Interaction of Singularities of Solutions to Semilinear Wave Equation at the Boundary

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Abstract

We shall study the reflection of singularities at the boundary for the semilinear wave equation $\Box u=F(x, u)$ in 3 dimensional space-time. We shall show that if a H^s-solution ($s>3/2$) is conormal in the past to a union of a characteristic plane and its reflected one, then singularities of the solution are reflected as in the linear case, and that if a H^s-solution is conormal to two characteristic planes and their reflected ones in the past, then the solution can be singular not only on the four planes but also on the light cone from the point at which the four planes intersest.

§ 1. Introduction

In this paper, we study the reflection and the interaction of singularities of solutions to the following semilinear wave equation,

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)u(t, x, y)=F(x, y, u) \qquad \text{in } \mathcal{O}, \tag{1.1}$$

with the boundary condition,

$$u=g(t, y)\in C^\infty \qquad \text{on } \{(t, x, y)\in \boldsymbol{R}^3 \mid x=0\}\cap\partial\mathcal{O}. \tag{1.2}$$

Here F and g are real valued C^∞ functions and $\mathcal{O}$ is an open neighborhood of the origin in $\boldsymbol{R}^3_+=\{(t, x, y)\in\boldsymbol{R}^3 \mid x>0\}$. We shall show that if a single singularity hits the boundary, it is reflected according to the usual law; but if two singularities hit it, a new singularity may imerge as a result of nonlinear interaction.

For stating our results precisely, we introduce some notation and some function spaces. For a domain Ω in $\boldsymbol{R}^n$ and $s\in\boldsymbol{R}$, $H^s(\Omega)$ is the Sobolev space of order s, and $H^s_{loc}(\Omega)=\{u\in\mathcal{D}(\Omega) \mid \varphi u\in H^s(\boldsymbol{R}^n)$ for any $\varphi\in\mathcal{D}(\Omega)\}$. When Σ is a C^∞ submanifold in Ω or a union of two hypersurfaces in Ω which intersect transversally, we define the space of distributions conormal to Σ as follows (cf. Hörmander [8]).

DEFINITION 1.1. *For $s\in\boldsymbol{R}$, we say*

$$u\in H^s(\Sigma, \infty) \text{ in } \Omega, \textit{ if } Z_1\circ Z_2\circ\cdots\circ Z_l u\in H^s_{loc}(\Omega)$$

for any choice of C^∞ vector fields $Z_1, Z_2, \cdots, Z_l$ which are tangent to Σ ($l=0, 1, 2, \cdots$). u is called conormal to Σ if $u\in\bigcup_{s\in\boldsymbol{R}} H^s(\Sigma, \infty)$.

We denote $\mathcal{O}^-=\mathcal{O}\cap\{t<0\}$ and $\mathcal{O}^+=\mathcal{O}\cap\{t>0\}\cap\{(t, x, y)\in\boldsymbol{R}^3|(C^-_{(t,x,y)}\cap\{t=0\}\cap\{x>0\})\subset(\mathcal{O}\cap\{t\leq 0\})\}$. Here and hereafter $\{t<0\}=\{(t, x, y)\in\boldsymbol{R}^3|t<0\}$ and so on. $C^-_{(t_0,x_0,y_0)}=\{(t, x, y)\in\boldsymbol{R}^3|(t-t_0)^2>(x-x_0)^2+(y-y_0)^2$ and $t<t_0\}$ is the backward light cone from the point (t_0, x_0, y_0).

THEOREM 1. *Let $\mathcal{O}$, $\mathcal{O}^-$ and $\mathcal{O}^+$ be as above, and $s>3/2$. Let Σ_1 be the plane $t=x_0x+y_0y$ with $x_0\leq 0$, $y_0\geq 0$ and $x_0^2+y_0^2=1$, and Σ_2 be its reflected plane $t=-x_0x+y_0y$. Suppose $u\in H^s(\mathcal{O})$ satisfies* (1.1) *and* (1.2), *and $u\in H^s(\Sigma_1\cup\Sigma_2, \infty)$ in $\mathcal{O}^-$. Then*

$$u\in C^\infty(\mathcal{O}^+\backslash\Sigma_1\cup\Sigma_2)\,. \tag{1.3}$$

THEOREM 2. *Let s, $\mathcal{O}$, $\mathcal{O}^-$, $\mathcal{O}^+$, Σ_1 and Σ_2 be as above. Let Σ_3 be the plane $t=x_1x+y_1y$ with $x_1, y_1\leq 0$ and $x_1^2+y_1^2=1$, and Σ_4 be its reflected plane $t=-x_1x+y_1y$. Suppose $u\in H^s(\mathcal{O})$ satisfies* (1.1) *and* (1.2), *and $u\in H^s(\Sigma_1\cup\Sigma_2\cup\Sigma_3\cup\Sigma_4, \infty)$ in $\mathcal{O}^-$. Then*

$$u\in C^\infty(\mathcal{O}^+\backslash\Sigma_1\cup\Sigma_2\cup\Sigma_3\cup\Sigma_4\cup\{t^2=x^2+y^2\})\,. \tag{1.4}$$

If the inhomogeneous term F of (1.1) is linear in u, it is well known that singulalities of u propagate along the null bicharacteristics for the d'Alembertian $\square=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}$ in the interior of $\mathcal{O}$ (Hörmander [7]). When $F(u)$ is nonlinear, most works are concerned with the solutions which belong to H^s ($s>3/2$) and H^r-singularities ($r>s$) were studied. This is because $u\in H^s$ implies $F(u)\in H^s$, which makes the problem much easier to handle. If F is C^∞ and $s<r\leq 2s-3/2+1$, H^r-singularities propagate in the interior of $\mathcal{O}$ as in the linear case (Rauch [11], Bony [4]). On the other hand, when $r>2s-3/2+1$, H^r-singularities do not propagate as in the linear case and additional nonlinear H^r-singularities can appear (Beals [3]). (For more general equations, see Bony [4]). For deeper understanding of nonlinear interaction, the notion of conormality is useful. If a solution to (1.1) is conormal in the past with respect to a smooth characteristic hypersurface, it remains conormal to it in the future (Bony [5]). The same is true for a pair of smooth characteristic hypersurfaces intersecting transversally in the future (Bony [5]). But when three progressing waves interact in $\boldsymbol{R}^3$, even if a solution u is conormal in the past, nonlinear interaction can produce a new singularity (Bony [6], Melrose and Ritter [9]). Indeed, Rauch and Reed [12] gave an example that demonstrated the appearance of a single nonlinear singularity on the surface of the light cone over the point of triple interaction.

The reflection of singularities at the boundary is also intensively studied. If F is linear in u, singularities of u are reflected at the boundary according to the law of geometrical optics (see Nirenberg [10]). If F is C^∞ and $s<r\leq 2s-3/2+1$, H^r-singularities are reflected as in the linear case (Ali Alabidi [1]). In this paper, when $r\geq 2s-3/2+1$, we study how H^r-singularities are reflected and interact each other at the boundary, under the condition that u is conormal in the past.

In § 2, we prove Theorem 1 and 2, and in § 3, we give an example that demonstrates the appearance of a nonlinear singularity at the boundary.

After completing this paper, the author learned that Sasaki [13] has given an example with a nonlinear singularity emerging at the boundary for the equation $(\partial_t^2-\partial_x^2-\partial_y^2)u=F(\partial_x u)$.

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§ 2. Proofs of Theorem 1 and 2

For proving Theorem 1 and 2, the vector field

$$M=t\partial_t+x\partial_x+y\partial_y$$

plays an important role. M is the generator of the dilation:

$$(e^{\theta M}f)(t, x, y)=f(e^\theta t, e^\theta x, e^\theta y).$$

We should remark that M was used by Beals in [2] to give another proof of the resuls of Bony [6], and Melrose and Ritter [9].

Lemma 2.1. *Let $\mathcal{O}$, $\mathcal{O}^-$ and $\mathcal{O}^+$ be as in* 1. *Let $s>3/2$ and $u\in H^s(\mathcal{O})$ satisfy* (1.1) *and* (1.2). *Suppose that $M^j u\in H^s_{loc}(\mathcal{O}^-)$ for all $j\in \boldsymbol{N}$. Then*

$$u\in C^\infty(\mathcal{O}^+\cap\{x^2+y^2<t^2\}). \tag{2.1}$$

Proof. First we show that u satisfies

$$M^j u\in H^s_{loc}(\mathcal{O}^+) \quad \text{for all } j\in \boldsymbol{N}. \tag{2.2}$$

Indeed, applying M to (1.1) and using $(\Box, M)=\Box M-M\Box=2\Box$, we have

$$\Box(Mu)=M\{F(x, y, u)\}+2F(x, y, u)\in H^{s-1}_{loc}(\mathcal{O}). \tag{2.3}$$

Since $u\in H^s_{loc}(\mathcal{O})(s>3/2)$ and $u|_{x=0}=g[t, y]\in C^\infty$,

$$Mu|_{x=0}=t\partial_t g+y\partial_y g\in C^\infty. \tag{2.4}$$

Hence the energy estimate for the mixed problem (cf. Hörmander [8] Chap.

XXIV 24.1) yields

$$Mu = H^s_{loc}(\mathcal{O}^+). \tag{2.5}$$

Repeating this argument, we obtain (2.2).

Now take $Q_0=(t_0, x_0, y_0)\in\mathcal{O}^+$ with $x_0^2+y_0^2<t_0^2$ arbitrarily and $(Q_0, P_0)=(t_0, x_0, y_0, \tau_0, \xi_0, \eta_0)\in T^*(\mathcal{O})\backslash 0$. If $t_0\tau_0+x_0\xi_0+y_0\eta_0\neq 0$, M is microlocal elliptic at (Q_0, P_0). Hence an application of the microlocal regularity theorem (cf. Taylor [14] Chap. VI Prop. 1.10) to (2.2) implies

$$u\in H^{s+j} \quad \text{at } (Q_0, P_0) \text{ for any } j=0, 1, 2, \cdots. \tag{2.6}$$

If $t_0\tau_0+x_0\xi_0+y_0\eta_0=0$, we have

$$\tau_0^2-\xi_0^2-\eta_0^2\neq 0 \quad \text{for } (\tau_0, \xi_0, \eta_0)\neq 0$$

and this implies

$$(Q_0, P_0)\notin \mathrm{Char}\,\square. \tag{2.7}$$

Hence again by the microlocal regularity theorem, we have

$$u\in H^{s+2} \quad \text{at } (Q_0, P_0). \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$u\in H^{s+2} \quad \text{at } Q_0. \tag{2.9}$$

Repeating this argument, we have the desired result

$$u\in C^\infty \quad \text{at } Q_0. \ \square \tag{2.10}$$

Proof of Theorem 1. We divide $\mathcal{O}^+\backslash(\Sigma_1\cup\Sigma_2)$ into the three regions $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$:

$$\begin{aligned}\mathcal{O}_1&=\{t<x_0x+y_0y\}\cap\mathcal{O}^+,\\ \mathcal{O}_2&=\{t>x_0x+y_0y\}\cap\{t<-x_0x+y_0y\}\cap\mathcal{O}^+,\\ \mathcal{O}_3&=\{t>-x_0x+y_0y\}\cap\mathcal{O}^+.\end{aligned}$$

We first consider in $\mathcal{O}_1$. By the property of finite propagation speed (cf. Taylor [14] Chap. IV §4), the values of u on $\mathcal{O}_1$ are determined by those of u on a set which does not intersect Σ_1 and Σ_2. Then by Theorem 3 of Ali Alabidi [1], we know

$$u\in C^\infty \quad \text{in } \mathcal{O}_1. \tag{2.11}$$

Next we consider in $\mathcal{O}_2$. By the commutator argument of Bony [5], it is easy to see that u is conormal to Σ_1 in $\mathcal{O}_1\cup\mathcal{O}_2$. This implies

$$u\in C^\infty \quad \text{in } \mathcal{O}_2. \tag{2.12}$$

Finally let us show

$$u\in C^\infty \quad \text{in } \mathcal{O}_3. \tag{2.13}$$

As the vector field

$$M_\varepsilon=(t-y_0\varepsilon)\partial_t+x\partial_x+(y-\varepsilon)\partial_y$$

is tangent to $\Sigma_1\cup\Sigma_2$ and $u\in H^s(\Sigma_1\cup\Sigma_2, \infty)$ in $\mathcal{O}^-$, we have

$$M_\varepsilon^j u\in H^s(\mathcal{O}^-) \quad \text{for all } j\in \boldsymbol{N}. \tag{2.14}$$

Then by an argument similar to the one used in the proof of Lemma 2.1, we obtain

$$u\in C^\infty(\{x^2+(y-\varepsilon)^2<(t-y_0\varepsilon)^2\}\cap\{t>y_0\varepsilon\}\cap\mathcal{O}^+). \tag{2.15}$$

Since

$$\bigcup(\{x^2+(y-\varepsilon)^2<(t-y_0\varepsilon)^2\}\cap\{t>y_0\varepsilon\}\cap\mathcal{O}^+)\supset\mathcal{O}_3$$

where the union is taken over ε with $(y_0\varepsilon, 0, \varepsilon)\in(\partial\mathcal{O}\cap\{x=0\})$, we conclude

$$u\in C^\infty \quad \text{in } \mathcal{O}_3. \quad \square \tag{2.16}$$

Proof of Theorem 2. As in the proof of Theorem 1, we divide $\mathcal{O}^+\backslash(\Sigma_1\cup\Sigma_2\cup\Sigma_2\cup\Sigma_3\cup\Sigma_4\cup\{x^2+y^2=t^2\})$ into the four regions $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$:

$$\mathcal{O}_1=(\{t<x_0x+y_0y\}\cup\{t<x_1x+y_1y\})\cap\mathcal{O}^+,$$
$$\mathcal{O}_2=(\{t<-x_0x+y_0y\}\cup\{t<-x_1x+y_1y\})\cap\mathcal{O}_1^c\cap\mathcal{O}^+,$$
$$\mathcal{O}_3=\{x^2+y^2>t^2\}\cap\{t>-x_0x+y_0y\}\cap\{t>-x_1x+y_1y\}\cap\mathcal{O}^+,$$
$$\mathcal{O}_4=\{x^2+y^2<t^2\}\cap\{t>-x_0x+y_0y\}\cap\{t>-x_1x+y_1y\}\cap\mathcal{O}^+.$$

The smoothness of u in $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ may be as in the first, second and third step of the proof of Theorem 1: The finiteness of propagation speed implies

$$u\in C^\infty \quad \text{in } \mathcal{O}_1. \tag{2.17}$$

The commutator argument of Bony [5] yields

$$u\in C^\infty \quad \text{in } \mathcal{O}_2. \tag{2.18}$$

If $(t, x, y)\in\mathcal{O}_3$, the backward light cone from (t, x, y) does not contain the origin. Using Lemma 2.1 for the vector field M_ε in the neighborhood of the cone and taking the union of $(\{x^2+(y-\varepsilon)^2<(t-y_0\varepsilon)^2\}\cap\{t>y_0\varepsilon\}\cap\mathcal{O}^+)$ over ε with $(y_0\varepsilon, 0, \varepsilon)\in(\partial\mathcal{O}\cap\{x=0\})$, we have

$$u\in C^\infty \quad \text{in } \mathcal{O}_3. \tag{2.19}$$

Since $u\in H^s(\Sigma_1\cup\Sigma_2\cup\Sigma_3\cup\Sigma_4, \infty)$ in $\mathcal{O}^-$ and M is tangent to Σ_j $(j=1, 2, 3, 4)$, we see

(2.20) $$M^j u \in H^s(\mathcal{O}^-) \quad \text{for all } j \in \boldsymbol{N}.$$

Hence by Lemma 2.1, we obtain

(2.21) $$u \in C^\infty \quad \text{in } \mathcal{O}_4.$$

Combining (2.17), (2.18), (2.19) and (2.21), we complete the proof of Theorem 2. □

§ 3. Appearance of nonlinear interaction at the boundary

In this section, we construct an example which demonstrates the appearance of the nonlinear singularity at the boundary. This is analogous to Rauch and Reed [11] who dealt with the nonlinear interaction in the interior. Let $\omega_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\omega_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\omega_3 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\omega_4 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

THEOREM 3. *Let ω_i $(i=1, 2, 3, 4)$ be as above and $F(s)$ be a C^∞ function such that*

$$F(s) \text{ is an odd function,}$$
$$-1 \leq F(s) \leq 1,$$
$$F(s) = 0 \qquad \text{for } 0 < s < \frac{4}{3} \text{ and } \frac{8}{3} < s, \quad \text{and}$$
$$F(s) = 1 \qquad \text{for } \frac{5}{3} < s < \frac{7}{3}.$$

Then there exists a solution u to the equation

(3.1) $$\Box u = F(u) \qquad -\delta < t < \delta \text{ and } x \in \boldsymbol{R}^2,$$

(3.2) $$u|_{x_1=0} = 0$$

such that

(3.3) $$\text{sing supp}\, u \cap \{-\delta < t < 0\} = \bigcup_{i=1}^{4} \{t - \omega_i \cdot x = 0\},$$

(3.4) $$\text{sing supp}\, u \cap \{0 < t < \delta\} = \bigcup_{i=1}^{4} \{t - \omega_i \cdot x = 0\} \cup \{t^2 = x^2 + y^2\}.$$

This theorem gives an example at which we aim. So we devote the rest of this section to the proof of this theorem.

Proof. Let h be the Heaviside function: $h(t) = 0$ for $t \leq 0$ and $h(t) = 1$ for $t > 0$. We set

(3.5) $$\chi_i = \prod_{\substack{i \neq j \\ 1 \leq j \leq 4}} h(t-\omega_j \cdot x) \qquad (i=1,2,3,4) \text{ and}$$

(3.6) $$\chi_5 = h(t-\omega_3 \cdot x)h(\omega_2 \cdot x - t) - h(t-\omega_4 \cdot x)h(\omega_1 \cdot x - t).$$

We need the following two lemmas.

LEMMA 3.1 *Let V_1 be the solution for the following initial value problem:*

(3.7) $$\Box V_1 = \chi_1 - \chi_2 - \chi_3 + \chi_4,$$

(3.8) $$V_1 \equiv 0 \qquad \textit{for } t<0.$$

Then

(3.9) $$\text{sing supp}\, V_1 \supset \{t = x_1^2 + x_2^2,\ t \geq 0\}.$$

LEMMA 3.2. *Let $a(t)$ be a C^∞ function such that $0 \leq a(t) \leq 1$, $a(t)=1$ for $t \geq -\delta$ and $a(t)=0$ for $t \leq -2\delta$, and V_2 be the solution for the initial value problem:*

(3.10) $$\Box V_2 = a(t)\chi_5,$$

(3.11) $$V_2 \equiv 0 \qquad \textit{for } t < -2\delta.$$

Then

(3.12) $$\text{sing supp}\, V_2 \subset \bigcup_{i=1}^{4} \{t - \omega_i \cdot x = 0\}$$

and if δ is sufficiently small,

(3.13) $$|V_1 + V_2| < \frac{1}{3} \qquad \textit{for } t \in (-\delta, \delta) \textit{ and } x \in \boldsymbol{R}^2.$$

Postponing the proof of the Lemma 3.1 and 3.2 to the end of this section, we continue the proof of Theorem 3. Let $V_3 = -h(t-\omega_1 \cdot x) + h(t-\omega_2 \cdot x) + 2h(t-\omega_3 \cdot x) - 2h(t-\omega_4 \cdot x)$ and $u = V_1 + V_2 + V_3$. As $\text{sing supp}\, V_3 \subset \bigcup_{i=1}^{4} \{t - \omega_i \cdot x = 0\}$, Lemma 3.1 and Lemma 3.2 imply (3.3) and (3.4). It is obvious that $\Box u = \chi$, where $\chi = \chi_1 - \chi_2 - \chi_3 + \chi_4 + \chi_5$. By the definition of u and χ, it is clear that $F(u) = \chi$. Hence u satisfies (3.1). Obviously we see $u|_{x_1=0} = 0$ from the facts $\chi|_{x_1=0} = 0$ and $V_3|_{x_1=0} = 0$, (3.7) and (3.10). □

Proof of Lemma 3.1. It suffices to show that V_1 is singular on $\{t^2 = x^2 + y^2$ and $t = 1\}$, because V_1 is homogeneous of degree 2 in (t, x). First we show that V_1 is singular at $(1, 1, 0)$. We set

$W_1 = \{t \geq \omega_2 \cdot x \text{ and } t \geq \omega_4 \cdot x\}$, $W_2 = \{t \geq \omega_1 \cdot x \text{ and } t \geq \omega_3 \cdot x\}$, $W_3 = \{t \geq \omega_1 \cdot x \text{ and } t \geq \omega_4 \cdot x\}$, $W_4 = \{t \geq \omega_1 \cdot x, t \geq \omega_2 \cdot x, t \leq \omega_3 \cdot x \text{ and } t \geq \omega_4 \cdot\}$.

We denote by χ_A the characteristic function of A. And for $\varepsilon > 0$, p and q are the points $(1, 1-\varepsilon, 0)$ and $(1, 1+\varepsilon, 0)$ respectively. We write the fundamental solution for the d'Alembertian $\square$ as

$$E(t, x)=\frac{1}{2\pi}(t^2-|x|^2)^{-1/2} \qquad \text{for } t^2>x^2 .$$

We set $z(t, x)=E*(2\chi_{W_3})-E*\chi_{W_1}-E*\chi_{W_2}+V_1$. By Hörmander's theorem on the propagation of singularities [7], we have

$$\text{sing supp } E*\chi_{W_i}\subset\partial W_i \qquad (i=1, 2, 3) .$$

In particular, $E*\chi_{W_i}$ is not singular at $(1, 1, 0)$. Therefore it suffices to show that z is singular at $(1, 1, 0)$. By the definition of z, we have

$$z(q)=0 , \tag{3.14}$$

$$z(p)=\frac{1}{\pi}\int_{P_\varepsilon}\frac{dt'dx'}{\sqrt{(t-t')^2-|x-x'|^2}} , \tag{3.15}$$

where $P_\varepsilon=W_4\cup C_p^-$ and C_p^- is the backward light cone from p. Following the calculation of Rauch and Reed [12], it is easy to see

$$z(p)\geq \text{Const. } \varepsilon^{5/2} . \tag{3.16}$$

(3.14) and (3.16) imply that z is not C^3 at $(1, 1, 0)$. Hence V_1 is not C^3 at $(1, 1, 0)$. Similarly we can show that V_1 is not C^3 at $(1, a, b)$ with $a^2+b^2=1$ and $a\neq 0$. Since sing supp V_1 is closed, we conclude (3.9).

Proof of Lemma 3.2. (3.12) follows by using again Hörmander's theorem [7]. To prove the latter half of the statement of Lemma 3.2, we represent $V_1 + V_2$ as

$$V_1+V_3=\frac{1}{2\pi}\int_{C^-_{(t,x)}}\frac{\chi(t, x)dt'dx'}{\sqrt{(t-t')^2-|x-x'|^2}} \tag{3.17}$$

using the fundamental solution. Here $C^-_{(t,x)}$ is the backward light cone from (t, x) and $\chi=\chi_1-\chi_2-\chi_3+\chi_4+\chi_5$. Estimating the right hand side of (3.17), we easily obtain (3.13). $\square$

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