac{1+x+y}{2}\right)\qquad((x,\ y)\in\overline{\Omega_2})\\[3mm]
L_i(x,\ y)=&\int_{\frac{1+x-y}{2}}^{x}f_i(s,\ s-x+y,\ L_1,\ L_2,\ L_3,\ L_4)ds+b_i\left(\frac{1+x-y}{2}\right)\\[2mm]
&\qquad\qquad\qquad\qquad\qquad\qquad((x,\ y)\in\overline{\Omega_2},\ i=3,\ 4).
\end{aligned}
\end{cases}
$$

Therefore we see that $\dfrac{\partial L_i}{\partial x}\in C^0(\overline{\Omega_2})$, and hence L_i $(1\leqq i\leqq 4)$ satisfy (V.4), (V.5) and (V.31) in $\overline{\Omega_2}$.

Thus we have constructed L_i $(1\leqq i\leqq 4)$ in the respective domains of Ω_1 and Ω_2, so that we see that there exists a unique solution L_i $(1\leqq i\leqq 4)$ to (V.4)-(V.6). In fact, to this end, we have only to verify that L_i $(1\leqq i\leqq 4)$ are actually C^1 on $\overline{\Omega_1}\cup\overline{\Omega_2}$. This can be seen from the fact that L_i $(1\leqq i\leqq 4)$ satisfy the integral equations (V.40) and (V.41) on $\overline{\Omega_2}$ and $\overline{\Omega_1}$, respectively:

(V.41)

$$
\begin{cases}
L_i(x,\ y)=r_i(x+y-1)+\int_1^x f_i(s,\ -s+x+y,\ L_1,\ L_2,\ L_3,\ L_4)ds\\[2mm]
\qquad\qquad\qquad\qquad\qquad\qquad\qquad((x,\ y)\in\overline{\Omega_1},\ i=1,\ 2)\\[3mm]
L_i(x,\ y)=r_i(-x+y+1)+\int_1^x f_i(s,\ s-x+y,\ L_1,\ L_2,\ L_3,\ L_4)ds\\[2mm]
\qquad\qquad\qquad\qquad\qquad\qquad\qquad((x,\ y)\in\overline{\Omega_1},\ i=3,\ 4).
\end{cases}
$$

Now we proceed to the proof of the estimates in Ω_2. By (V.36), (V.37), (V.39) and (V.4), for $1\leqq i\leqq 4$, we see

(V. 42) $$\|L_i\|_{C^0(\bar{\Omega}_2)} \leqq M_{25}(M,\ \|P\|_{[C^1[0,1]]^4},\ h) \times \max_{j=3,4} \|b_j\|_{C^0}$$

and

(V. 43) $$\|L_i\|_{C^1(\bar{\Omega}_2)} \leqq M_{25}(M,\ \|P\|_{[C^1[0,1]]^4},\ h) \times \max_{j=3,4} \|b_j\|_{C^1}.$$

Combining (V. 42), (V. 43) with (V. 22)–(V. 24) and recalling (V. 32), for $1 \leqq i \leqq 4$, we obtain

(V. 44) $$\|L_i\|_{C^0(\bar{\Omega})} \leqq M_{25}(M,\ \|P\|_{[C^1[0,1]]^4},\ h) \times \max_{1 \leqq j \leqq 4} \|r_j\|_{C^0[0,1]}$$

and

(V. 45) $$\|L_i\|_{C^1(\bar{\Omega})} \leqq M_{25}(M,\ \|P\|_{[C^1[0,1]]^4},\ h) \times \max_{1 \leqq j \leqq 4} \|r_j\|_{C^1[0,1]},$$

which imply (2.37) and (2.38), respectively.

Finally, in order to complete the proof of Lemma 5, in view of (V. 25), we have to prove

(V. 46) $$\|L_i(\cdot,\ \cdot,\ q_1^{(1)},\ q_2^{(1)},\ D_1) - L_i(\cdot,\ \cdot,\ q_1^{(2)},\ q_2^{(2)},\ D_2)\|_{C^0(\bar{\Omega}_2)}$$

$$\leqq M_{26}\Big(\max_{1 \leqq j \leqq 4} \|r_j^{(2)}\|_{C^0[0,1]} \times (\|q_1^{(1)} - q_1^{(2)}\|_{C^0[0,1]} + \|q_2^{(1)} - q_2^{(2)}\|_{C^0})$$

$$+ \|b_3^{(1)} - b_3^{(2)}\|_{C^0[\frac{1}{2},1]} + \|b_4^{(1)} - b_4^{(2)}\|_{C^0[\frac{1}{2},1]}\Big) \qquad (1 \leqq i \leqq 4)$$

for each $(q_1^{(m)},\ q_2^{(m)}) \in \mathcal{A}_M$ ($m=1,\ 2$). Here and henceforth we put $L_{m,i}(x,\ y) = L_i(x,\ y,\ q_1^{(m)},\ q_2^{(m)},\ D_m)$ $((x,\ y) \in \bar{\Omega},\ 1 \leqq i \leqq 4,\ m=1,\ 2)$ and $b_i^{(m)}(x) = L_{m,i}(x,\ 1-x)$ $(\frac{1}{2} \leqq x \leqq 1,\ i=3,\ 4,\ m=1,\ 2)$ (cf. (V. 26)).

We show

LEMMA V.2. *Let $f(x,\ y)$ and $\dfrac{\partial f(x,\ y)}{\partial y}$ be continuous functions in $\bar{\Omega}_2$ and satisfy*

(V. 47) $$|f(x,\ y)| \leqq g(x) \qquad\qquad ((x,\ y) \in \bar{\Omega}_2)$$

for some $g \in C^0[0,\ 1]$. Then

(I) *Let $a \in C^1\left[\dfrac{1}{2},1\right]$ and let $u \in C^1(\bar{\Omega}_2)$ be the solution to*

(V. 48) $$\frac{\partial u(x,\ y)}{\partial x} + \frac{\partial u(x,\ y)}{\partial y} = f(x,\ y) \quad ((x,\ y) \in \bar{\Omega}_2),\ u(x,1-x) = a(x)$$

$$\left(\frac{1}{2} \leqq x \leqq 1\right).$$

Then we have

(V.49) $|u(x, y)| \leqq ||a||_{C^0[\frac{1}{2}, 1]} + \int_x^1 g(s) ds \qquad ((x, y) \in \overline{\Omega}_2)$.

(II) *Let $a \in C^1[0, 1]$ and let $v \in C^1(\overline{\Omega}_2)$ be the solution to*

(V.50) $\dfrac{\partial v(x, y)}{\partial x} - \dfrac{\partial v(x, y)}{\partial y} = f(x, y) \quad ((x, y) \in \overline{\Omega}_2), \ v(x, 0) = a(x) \qquad (0 \leqq x \leqq 1)$.

Then we have

(V.51) $|v(x, y)| \leqq ||a||_{C^0[1, 0]} + \int_{x_2^-}^1 g(s) ds \qquad ((x, y) \in \overline{\Omega}_2)$.

Proof of Lemma V.2. By integrating the equations along the characteristic curves, the solutions u to (V.48) and v to (V.50) are represented in the forms

$$u(x, y) = a\left(\frac{1 + x - y}{2}\right) + \int_{\frac{1+x-y}{2}}^x f(s, s - x + y) ds \qquad ((x, y) \in \overline{\Omega}_2)$$

and

$$v(x, y) = a(x + y) + \int_{x+y}^x f(s, -s + x + y) ds \qquad ((x, y) \in \overline{\Omega}_2) .$$

Hence, noting that $\dfrac{1 + x - y}{2} \leqq 1$ and $x + y \leqq 1$ for $(x, y) \in \overline{\Omega}_2$, we can immediately see this lemma.

We return to the proof of (V.46). We note that on $\overline{\Omega}_2$, the functions $L_{1,i} - L_{2,i}$ $(i = 3, 4)$ satisfy the equations (V.27) and

(V.52) $L_{1,i}(x, 1-x) - L_{2,i}(x, 1-x) = b_i^{(1)}(x) - b_i^{(2)}(x) \qquad \left(\dfrac{1}{2} \leqq x \leqq 1, \ i = 3, 4\right)$.

Since $(x, y) \in \overline{\Omega}_2$ implies $0 \leqq y \leqq \min\{x, 1-x\}$, we have

$|[\text{the right hand side of (V.27)}]|$

$\leqq \max\limits_{0 \leqq y \leqq \min\{x, 1-x\}} \sum\limits_{j=1}^4 |a_{ij}(x, y, q_1^{(1)}, q_2^{(1)})| \times |(L_{1,j} - L_{2,j})(x, y)|$

$\quad + \max\limits_{0 \leqq y \leqq \min\{x, 1-x\}} \sum\limits_{j=1}^4 |a_{ij}(x, y, q_1^{(1)}, q_2^{(1)}) - a_{ij}(x, y, q_1^{(2)}, q_2^{(2)})| \times |L_{2,j}(x, y)|$

$\leqq M_{27} \max\limits_{1 \leqq j \leqq 4} \max\limits_{0 \leqq y \leqq \min\{x, 1-x\}} |(L_{1,j} - L_{2,j})(x, y)|$

$\quad + M_{27} \max\limits_{1 \leqq j \leqq 4} ||r_j||_{C^0}(||q_1^{(1)} - q_1^{(2)}||_{C^0[0,1]} + ||q_2^{(1)} - q_2^{(2)}||_{C^0[0,1]}) \qquad \text{(by (V.44))}$

$\equiv g(x)$.

Hence, by Lemma V.2 (I), we get

$$|(L_{1,i}-L_{2,i})(x,y)| \leq \max_{j=3,4} ||b_j^{(1)}-b_j^{(2)}||_{C^0[\frac{1}{2},1]} + \int_x^1 g(s)ds .$$

Taking the maxima of the both hand sides with respect to $y\in[0, \min\{x,1-x\}]$, we have

(V.53)
$$\max_{i=3,4} \max_{0\leq y\leq \min(x,1-x)} |(L_{1,i}-L_{2,i})(x,y)|$$

$$\leq \max_{j=3,4} ||b_j^{(1)}-b_j^{(2)}||_{C^0[\frac{1}{2},1]}$$

$$+ M_{27} \int_x^1 \max_{1\leq j\leq 4} \max_{0\leq t\leq \min(s,1-s)} |(L_{1,j}-L_{2,j})(s,t)|ds$$

$$+ M_{27} \max_{1\leq j\leq 4} ||r_j^{(2)}||_{C^0}(||q_1^{(1)}-q_1^{(2)}||_{C^0[0,1]} + ||q_2^{(1)}-q_2^{(2)}||_{C^0}) .$$

Henceforth we set

(V.54)
$$\begin{cases} \eta_1(x) = \max_{i=3,4} \max_{0\leq y\leq \min(x,1-x)} |(L_{1,i}-L_{2,i})(x,y)| \\ \eta(x) = \max_{1\leq i\leq 4} \max_{0\leq y\leq \min(x,1-x)} |(L_{1,i}-L_{2,i})(x,y)| . \end{cases}$$

Then we can rewrite (V.53) as

(V.55)
$$\eta_1(x) \leq \max_{j=3,4} ||b_j^{(1)}-b_j^{(2)}||_{C^0[\frac{1}{2},1]}$$

$$+ M_{27} \max_{1\leq j\leq 4} ||r_j^{(2)}||_{C^0}(||q_1^{(1)}-q_1^{(2)}||_{C^0} + ||q_2^{(1)}-q_2^{(2)}||_{C^0})$$

$$+ M_{27} \int_x^1 \eta(s)ds \qquad\qquad (0\leq x\leq 1) .$$

Next, in $\overline{\Omega_2}$, the functions $L_{1,i}-L_{2,i}$ $(i=1,2)$ satisfy the equations (V.27) and

(V.56)
$$\begin{cases} (L_{1,1}-L_{2,1})(x,0) \\ = -k(L_{1,3}-L_{2,3})(x,0) - l(L_{1,4}-L_{2,4})(x,0) \\ (L_{1,2}-L_{2,2})(x,0) \\ = l(L_{1,3}-L_{2,3})(x,0) + k(L_{1,4}-L_{2,4})(x,0) \qquad (0\leq x\leq 1) . \end{cases}$$

Therefore we proceed in a way similar to (V.53), in view of Lemma V.2 (II) and (V.44), so that we get for $i=1,2$

(V.57)
$$\max_{0\leq y\leq \min(x,1-x)} |(L_{1,i}-L_{2,i})(x,y)|$$

$$\leq (|k|+|l|) \max_{j=3,4} |(L_{1,j}-L_{2,j})(x+y,0)|$$

$$+ M_{27} \int_x^1 \max_{1\leq j\leq 4} \max_{0\leq t\leq \min(s,1-s)} |(L_{1,j}-L_{2,j})(s,t)|ds$$

$$+M_{27}\max_{1\le j\le 4}||r_j^{(2)}||_{C^0[0,1]}(||q_1^{(1)}-q_1^{(2)}||_{C^0[0,1]}+||q_2^{(1)}-q_2^{(2)}||_{C^0})$$

$$\le(|k|+|l|)\eta_1(x+y)$$

$$+M_{27}\max_{1\le j\le 4}||r_j^{(2)}||_{C^0[0,1]}(||q_1^{(1)}-q_1^{(2)}||_{C^0}+||q_2^{(1)}-q_2^{(2)}||_{C^0})$$

$$+M_{27}\int_x^1\eta(s)ds \qquad\qquad\qquad\text{(by (V.54))}$$

$$\le(|k|+|l|)\max_{j=3,4}||b_j^{(1)}-b_j^{(2)}||_{C^0[0,1]}$$

$$+M_{27}(|k|+|l|+1)\max_{1\le j\le 4}||r_j^{(2)}||_{C^0}(||q_1^{(1)}-q_1^{(2)}||_{C^0}+||q_2^{(1)}-q_2^{(2)}||_{C^0})$$

$$+M_{27}(|k|+|l|+1)\int_x^1\eta(s)ds \qquad\qquad (0\le x\le 1).$$

In the last inequality, we use

$$\eta_1(x+y)\le\max_{j+4}||b_j^{(1)}-b_j^{(2)}||_{C^0}+M_{27}\int_{x+y}^1\eta(s)ds$$

$$+M_{27}\max_{1\le j\le 4}||r_j^{(2)}||_{C^0}(||q_1^{(1)}-q_1^{(2)}||_{C^0}+||q_2^{(1)}-q_2^{(2)}||_{C^0}) \qquad\text{(by (V.55))}$$

$$\le\max_{j=3,4}||b_j^{(1)}-b_j^{(2)}||_{C^0}+M_{27}\int_x^1\eta(s)ds$$

$$+M_{27}\max_{1\le j\le 4}||r_j^{(2)}||_{C^0}(||q_1^{(1)}-q_1^{(2)}||_{C^0}+||q_2^{(1)}-q_2^{(2)}||_{C^0})$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (\text{by }x+y\ge x).$$

By (V.57) and (V.55), we reach

$$\eta(x)\le M_{28}\max_{j=3,4}||b_j^{(1)}-b_j^{(2)}||_{C^0[0,1]}+M_{28}\int_x^1\eta(s)ds$$

$$+M_{28}\max_{1\le j\le 4}||r_j^{(2)}||_{C^0}(||q_1^{(1)}-q_1^{(2)}||_{C^0}+||q_2^{(1)}-q_2^{(2)}||_{C^0}) \qquad (0\le x\le 1),$$

which implies

(V.58) $\quad\eta(x)\le M_{28}e^{M_{28}}\Big(\max_{j=3,4}||b_j^{(1)}-b_j^{(2)}||_{C^0}$

$$+\max_{1\le j\le 4}||r_j^{(2)}||_{C^0}(||q_1^{(1)}-q_1^{(2)}||_{C^0}+||q_2^{(1)}-q_2^{(2)}||_{C^0})\Big) \qquad (0\le x\le 1),$$

by Gronwall's inequality. Since

$$\max_{0\le x\le 1}\eta(x)=\max_{1\le i\le 4}||L_i(\cdot,\cdot,q_1^{(1)},q_2^{(1)},D_1)-L_i(\cdot,\cdot,q_1^{(2)},q_2^{(2)},D_2)||_{C^0(\bar\omega_2)},$$

the inequality (V.58) means (V.46), our conclusion. Thus we complete the proof of Lemma 5.

Appendix VI. Proof of Lemma 6

Let $\Phi(x)$ be a fundamental matrix for the linear homogeneous system

(VI.1)
$$\frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = A(x)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \quad (0 \le x \le 1).$$

That is, $\Phi(x)$ is a 2×2 matrix and satisfies

$$\frac{d\Phi(x)}{dx} = A(x)\Phi(x) \;\; (0 \le x \le 1) \quad \text{and} \quad \det \Phi(x) \ne 0 \;\; (0 \le x \le 1)$$

(Coddington and Levinson [1, p. 69], for example). Here let us recall that

$$A(x) = \frac{1}{2}\begin{pmatrix} -a(x)-b(x)+p_1(x)+p_2(x) & a(x)+b(x)-p_1(x)-p_2(x) \\ -a(x)+b(x)-p_1(x)+p_2(x) & a(x)-b(x)+p_1(x)-p_2(x) \end{pmatrix} \quad (0 \le x \le 1).$$

Then the solution $\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ to (2.42) with (2.43) is given by

(VI.2)
$$\begin{pmatrix} u(x)-1 \\ v(x)-1 \end{pmatrix} = \Phi(x)\int_0^x \Phi^{-1}(y)\begin{pmatrix} K_{11}(y,y)-K_{22}(y,y)+K_{12}(y,y)-K_{21}(y,y) \\ K_{11}(y,y)-K_{22}(y,y)+K_{21}(y,y)-K_{12}(y,y) \end{pmatrix} dy$$

$$(0 \le x \le 1),$$

([1, p. 74], for instance).
Now we proceed to

Proof of the part (I) of Lemma 6. By (VI.2), we have

(VI.3) $|u(x)-1|, \; |v(x)-1| \le 16 \|\Phi\|_{[C^0[0,1]]^4} \|\Phi^{-1}\|_{[C^0[0,1]]^4} \times \max\limits_{\substack{1 \le i,j \le 2 \\ 0 \le x \le 1}} |K_{ij}(x,x)|$

$$(0 \le x \le 1).$$

On the other hand, in view of (2.27) and (2.37), we have

(VI.4)
$$\|K\|_{[C^0(\bar{D})]^4} \le M_4 M_5 \delta_0.$$

Therefore, if

(VI.5)
$$\delta_0 \le (32 M_4 M_5 (\|\Phi\|_{[C^0[0,1]]^4} \times \|\Phi^{-1}\|_{[C^0[0,1]]^4} + 1))^{-1},$$

then we obtain

(VI.6)
$$|u(x)-1|, \; |v(x)-1| \le \frac{1}{2} \quad (0 \le x \le 1).$$

which is the conclusion in the part (I).

Proof of the part (II) of Lemma 6. Since by

$$A(x)\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad (0\leqq x\leqq 1),$$

we have

(VI.7) $\dfrac{d}{dx}\begin{pmatrix}u(x)\\v(x)\end{pmatrix}=A(x)\begin{pmatrix}u(x)-1\\v(x)-1\end{pmatrix}$

$$+\begin{pmatrix}K_{11}(x,\,x)-K_{22}(x,\,x)+K_{12}(x,\,x)-K_{21}(x,\,x)\\K_{11}(x,\,x)-K_{22}(x,\,x)+K_{21}(x,\,x)-K_{12}(x,\,x)\end{pmatrix} \quad (0\leqq x\leqq 1).$$

Therefore we get

(VI.8) $||du/dx||_{C^0[0,1]}+||dv/dx||_{C^0[0,1]}$

$$\leqq M_{29}(\max\{||u-1||_{C^0},\,||v-1||_{C^0}\}+\max_{\substack{1\leqq i,j\leqq 2\\0\leqq x\leqq 1}}|K_{ij}(x,\,x)|)$$

$$\leqq M_{30}\max_{\substack{1\leqq i,j\leqq 2\\0\leqq x\leqq 1}}|K_{ij}(x,\,x)| \qquad\qquad \text{by (VI.3).}$$

Next, differentiating the both hand sides of (VI.7) with respect to x, we get

$$\frac{d^2}{dx^2}\begin{pmatrix}u(x)\\v(x)\end{pmatrix}=\frac{dA(x)}{dx}\begin{pmatrix}u(x)-1\\v(x)-1\end{pmatrix}+A(x)\frac{d}{dx}\begin{pmatrix}u(x)\\v(x)\end{pmatrix}$$

$$+\frac{d}{dx}\begin{pmatrix}K_{11}(x,\,x)-K_{22}(x,\,x)+K_{12}(x,\,x)-K_{21}(x,\,x)\\K_{11}(x,\,x)-K_{22}(x,\,x)+K_{21}(x,\,x)-K_{12}(x,\,x)\end{pmatrix} \quad (0\leqq x\leqq 1).$$

Therefore by a way similar to the one in getting (VI.8), we have

(VI.9) $||d^2u/dx^2||_{C^0}+||d^2v/dx^2||_{C^0}$

$$\leqq M_{31}\max\left\{\max_{\substack{1\leqq i,j\leqq 2\\0\leqq x\leqq 1}}|K_{ij}(x,\,x)|,\quad \max_{\substack{1\leqq i,j\leqq 2\\0\leqq x\leqq 1}}\left|\frac{dK_{ij}(x,\,x)}{dx}\right|\right\}.$$

Estimating p_1-r_1 and p_2-r_2 in (2.46) by using (VI.6), (VI.8) and (VI.9), we reach (2.47) and (2.48). In obtaining (2.48), we note also that

$$||du/dx||_{C^0}^2+||dv/dx||_{C^0}^2\leqq M_{30}^2\max_{\substack{1\leqq i,j\leqq 2\\0\leqq x\leqq 1}}|K_{ij}(x,\,x)|^2\leqq M_{30}^2\max_{\substack{1\leqq i,j\leqq 2\\0\leqq x\leqq 1}}|K_{ij}(x,\,x)|,$$

which is seen from $|K_{ij}(x,\,x)|\leqq 1$ $(0\leqq x\leqq 1,\,1\leqq i,\,j\leqq 2)$ by (VI.4) and (VI.5).

Proof of the part (III) of Lemma 6. We have only to prove

(VI.10) $\|du^{(i)}/dx\|_{C^0}, \|dv^{(i)}/dx\|_{C^0} \leqq M_{32}$ $(i=1, 2)$

and

(VI.11) $\|u^{(1)}-u^{(2)}\|_{C^1[0,1]}, \|v^{(1)}-v^{(2)}\|_{C^1[0,1]} \leqq M_{32} \max_{\substack{1\leqq i,j\leqq 2 \\ 0\leqq x\leqq 1}} |K_{ij}^{(1)}(x, x)-K_{ij}^{(2)}(x, x)|$.

If (VI.10) and (VI.11) are proved, then, from (2.45), we can derive (2.52), the conclusion.

Verification of (VI.10).   For $q^{(i)} \in \mathcal{A}_M$ $(i=1, 2)$, we get $\|K^{(i)}\|_{(C^0(\bar{D}))^4} \leqq M_4 M_5 \delta_0$ by (VI.4).   Therefore, in a manner similar to (VI.8), we obtain (VI.10).

Verification of (VI.11).   Since $(u^{(i)}, v^{(i)})$ is the solution to (2.49) and (2.50) $(i=1, 2)$, it follows that $(u^{(1)}-u^{(2)}, v^{(1)}-v^{(2)})$ satisfies

(VI.12) $\dfrac{d}{dx}\begin{pmatrix}(u^{(1)}-u^{(2)})(x)\\(v^{(1)}-v^{(2)})(x)\end{pmatrix} = A(x)\begin{pmatrix}(u^{(1)}-u^{(2)})(x)\\(v^{(1)}-v^{(2)})(x)\end{pmatrix} + d(x)$ $(0\leqq x\leqq 1)$

and $\begin{pmatrix}(u^{(1)}-u^{(2)})(0)\\(v^{(1)}-v^{(2)})(0)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.   Here we set

$$d(x) = \begin{pmatrix} K_{11}^{(1)}(x, x)-K_{11}^{(2)}(x, x)-(K_{22}^{(1)}(x, x)-K_{22}^{(2)}(x, x))\\ K_{11}^{(1)}(x, x)-K_{11}^{(2)}(x, x)-(K_{22}^{(1)}(x, x)-K_{22}^{(2)}(x, x)) \end{pmatrix}$$
$$+ \begin{pmatrix} K_{12}^{(1)}(x, x)-K_{12}^{(2)}(x, x)-(K_{21}^{(1)}(x, x)-K_{21}^{(2)}(x, x))\\ K_{21}^{(1)}(x, x)-K_{21}^{(2)}(x, x)-(K_{12}^{(1)}(x, x)-K_{12}^{(2)}(x, x)) \end{pmatrix}$$ $(0\leqq x\leqq 1)$.

By the fundamental matrix $\Phi(x)$, we have

$$\begin{pmatrix}(u^{(1)}-u^{(2)})(x)\\(v^{(1)}-v^{(2)})(x)\end{pmatrix} = \Phi(x)\int_0^x \Phi^{-1}(y)d(y)dy$$ $(0\leqq x\leqq 1)$,

so that we can easily reach

(VI.13) $\|u^{(1)}-u^{(2)}\|_{C^0}+\|v^{(1)}-v^{(2)}\|_{C^0} \leqq M_{32} \max_{\substack{1\leqq i,j\leqq 2 \\ 0\leqq x\leqq 1}} |K_{ij}^{(1)}(x, x)-K_{ij}^{(2)}(x, x)|$.

Using (VI.13) in (VI.12) and noting that

$$\|d\|_{C^0} \leqq 4 \max_{\substack{1\leqq i,j\leqq 2 \\ 0\leqq x\leqq 1}} |K_{ij}^{(1)}(x, x)-K_{ij}^{(2)}(x, x)| ,$$

we can obtain a similar estimate for $\|d(u^{(1)}-u^{(2)})/dx\|_{C^0}+\|d(v^{(1)}-v^{(2)})/dx\|_{C^0}$.

Thus (VI.11) is proved, and the proof of Lemma 6 is completed.

Appendix VII.   Proof of Lemma 8

Since $\mu_n \in \sigma(A_{Q,h,J})$ $(n \in \mathbf{Z})$ by (2.59) and $\overline{\sigma(A_{Q,h,J})} = \sigma(A^*_{Q,h,J})$ and $\psi^*(\cdot, \lambda)$ satisfies (2.77), we see that $\psi^*(\cdot, \overline{-\mu_n})$ is an eigenvector of $A^*_{Q,h,J}$ associated with the eigenvalue $\overline{\mu_n}$ $(n \in \mathbf{Z})$. That is,

(VII.1) $$A^*_{Q,h,J}\psi^*(\cdot, \overline{-\mu_n}) = \overline{\mu_n}\psi^*(\cdot, \overline{-\mu_n}).$$

Here we recall that

$$(A^*_{Q,h,J}u)(x) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{du(x)}{dx} + {}^tQ(x)u(x)$$

and

$$\mathscr{D}(A^*_{Q,h,J}) = \left\{ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \{H^1(0,1)\}^2 ; \; u_2(0) - hu_1(0) = u_2(1) - Ju_1(1) = 0 \right\}$$

(cf. (III.5)).   Thus the part (I) of this lemma is proved.

Next we will prove the part (II) of this lemma.   To this end, we define a transformation T in $\{L^2(0,1)\}^2$ by

(VII.2) $$(Tf)(x) = R(x)f(x) + \int_0^x K(x,y)f(y)dy \qquad (0 \le x \le 1).$$

Here $R(x)$ is given by (2.64), (2.65) and $K(x,y)$ is the solution to the problem (2.34), (2.35), (2.62), (2.63).

Since $R(x)^{-1}$ exists for $0 \le x \le 1$ and $K \in \{C^0(\bar{\Omega})\}^4$, by applying the routine argument for Volterra's integral equations of the second kind (for example, Yosida [14]), we can show that T^{-1} exists and is bounded in $\{L^2(0,1)\}^2$. In fact, setting

$$K^{(1)}(x,y) = -R(x)^{-1}K(x,y)$$

and

$$K^{(n)}(x,y) = -\int_y^x R(x)^{-1}K(x,z)K^{(n-1)}(z,y)dz \qquad (n \ge 2),$$

we see that the series $\sum_{n=1}^{\infty} K^{(n)}(x,y)$ converges absolutely and uniformly with respect to $(x,y) \in \bar{\Omega}$.   Let us put

$$\Gamma(x,y) = \sum_{n=1}^{\infty} K^{(n)}(x,y) \qquad ((x,y) \in \bar{\Omega}).$$

Then we have $\Gamma \in \{C^0(\bar{\Omega})\}^4$ and

$$(T^{-1}f)(x) = R(x)^{-1}f(x) + \int_0^x \Gamma(x,y)R(y)^{-1}f(y)dy \qquad (0 \leqq x \leqq 1).$$

On the other hand, since (2.66) is nothing but $\phi(\cdot,\mu_n) = T\phi(\cdot,\mu_n)$ $(n \in \mathbf{Z})$ and $\{\phi(\cdot,\mu_n)\}_{n \in \mathbf{Z}}$ is a Riesz basis in $\{L^2(0,1)\}^2$ by Lemma 3 (I), it follows from the result in Gohberg and Kreĭn [2, p.309] that also $\{\psi(\cdot,\mu_n)\}_{n \in \mathbf{Z}}$ forms a Riesz basis in $\{L^2(0,1)\}^2$.

Therefore we have only to prove the expression (2.79). The function $\phi(\cdot,\mu_n)$ satisfies (2.67) with $\lambda = \mu_n$ and $\mu_n \in \sigma(A_{Q,h,J})$ is a simple eigenvalue, so that we get

(VII.3) $A_{Q,h,J}\phi(\cdot,\mu_n) = \mu_n\phi(\cdot,\mu_n) \qquad (n \in \mathbf{Z}).$

Thus we can obtain

(VII.4) $(\phi(\cdot,\mu_m),\ \phi^*(\cdot,\overline{-\mu_n}))_{\{L^2(0,1)\}^2} = 0,\qquad \text{if } n \neq m.$

In fact, we have

$$\mu_m(\phi(\cdot,\mu_m),\ \phi^*(\cdot,\overline{-\mu_n})) = (A_{Q,h,J}\phi(\cdot,\mu_m),\ \phi^*(\cdot,\overline{-\mu_n})) \qquad \text{(by (VII.3))}$$

$$= (\phi(\cdot,\mu_m),\ A^*_{Q,h,J}\phi^*(\cdot,\overline{-\mu_n})) \qquad\qquad\qquad \text{(by (III.6))}$$

$$= (\phi(\cdot,\mu_m),\ \overline{\mu_n}\phi^*(\cdot,\overline{-\mu_n})) \qquad\qquad\qquad\quad \text{(by (VII.1))}$$

$$= \mu_n(\phi(\cdot,\mu_m),\ \phi^*(\cdot,\overline{-\mu_n})),$$

so that in virtue of $\mu_n \neq \mu_m$ $(m \neq n)$, we get (VIII.4).

Now we will complete the proof of (2.79). First we have

$$\alpha_n \equiv (\phi(\cdot,\mu_n),\ \phi^*(\cdot,\overline{-\mu_n})) \neq 0 \qquad (n \in \mathbf{Z}).$$

(In fact, contrarily assume that $(\phi(\cdot,\mu_{n_0}),\ \phi^*(\cdot,\overline{-\mu_{n_0}})) = 0$ for some $n_0 \in \mathbf{Z}$. Then, by (VII.4), we have $(\phi(\cdot,\mu_n),\ \phi^*(\cdot,\overline{-\mu_{n_0}})) = 0$ for each $n \in \mathbf{Z}$, which implies $\phi^*(\cdot,\overline{-\mu_{n_0}}) = 0$ by the completeness of $\{\phi(\cdot,\mu_n)\}_{n \in \mathbf{Z}}$. This contradicts $\phi^*(0,\overline{-\mu_{n_0}}) = \binom{1}{h}$ in (2.77). Thus we see $\alpha_n \neq 0$ $(n \in \mathbf{Z})$.)

Since $\{\phi(\cdot,\mu_n)\}_{n \in \mathbf{Z}}$ is a Riesz basis, for each $u \in \{L^2(0,1)\}^2$, we get $u = \sum_{n=-\infty}^{\infty} c_n\phi(\cdot,\mu_n)$ with appropriate $c_n \in \mathbf{C}$ $(n \in \mathbf{Z})$. Applying (VII.4), we obtain $c_n = (u,\ \phi^*(\cdot,\overline{-\mu_n}))\alpha_n^{-1}$ $(n \in \mathbf{Z})$, which imply (2.79), our conclusion. Thus we complete the proof of the part (II) of Lemma 8.

For the system $\{\phi(\cdot,\mu_n^*)\}_{n \in \mathbf{Z}}$, we can proceed similarly. Thus Lemma 8 is proved.

References

[1]   Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill (New York, 1955).

[2]   Gohberg, I. C. and Krein, M. G., *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs Vol. 18, Amer. Math. Soc. (Providence, R. I., 1969).

[3]   Hochstadt, H., On the well-posedness of the inverse Sturm-Liouville problem, *J. Differential Equations* **23**, 402–413 (1977).

[4]   Iwasaki, K., On the inverse Sturm-Liouville problem with spatial symmetry, *Funkcial. Ekvac.* **31**, 25–74 (1988).

[5]   Kato, T., *Perturbation Theory for Linear Operators*, second ed., Springer (Berlin-Heidelberg-New York, 1976).

[6]   Kolmogorov, A. N. and Fomin, S.V., *Introductory Real Analysis* (English translation), Prentice Hall, Inc. (Englewood Cliffs, New Jersey, 1970).

[7]   Mizutani, A., On the inverse Sturm Liouville problem, *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.* **31**, 319–350 (1984).

[8]   Nagumo, M., *Partial Differential Equations* I (in Japanese), Iwanami (Tokyo, 1957).

[9]   Petrovsky, I. G., *Lectures on Partial Differential Equations* (English translation), Interscience (New York, 1954).

[10]   Russell, D. L., Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory, *J. Math. Anal. Appl.* **40**, 336–368 (1972).

[11]   Russell, D. L., Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems, *J. Math. Anal. Appl.* **62**, 186–225 (1978).

[12]   Yamamoto, M., Inverse spectral problem for systems of ordinary differential equations of first order, I., *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.* **35**, 519–546 (1988).

[13]   Yamamoto, M., Inverse eigenvalue problem for a vibration of a string with viscous drag (preprint).

[14]   Yosida, K., *Lectures on Differential and Integral Equations*, Interscience (New York, 1960).

[15]   Yosida, K., *Functional Analysis*, third ed., Springer (Berlin-Göttingen-Heidelberg, 1971).