The Topology of the Configuration of Projective Subspaces in a Projective Space I

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Abstract

We assume that there is given a finite family of projective subspaces in certain projective space. Our aim is to prove that the simple homotopy type of the union of all the subspaces in question is completely determined by the complex of the nerves resulting from the family, equipped with the filtration obtained by assigning to each simplex the Krull dimension of the corresponding intersection. This fact enables us to compute the homology groups of the union and its complement in principle.

§1. Introduction

Let $\boldsymbol{F}$ be one of the following fields: $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$. Let $\mathcal{V}=\{V_i|i\in I\}$ be a finite family of euclidean subspaces in $\boldsymbol{F}^n$ passing through 0 endowed with the usual metric. We know that the collection of all the subsets of I forms a simplicial complex $\Delta(I)$ called the *standard simplex* with the set of vertices I. If ρ is a simplex of $\Delta(I)$, its *geometric realization* $|\rho|$ means the convex hull of the set I embedded in $\boldsymbol{R}^I$ and the *geometric realization* $|K|$ of a subcomplex K of $\Delta(I)$ stands for the polyhedron being the union $\cup\{|\rho||\rho\in K\}$. Let ρ be a simplex of $\Delta(I)$. Then we have an affine subspace V_ρ of $\boldsymbol{F}^n$ defined by $V_\rho=\cap\{V_i|i\in\rho\}$. We write $d(\rho)$ for $\dim_{\boldsymbol{F}} V_\rho$. We adopt the convention here that the dimension of the vacuous set is $-\infty$. For each i in I, we define

$$V_i^*=V_i\backslash\{0\},\ SV_i=\{x\in V_i||x|=1\},\ PV_i=\{V_i^*/F^*\}$$

which are naturally the subspaces of $(\boldsymbol{F}^n)^*$, $S\boldsymbol{F}^n$, $P\boldsymbol{F}^n$ respctively. Further we put

$$\mathcal{V}^*=\{V_i^*|i\in I\},\qquad S\mathcal{V}=\{SV_i|i\in I\},\qquad P\mathcal{V}=\{PV_i|i\in I\}.$$

For each of the families above mentioned, we consider the union of the spaces belonging to it, that is:

$$\tilde{Y}(\mathcal{V}^*)=\cup\{V_i^*\subset(\boldsymbol{F}^n)^*|i\in I\},\quad \tilde{Y}(S\mathcal{V})=\cup\{SV_i\subset S\boldsymbol{F}^n|i\in I\},$$
$$\tilde{Y}(P\mathcal{V})=\cup\{PV_i\subset P\boldsymbol{F}^n|i\in I\}.$$

Now we define the *half spaces* to be the spaces given as follows:

$$E(\mathcal{V}^*)=\cup\{|\rho|\times(\boldsymbol{F}^{d(\rho)})^*\subset|\Delta(I)|\times(\boldsymbol{F}^n)^*\,|\,\rho\in\Delta(I)\}$$
$$E(S\mathcal{V})=\cup\{|\rho|\times S\boldsymbol{F}^{d(\rho)}\subset|\Delta(I)|\times S\boldsymbol{F}^n\,|\,\rho\in\Delta(I)\}$$
$$E(P\mathcal{V})=\cup\{|\rho|\times P\boldsymbol{F}^{d(\rho)}\subset|\Delta(I)|\times P\boldsymbol{F}^n\,|\,\rho\in\Delta(I)\}$$

where $\boldsymbol{F}^l$ is identified with the subspace of $\boldsymbol{F}^m$ consisting of vectors whose i-th component is 0 for each integer i satisfying $l+1\leqq i\leqq m$ whenever $l\leqq m$.

The main result of this paper is the following.

THEOREM 1.1. $\tilde{Y}(\mathcal{V}^*)\underset{s}{\simeq}E(\mathcal{V}^*)$, $\tilde{Y}(S\mathcal{V})\underset{s}{\simeq}E(S\mathcal{V})$, $\tilde{Y}(P\mathcal{V})\underset{s}{\simeq}E(P\mathcal{V})$.

The above result renders us a method in computing the homology groups of the space $P\boldsymbol{F}^n\backslash\tilde{Y}(P\mathcal{V})$. The simplest case is when the identity $\dim_{\boldsymbol{F}}V_\rho=n-\dim\rho-1$ holds for every ρ in $\Delta(I)$. Then $\mathcal{V}$ is called in general position. Now the following Corollary is an easy exercise except the case where $\boldsymbol{F}=\boldsymbol{C}$, $n\leqq 2$.

COROLLARY 1.2. *Let $\mathcal{V}$ be in general position. Then we have a homotopy equivalence*

$$P\boldsymbol{F}^n\backslash\tilde{Y}(P\mathcal{V})\simeq(\overset{m}{\times}S^{d-1})^{(d-1)n}$$

where $m=\#I-1$, $d=\dim_{\boldsymbol{R}}\boldsymbol{F}$ and $(\overset{m}{\times}S^{d-1})^p$ denotes the p-dimensional skeleton of the most economical cellular decomposition of $(\overset{m}{\times}S^{d-1})$.

In the case $\boldsymbol{F}=\boldsymbol{R}$, this follows from the direct computation. In the case $\boldsymbol{F}=\boldsymbol{C}$, the existence of the above homotopy equivalence is due to Akio Hattori [3].

The content of this series of papers was already announced in [4].

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§ 2. Filtered simplicial complex

A *filtration* F of a topological space B is a family of closed subspaces ${}_lB$, one for each l in $\boldsymbol{Z}$, with

$$\cdots\subset{}_{l-1}B\subset{}_lB\subset{}_{l+1}B\subset\cdots.$$

If F, F' are filtrations of B, B' respectively, a map $f\colon\ B\longrightarrow B'$ of filtered spaces is a continuous map with the property $f({}_lB)\subset{}_lB'$.

A filtration F of B defines a function $F\colon\ B\longrightarrow \boldsymbol{Z}$ given by the formula $F(b)=\inf\{l|b\in{}_lB\}$ for which $F^{-1}(]-\infty,\ l])$ is a closed subspace of B. Conversely a function $F\colon\ B\longrightarrow \boldsymbol{Z}$ on a topological space B with $F^{-1}(]-\infty,\ l])$ being a closed subspace for each l, defines a filtration of B.

Let B' be a closed subspace of B and let $i\colon\ B'\longrightarrow B$ be an injection. If $F\colon\ B\longrightarrow \boldsymbol{Z}$ is a filtration of B, then $F\circ i\colon\ B'\longrightarrow \boldsymbol{Z}$ defines a filtration of B' called the restriction of F, explicitly $F'=F\circ i$ is defined by ${}_lB'=B'\cap{}_lB$.

Let B be a topological space and let I be a unit interval $[0,\ 1]$. Let $p\colon\ B\times I\longrightarrow B$ be a projection to the first factor B, then $F\circ p\colon\ B\times I\longrightarrow \boldsymbol{Z}$ is by definition a filtration of $B\times I$ induced from F, in other words $F=F\circ p$ is defined by ${}_l(B\times I)={}_lB\times I$.

Let F, F' be filtrations of topological spaces B, B' respectively. Let $f, g\colon B\longrightarrow B'$ be two maps of filtered spaces. A homotopy between f and g is a map $h\colon B\times I\longrightarrow B$ of filtered spaces satisfying the property $f=h\circ i_0$, $g=h\circ i_1$ where $i_t\colon B\longrightarrow B\times I$ denotes the inclution defined by $i_t(b)=(b,\ t)$.

Analogously a *filtration* F of a simplicial complex K is defined to be a family of subcomplexes ${}_lK$, one for each l in Z with

$$\cdots\subset{}_{l-1}K\subset{}_lK\subset{}_{l+1}\subset\cdots.$$

If F, F' are filtrations of complexes K, K' respectively, a map $f\colon\ K\longrightarrow K'$ of filtered complexes is a simplicial map with the property $f({}_lK)\subset{}_lK'$.

Also in this case, there is a one-to-one correspondence between a filtration F of a simplicial complex K and a function $F\colon\ K\longrightarrow \boldsymbol{Z}$ satisfying $F(\rho)\leqq F(\rho')$ whenever $\rho\prec\rho'$.

The restriction of the filtration on a subcomplex is likewise defined.

Clearly a filtration F of a simplicial complex K defines a filtration $|F|$ of the corresponding polyhedron $|K|$ by putting $|F|\ {}_l(|K|)=|{}_lK|$. Let F, F' be filtrations of simplicial complexes K, K respectively. If $f\colon\ K\longrightarrow K'$ is a map of filtered complexes, then the induced map $|f|\colon\ |K|\longrightarrow|K'|$ gives rise to a map of filtered spaces.

Example. Let I be a finite set of indices and let $\mathcal{V}=\{V_i|i\in I\}$ be a family of affine subspaces of $\boldsymbol{F}^n$ passing through 0. Let $\Delta(I)$ be the standard simplex with the set of vertices I. Then for each simplex ρ in $\Delta(I)$, there is determined a subspace $V_\rho=\cap\{V_i|i\in\rho\}$. We define a function $F(\mathcal{V})\colon\ \Delta(I)\longrightarrow \boldsymbol{Z}$ by $F(\mathcal{V})(\rho)=-\dim V_\rho$. As essily seen, $F(\mathcal{V})$ defines a filtration of $\Delta(I)$.

In fact, the function $F=F(\mathcal{V})$ has the folloing properties:

0) $F(\emptyset)=-\infty$, $-n\leqq F(\{i\})\leqq 0$ for any $i\in I$;

i) $\rho\prec\rho'$ implies $F(\rho)\leqq F(\rho')$;

ii) $F(\bigcup_{1\leqq i\leqq l}\rho_i)\leqq\sum_{1\leqq i\leqq l}F(\rho_i)-(l-1)F(\bigcap_{1\leqq i\leqq l}\rho_i)$.

However we are ignorant of when a function $F\colon\ \Delta(I)\longrightarrow \boldsymbol{Z}$ having the properties listed above can be realized by a family $\mathcal{V}$ so as to satisfy $F=F(\mathcal{V})$.

A simplicial complex is called ordered if for each simplex, a simple order of its vertices is given so that the order of each simplex agrees with the order of its faces.

A *filtration* F of an ordered simplicial complex K is a function $F: K \longrightarrow \boldsymbol{Z}$ satisfying the conditions:

i) $F|K^0: K^0 \longrightarrow \boldsymbol{Z}$ is nondecreasing with respect to the order of K^0, that is $\sigma \leqq \sigma'$ implies $F(\sigma) \leqq F(\sigma')$;

ii) If σ is a simplex of K with the set of vertices $\sigma_0, \cdots, \sigma_q$ numbered so that $\sigma_0 < \cdots < \sigma_q$, then we have $F(\sigma) = F(\sigma_q) = \max\{F(\sigma_i) | 0 \leqq i \leqq q\}$.

A map $f: K \longrightarrow K'$ of filtered ordered simplicial complexes is assumed to be monotone non-decreasing in the sense that $f(\sigma) \leqq f(\sigma')$ whenever $\sigma \leqq \sigma'$.

Let F be a filtration of a simplicial complex K. $'K$ denotes the derived complex of K: that is the simplicial complex whose set of vertices are the set of barycenters $\hat{\rho}$ taken once for each simplex ρ in K and for each sequence such that $\rho_0 \prec \cdots \prec \rho_q$, the sequence of the corresponding barycenters $\hat{\rho}_0, \cdots, \hat{\rho}_q$ is the set of vertices of a simplex σ. We write $\hat{\rho} \leqq \hat{\rho}'$ if $\rho \prec \rho'$. Then $'K$ becomes an ordered simplicial complex with respect to the relation $<$. Now the *derived filtration* $'F$ of $'K$ is defined as follows: If σ is a simplex with vertices $\hat{\rho}_0, \cdots, \hat{\rho}_q$, then $'F$ maps σ to $F(\rho_q) = \max\{F(\rho_i) | 0 \leqq i \leqq q\}$. It is easily checked that $'F$ gives rise to a filtration of the ordered simplicial complex $'K$.

Here we should remark that $|F|$ and $|'F|$ satisfy the identity $|'F| \circ i = |F|$ with the natural identification $i: |'K| \longrightarrow |K|$.

EXAMPLE Let $\mathcal{V} = \{V_i | i \in I\}$ be as before. Then the derived filtration $'F(\mathcal{V})$ of the derived complex is defined as follows: If σ is a simplex in $'\Delta(I)$ with vertices $\hat{\rho}_0, \cdots, \hat{\rho}_q$ such that $\rho_0 \prec \cdots \prec \rho_q$, we have $'F(\mathcal{V})(\sigma) = F(\mathcal{V})(\rho_q) = -\dim_F V_{\rho_q}$.

Here we should remark that for any two vertices $\hat{\rho}, \hat{\rho}'$ contained in some simplex in $'\Delta(I)$, the condition $'F(\mathcal{V})(\hat{\rho}) = {}'F(\mathcal{V})(\hat{\rho}')$ implies $V_\rho = V_{\rho'}$.

Throughout the rest of this paragraph, we are only concerned with an ordered simplicial complex K endowed with a filtration.

Now we define a category K as follows: Take the *objects* to be a set of all simplices in K. An *elementary morphism* $\varphi: \sigma \Longrightarrow \sigma'$ is by difinition a filtration preserving simplicial map $\varphi: \mathrm{Cl}\,\sigma \longrightarrow \mathrm{Cl}\,\sigma'$ of ordered simplicial complexes belenging to some simplex in commen: more precisely a simplicial map $\varphi: \mathrm{Cl}\,\sigma \longrightarrow \mathrm{Cl}\,\sigma'$ is written as $\varphi: \sigma \Longrightarrow \sigma'$ if both of σ and σ' lie in some simplex σ'' in K and for each pair of vertices σ_i, σ_j of σ satisfying $\sigma_i \leqq \sigma_j$ we have $\varphi(\sigma_i) \leqq \varphi(\sigma_j)$ and for every vertex σ_i of σ we have $F(\varphi(\sigma_i)) = F(\sigma_i)$. Further a *morphism* $\varphi: \sigma \longrightarrow \sigma'$ is a composite of elementary morphisms φ_i expressed in the form $\varphi = \varphi_k \circ \cdots \circ \varphi_1$.

A morphism $\varphi: \sigma \longrightarrow \sigma'$ is called a *monomorphism* and written as $\varphi: \sigma \rightarrowtail \sigma'$, if for any simplex σ'' and any two morphisms $\varphi_0, \varphi_1: \sigma'' \longrightarrow \sigma$, $\varphi \circ \varphi_0 = \varphi \circ \varphi_1$ implies $\varphi_0 = \varphi_1$.

A morphism $\varphi: \sigma \longrightarrow \sigma'$ is called an *epimorphism* and written as $\varphi: \sigma \twoheadrightarrow \sigma'$, if for any simplex σ'' and any two morphisms $\varphi_0, \varphi_1: \sigma' \longrightarrow \sigma''$,

$\varphi_0\circ\varphi=\varphi_1\circ\varphi$ implies $\varphi_0=\varphi_1$.

Given a morphism $\varphi:\ \sigma\longrightarrow\sigma'$, we denote by $|\varphi|:\ |\sigma|\longrightarrow|\sigma'|$ the continuous map induced from φ.

If $|\varphi|$ is an injection, then φ is a monomorphism and vice versa. If $|\varphi|$ is a surjection, then φ is an epimorphism, however the converse is not necessarily true except in the case $\sigma'\prec\sigma$.

A simplex σ is called *nondegenerate*, if for any two morphisms $\varphi_0,\ \varphi_1:\ \sigma'\longrightarrow\sigma$ we obtain $\varphi_0=\varphi_1$. Let σ be a simplex with vertices $\sigma_0<\cdots<\sigma_q$. Then σ is nondegenerate if and only if we have $F(\sigma_0)<\cdots<F(\sigma_q)$.

Let σ be a simplex with vertices $\sigma_0<\cdots<\sigma_q$. Let $\varphi:\ \sigma\longrightarrow\sigma'$ be a morphism. Then there exists an increasing sequence of integers $0=i_0<i_1<\cdots<i_p\leqq q<i_{p+1}=q+1$ satisfying the conditions

$$\begin{cases}\varphi(\sigma_{i_h})=\varphi(\sigma_i) & i_h\leqq i\leqq i_{h+1}-1 \quad 0\leqq h\leqq p\\ \varphi(\sigma_{i_h})<\varphi(\sigma_{i_{h+1}}) & \qquad\qquad\qquad 0\leqq h\leqq p-1.\end{cases}$$

Now the definition of a filtration implies

$$\begin{cases}F\varphi(\sigma_{i_h})=F\varphi(\sigma_i) & i_h\leqq i\leqq i_{h+1}-1 \quad 0\leqq h\leqq p\\ F\varphi(\sigma_{i_h})\leqq F\varphi(\sigma_{i_{h+1}}) & \qquad\qquad\qquad 0\leqq h\leqq p-1\end{cases}$$

and further the definition of a morphism implies

$$\begin{cases}F(\sigma_{i_h})=F(\sigma_i) & i_h\leqq i\leqq i_{h+1}-1 \quad 0\leqq h\leqq p\\ F(\sigma_{i_h})\leqq F(\sigma_{i_{h+1}}) & \qquad\qquad\qquad 0\leqq h\leqq p-1.\end{cases}$$

Here we define $\operatorname{Coim}\varphi$, $\operatorname{Im}\varphi$ to be the simplices of the form

$$\operatorname{Coim}\varphi=|\sigma_{i_0},\ \sigma_{i_1},\ \cdots,\ \sigma_{i_p}|,\qquad \operatorname{Im}\varphi=|\varphi(\sigma_{i_0}),\ \varphi(\sigma_{i_1}),\ \cdots,\ \varphi(\sigma_{i_p})|.$$

These conventions enable us to determine unique morphisms $\operatorname{coim}\varphi$, $\operatorname{eq}\varphi$ which render the two diagrams

$$\begin{tikzcd}
\sigma \arrow[r, "\varphi"] \arrow[d, two heads, "\operatorname{coim}\varphi"'] & \sigma' \\
\operatorname{Coim}\varphi \arrow[r, tail, two heads, "\operatorname{eq}\varphi"'] & \operatorname{Im}\varphi \arrow[u, "\operatorname{im}\varphi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\sigma \arrow[r, "\varphi"] & \sigma' \\
\operatorname{Coim}\varphi \arrow[u, "\iota"] \arrow[r, tail, two heads, "\operatorname{eq}\varphi"'] & \operatorname{Im}\varphi \arrow[u, "\operatorname{im}\varphi"']
\end{tikzcd}$$

commutative where ι and $\operatorname{im}\varphi$ denote the inclusion. In fact, we obtain $(\operatorname{coim}\varphi)(\sigma_i)=\sigma_{i_h}$ $i_h\leqq i\leqq i_{h+1}$ and $\operatorname{eq}\varphi$ is an equivalence being the restriction of φ on $\operatorname{Coim}\varphi$.

Let σ be a simplex with vertices $\sigma_0<\cdots<\sigma_q$. Then there exists an increasing sequence $0=i_0<i_1<\cdots<i_p\leqq q<i_{p+1}=q+1$ satisfying the conditions

$$\begin{cases} F(\sigma_{i_h})=F(\sigma_i) & i_h\leqq i\leqq i_{h+1}-1 \quad 0\leqq h\leqq p \\ F(\sigma_{i_h})<F(\sigma_{i_{h+1}}) & \qquad\qquad\quad 0\leqq h\leqq p-1. \end{cases}$$

We define the *normalization* $\nu(\sigma)$ of σ to be a nondegenerate simplex of the vertices $\sigma_{i_0}<\sigma_{i_1}<\cdots<\sigma_{i_p}$. Now we have a morphism $\eta\colon\ \sigma\longrightarrow\nu(\sigma)$ which maps σ_i to σ_{i_h} for each i satisfying $i_h\leqq i\leqq i_{h+1}-1$ with $0\leqq h\leqq p$. This morphism will be called the *degeneracy* of σ.

Suppose given a morphism $\varphi\colon\ \sigma\longrightarrow\sigma'$ with σ' being nondegenerate. Then the simplex $\mathrm{Coim}\,\varphi$ and the morphism $\mathrm{coim}\,\varphi$ depend only on σ, not on the choice of φ. In fact we can show that $\mathrm{Coim}\,\varphi$ and $\mathrm{coim}\,\varphi$ coincide with $\nu(\sigma)$ and η respectively.

Next we introduce an equivalence relation in the set of all nondegenerate simplices in K. An equivalence $\alpha\colon\ \sigma\longrightarrow\sigma'$ is called a *strong congruence*, if there exists a simplex σ'' in K and a degeneracy map $\eta''\colon\ \sigma''\longrightarrow\nu(\sigma'')$ so as to make the diagram

$$\begin{tikzcd} \sigma'' \arrow[r, "\eta''"] & \nu(\sigma'') \\ \sigma \arrow[u, tail] \arrow[ur, "\eta''|\sigma"] \arrow[r, tail, two heads, "\alpha"'] & \sigma' \arrow[u, tail] \end{tikzcd}$$

commutative where the vertical arrows denote the inclusions.

LEMMA 2.1. *Let $\eta\colon\ \sigma\longrightarrow\nu(\sigma)$ be a degeneracy of σ. Suppose given a face $\sigma'<\sigma$. We write $\eta(\sigma')=\mathrm{Im}(\eta|\sigma')$. Let $\varphi\colon\ \sigma'\longrightarrow\eta(\sigma')$ be a morphism such that the diagram*

$$\begin{tikzcd} \sigma \arrow[r, "\eta"] & \nu(\sigma) \\ \sigma' \arrow[u, tail] \arrow[ur, "\eta|\sigma'"] \arrow[r, "\varphi"] & \eta(\sigma') \arrow[u, tail] \end{tikzcd}$$

is commutative where the vertical arrows denote the inclutions.

Then there exist a degeneracy $\eta'\colon\ \sigma'\longrightarrow\nu(\sigma')$ and a strong congruence $\alpha\colon\ \nu(\sigma')\longrightarrow\eta(\sigma')$ satisfying the relation $\varphi=\alpha\circ\eta'$.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
\sigma \arrow[r, "\eta"] & \nu(\sigma) \\
\sigma' \arrow[u, tail] \arrow[r, "\varphi"] \arrow[ur, "\eta|\sigma'"] \arrow[d, "\eta'"'] & \eta(\sigma') \arrow[u, tail] \arrow[d, equal] \\
\nu(\sigma') \arrow[r, tail, two heads, "\mathrm{eq}\,\varphi"] & \eta(\sigma')
\end{tikzcd}$$

Since $\mathrm{Im}\,\varphi$ is equal to $\eta(\sigma')$ and it is nondegenerate, we have $\nu(\sigma')=\mathrm{Coim}\,(\varphi)$, $\eta(\sigma')=\mathrm{Im}\,(\eta|\sigma')$.

LEMMA 2.2. *Let* α: $\sigma \longrightarrow \alpha(\sigma)$ *be a strong congruence. Suppose given a simplex* $\sigma' \lessdot \sigma$. *We write* $\alpha(\sigma')=\mathrm{Im}\,(\alpha|\sigma)$. *Let* α': $\sigma' \longrightarrow \alpha(\sigma')$ *be a morphism such that the diagram*

$$\begin{tikzcd}
\sigma \arrow[r, tail, two heads, "\alpha"] & \alpha(\sigma) \\
\sigma' \arrow[u, tail] \arrow[r, tail, two heads, "\alpha'"] \arrow[ur, "\alpha|\sigma'"] & \alpha(\sigma') \arrow[u, tail]
\end{tikzcd}$$

is commutative where the vertical arrows denote the inclusions.

Then α': $\sigma' \longrightarrow \alpha(\sigma')$ *is a strong congruence.*

Proof. We have a simplex σ'' and the degeneracy η'': $\sigma'' \longrightarrow \nu(\sigma'')$ such that the diagram

$$\begin{tikzcd}
\sigma'' \arrow[r, "\eta''"] & \nu(\sigma'') \\
\sigma \arrow[u, tail] \arrow[r, tail, two heads, "\alpha"] & \alpha(\sigma) \arrow[u, tail] \\
\sigma' \arrow[u, tail] \arrow[r, tail, two heads, "\alpha'"] \arrow[ur, "\alpha|\sigma'"] & \alpha(\sigma') \arrow[u, tail]
\end{tikzcd}$$

is commutative where η'': $\sigma'' \longrightarrow \nu(\sigma'')$ denotes the degeneracy of σ'' and vertical arrows mean the inclusions. Since α is a monomorphism, $\operatorname{coim}(\alpha|\sigma')=1_{\sigma'}$ and hence $\alpha'=\operatorname{eq}(\alpha|\sigma')$.

An equivalence $\alpha:\ \sigma \longrightarrow \sigma'$ is called a *(weak) congruence* and denoted by $\alpha:\ \sigma\equiv\sigma'$ if there are given an integer k and $2k$ strong congruences

$$\sigma=\sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xleftarrow{\alpha_2} \sigma_2 \longrightarrow \cdots \longleftarrow \sigma_{2k-2} \xrightarrow{\alpha_{2k-1}} \sigma_{2k-1} \xleftarrow{\alpha_{2k}} \sigma_{2k}=\sigma'$$

running alternately to the left and to the right, and α has a factorization $\alpha=\alpha_{2k}^{-1}\circ\alpha_{2k-1}\circ\cdots\circ\alpha_2^{-1}\circ\alpha_1$.

REMARK Let σ, σ' be two simplices in K. If α_0, $\alpha_1:\ \sigma\equiv\sigma'$ are two congruences, then we have $\alpha_0=\alpha_1$.

LEMMA 2.3. *Let $\alpha:\ \sigma\equiv\alpha(\sigma)$ be a congruence. For a face σ' of σ, we put $\alpha(\sigma')=\operatorname{Im}(\alpha|\sigma')$. Let $\alpha':\ \sigma'\longrightarrow\alpha(\sigma')$ be a morphism such that the diagram*

$$\begin{tikzcd}
\sigma \arrow[r, "\alpha"] & \alpha(\sigma) \\
\sigma' \arrow[u, hook] \arrow[ur, "\alpha|\sigma'"] \arrow[r, "\alpha'"'] & \alpha(\sigma') \arrow[u, hook]
\end{tikzcd}$$

is commutative where the vertical arrows mean the inclusions.

Then $\alpha':\ \sigma'\equiv\alpha(\sigma')$ is a congruence.

§ 3. Halfspace

Let $F_B:\ B\longrightarrow \boldsymbol{Z}$ and $F_X:\ X\longrightarrow \boldsymbol{Z}$ be filtrations of spaces B and X respectively. Then the *sum* $F_B+F_X:\ B\times X\longrightarrow \boldsymbol{Z}$ is by definition the filtration of $B\times X$ given by $(F_B+F_X)(b,\ x)=F_B(b)+F_X(x)$. New we write E for the subspace $(F_B+F_X)^{-1}([-\infty,\ 0])={}_0(B\times X)$, that is

$$\begin{aligned} E&=\{(b,\ x)\in B\times X|F_B(b)+F_X(x)\leqq 0\} \\ &=\cup\{{}_{-l}B\times{}_lX|l\in Z\}. \end{aligned}$$

The space E will be called the *half space* of $B\times X$ with respect to F_B+F_X.

Let F_B, F_X be filtrations of spaces B, X respectively. Let $B\times X$ be eqipped with the filtration F_B+F_X and let E be its half space. Let $F_{B'}$, $F_{X'}$ be filtrations of spaces B', X' respectively. Let $B'\times X'$ be eqipped with the filtration $F_{B'}$ $F_{X'}$ and let E' be its half space. If $f_B:\ B\longrightarrow B'$, $f_X:\ X\longrightarrow X'$ are maps of filtered spaces, then $f_B\times f_X:\ B\times X\longrightarrow B'\times X'$ also a map of filtered spaces and so it induces a map $\tilde{f}:\ E\longrightarrow E'$.

Let F_B, F_X be filtrations of spaces B, X respectively. Let E be a half space of $B\times X$ with respect to F_B+F_X. If i: $B'\longrightarrow B$ be an inclusion, then the *restriction* $F_{B'}=F_B\circ i$: $B'\longrightarrow \boldsymbol{Z}$ of F_B on B' is defined. Let E' be a half space of $B'\times X$ with respect to $F_{B'}+F_X$. Now the inclusion i induces an inclusion $\tilde{i}$: $E'\longrightarrow E$ through which E' can be embedded onto $(B'\times X)\cap E$.

Let F_B, F_X be the filtrations of B, X respectively. We write also E for the resulting half space. If p: $B\times I\longrightarrow B$ be a projection, then F_B induces the filtration $F_B\circ p$: $B\times I\longrightarrow \boldsymbol{Z}$ of $B\times I$ which will be denoted by the same notation F_B. Let E' be a half space of $(B\times I)\times X$ with respect to F_B+F_X. Then the projection p yields a projection $\tilde{p}$: $E'\longrightarrow E$ in terms of which E' can be shown to be identified with $E\times I$ in such a way that $\tilde{p}$ corresponds with the projection.

Let F_B and $F_{B'}$ be filtrations of spaces B and B' respectively. Let F_X be a filtration of a space X. We write E and E' be the corresponding half spaces of $B\times X$ and $B'\times X$ respectively. Let f, g: $B\longrightarrow B'$ be two maps of filtered spaces and let h: $B\times I\longrightarrow B'$ be a homotopy from f to g. Then the maps $\tilde{f}$, $\tilde{g}$: $E\longrightarrow E'$ are homotopic and the map $\tilde{h}$: $E\times I\longrightarrow E'$ gives rise to a homotopy from $\tilde{f}$ to $\tilde{g}$.

Let F_K: $K\longrightarrow \boldsymbol{Z}$ be a filtration of a simplical complex K. Then we have the filtration $|F_K|$: $|K|\longrightarrow \boldsymbol{Z}$ of a polyhedron $|K|$. Let F_X: $X\longrightarrow \boldsymbol{Z}$ be a filtration of a space X. This filtration allows us to construct a half space E of $|K|\times X$ with respect to $|F_K|+F_X$ which can be expressed in the form

$$E=\cup\{|\rho|\times_{d(\rho)}X|\rho\in K\}$$

where $|\rho|$ means the geometric realization of a simplex ρ in K and $d(\rho)$ stands for the integer $-F_K(\rho)$.

On the other hand, we know that F defines a derived filtration $'F_K$: $'K\longrightarrow \boldsymbol{Z}$ of the derived complex $'K$. Let σ be a simplex of $'K$ with vertices $\hat{\rho}_0,\cdots,\hat{\rho}_q$ such that $\rho_0<\cdots<\rho_q$. By the definition, $'F_K(\sigma)={}'F_K(\hat{\rho}_q)$ is equal to $F_K(\rho_q)$. As observed above $'F_K$ induces a filtration $|'F_K|$: $|'K|\longrightarrow \boldsymbol{Z}$. Let F_X: $X\longrightarrow \boldsymbol{Z}$ be a filtration of a space X as above. Then we obtain the half space $'E$ of $|'K|\times X$ with respect to $|'F_K|+F_X$ too. Now the half space can be rewritten as

$$'E=\cup\{|\sigma|\times_{'d(\sigma)}X|\sigma\in{}'K\}$$

where $'d(\sigma)={}'d(\hat{\rho}_q)$ represents $d(\rho_q)$.

Since the filtrations $|F|$ and $|'F|$ satisfy the identity $|'F|\circ i=|F|$ with the natural identification i: $|'K|\longrightarrow|K|$, we can easily prove the following Proposition.

PROPOSITION 3.1. *Let F_K be a filtration of a simplicial complex K and let F_X be a filtration of a space X. Then the natural identification i: $|'K|\longrightarrow|K|$ provides us with an identification*

$$\tilde{i}:\ 'E\longrightarrow E.$$

Here we remark that for an arbitrary filtration F of a simplicial complex K, the derived filtration $'F$ is a filtration of an ordered simplicial complex $'K$. So the above Proposition says that we only need to consider filtrations of an ordered simplicial complex.

EXAMPLE Let $\mathcal{V}=\{V_i | i\in I\}$ be a family of affine subspaces V_i of $\boldsymbol{F}^n$ passing through 0. The filtration $F(\mathcal{V})\colon \Delta(I) \longrightarrow \boldsymbol{Z}$ is defined by the formula $F(\mathcal{V})(\rho)=-\dim_{\boldsymbol{F}} V_\rho$ with $V_\rho=\cap\{V_i | i\in\rho\}$. $F(\mathcal{V})$ produces a filtration $|F(\mathcal{V})|\colon |\Delta(I)| \longrightarrow Z$ of a space $|\Delta(I)|$. Let $F_X\colon X \longrightarrow \boldsymbol{Z}$ be a filtration of a space X. Now the half space E of $|\Delta(I)|\times X$ with respect to $|F(\mathcal{V})|+F$ is

$$E=\cup\{|\rho|\times_{d(\rho)}X | \rho\in\Delta(I)\}$$

where $d(\rho)$ denotes $-F(\mathcal{V})(\rho)=\dim_{\boldsymbol{F}} V_\rho$.

Let us cosider the derived filtration $'F(\mathcal{V})\colon\ '\Delta(I) \longrightarrow \boldsymbol{Z}$. If $\sigma\in{}'\Delta(I)$ be a simplex with vertices $\hat{\rho}_0, \cdots, \hat{\rho}_q$ such that $\rho_0 \prec\cdots\prec\rho_q$, $'F(\mathcal{V})$ is given by $'F(\mathcal{V})(\sigma)=-\dim_{\boldsymbol{F}} V_{\rho_q}$. Now the half space $'E$ of $|'K|\times X$ with respect to $|'F(\mathcal{V})|+F_X$ can be written in the form

$$'E=\cup\{|\sigma|\times_{'d(\sigma)}X | \sigma\in{}'\Delta(I)\}$$

where $'d(\sigma)$ represents $d(\rho_q)$.

In what follows, we shall be mainly concerned with particular filtrations $F\colon X \longrightarrow \boldsymbol{Z}$ as listed below:

0) Let $F_{\boldsymbol{F}^n}\colon \boldsymbol{F}^n \longrightarrow \boldsymbol{Z}$ be the filtration defined by

$${}_{-1}\boldsymbol{F}^n=\emptyset\subset{}_0\boldsymbol{F}^n=\{0\}\subset{}_1\boldsymbol{F}^n=\boldsymbol{F}^1\subset\cdots\subset{}_l\boldsymbol{F}^n=\boldsymbol{F}^l\subset\cdots\subset{}_n\boldsymbol{F}^n=\boldsymbol{F}^n$$

where $\boldsymbol{F}^l$ is identified with the subspace of $\boldsymbol{F}^n$ consisting of vecters whose i-th component vanishes for i larger than l.

Let us denote by $E(\mathcal{V})$ the corresponding half space, that is

$$E(\mathcal{V})=\cup\{|\rho|\times\boldsymbol{F}^{d(\rho)} | \rho\in\Delta(I)\}.$$

i) Let us denote by $(\boldsymbol{F}^n)^*$ the space $\boldsymbol{F}^n\backslash\{0\}$. Let $F_{(\boldsymbol{F}^n)^*}$ be the filtration of $(\boldsymbol{F}^n)^*$ defined by ${}_l(\boldsymbol{F}^n)^*=(\boldsymbol{F}^l)^*$.

We write $E(\mathcal{V}^*)$ for the corresponding half space, that is

$$E(\mathcal{V}^*)=\cup\{|\rho|\times(\boldsymbol{F}^{d(\rho)})^* | \rho\in\Delta(I)\}.$$

ii) Let us denote by $S\boldsymbol{F}^n$ the unit sphere of $\boldsymbol{F}^n$ with respect to the usual norm on $\boldsymbol{F}^n$. Let $F_{S\boldsymbol{F}^n}$ be the filtration of $S\boldsymbol{F}^n$ defined by ${}_l(S\boldsymbol{F}^n)=S\boldsymbol{F}^l$.

We write $E(S\mathcal{V})$ for the corresponding half space, that is

$$E(S\mathcal{V})=\cup\{|\rho|\times S\boldsymbol{F}^{d(\rho)} | \rho\in\Delta(I)\}.$$

iii) Let us denote by $P\boldsymbol{F}^n$ the projective space associated with $\boldsymbol{F}^n$. Let $F_{P\boldsymbol{F}^n}$ be the filtration of $P\boldsymbol{F}^n$ defined by ${}_l(P\boldsymbol{F}^n)=P\boldsymbol{F}^l$.

We write $E(P\subset V)$ for the corresponding half space, that is

$$E(P\subset V)=\cup\{|\rho|\times PF^{d(\rho)}|\rho\in\Delta(I)\}.$$

§ 4. Normalized maps from half spaces

Let F_K be a filtration of an ordered simplicial complex K and let F_X be a filtration of a space X. Now we can construct a half space E of $|K|\times X$ with respect to $|F_K|+F_X$.

For later convenience, we want to write $\tilde{K}$ for the half space E above mentioned and use the notation p for the projection from E to K. Without fear of confusion, for a subcomplex L of K we also use the notation $\tilde{L}$ to express the restriction E on L.

Here we introduce the equivalence relation among all nondegenerate q-simplices by congruence as defined in § 2 and we choose one representative in each equivalence class. The resulting complete set of representatives will be denoted by N_q.

Let K^q denote the q-skeleton of K. We define a subspace $\tilde{N}_q$ of $\tilde{K}^q$ by

$$\tilde{N}_q=\cup\{|\sigma|\times_{d(\sigma)}X|\sigma\in N_q\}$$

where $d(\sigma)$ denotes $-F_K(\sigma)$. This space lies in $\tilde{K}^q\cap p^{-1}(N_q)$.

Let $\tilde{Y}$ be a topological space and let f be a continuous map from $\tilde{K}$ to $\tilde{Y}$. Then f is called *normalised*, if f satisfies the following two conditions:

i) If η: $\sigma\longrightarrow\nu(\sigma)$ is a nondegeneracy, then we have

$$f\,\big|\,|\sigma|\times_{d(\sigma)}X=(f\,\big|\,|\nu(\sigma)|\times_{d(\sigma)}X)\circ(\eta\times 1).$$

ii) If α: $\sigma\longrightarrow\alpha(\sigma)$ is a congruence, then we have

$$f\,\big|\,|\sigma|\times_{d(\sigma)}X=(f\,\big|\,|\alpha(\sigma)|\times_{d(\sigma)}X)\circ(\alpha\times 1).$$

In these two conditions, we employ the notation $d(\sigma)$ to denote $-F_K(\sigma)$.

Proposition 4.1. *Let q be a nonnegative integer. Assume that there is given a normalized map f_{q-1}: $\tilde{K}^{q-1}\longrightarrow\tilde{Y}$ and a continuous map f'_q: $\tilde{N}_q\longrightarrow\tilde{Y}$ such that*

$$f'_q\big|\tilde{K}^{q-1}\cap\tilde{N}_q=f_{q-1}\big|\tilde{K}^{q-1}\cap\tilde{N}_q.$$

Then there exists a unique normalized map

$$f_q:\ \tilde{K}^{q-1}\cup\tilde{N}_q\longrightarrow\tilde{Y}$$

such that

$$f_q\big|\tilde{K}^{q-1}=f_{q-1},\qquad f_q\big|\tilde{N}_q=f'_q.$$

Proof. Let σ be a q-simplex of K not belonging to N_q. We need to construct a map

$$f_\sigma:\ |\sigma|\times_{d(\sigma)}X \longrightarrow \tilde{Y}$$

satisfying the compatibility condition

$$f_\sigma\big||\partial\sigma|\times_{d(\sigma)}X=f_{q-1}\big||\partial\sigma|\times_{d(\sigma)}X.$$

Suppose first that σ is degenerate. Let $\eta:\ \sigma\longrightarrow\nu(\sigma)$ be the degeneracy of σ. Then we define f_σ by

$$f_\sigma=(f_{q-1}\big||\nu(\sigma)|\times_{d(\sigma)}X)\circ(\eta\times 1).$$

For σ nondegenerate, we choose a unique congruence $\alpha:\ \sigma\longrightarrow\alpha(\sigma)$ such that $\alpha(\sigma)$ is contained in N_q. Now we define f_σ by

$$f_\sigma=(f'_q\big||\alpha(\sigma)|\times_{d(\sigma)}X)\circ(\alpha\times 1).$$

Now we have to check that the compatibility condition is satisfied.

Let σ' be a $(q-1)$-simplex being a face of the q-simplex σ. From the definition of filtration, we have $F_K(\sigma')\leqq F_K(\sigma)$ and so obtain $d(\sigma')\geqq d(\sigma)$. Hence we can form the inclusion $j:\ {}_{d(\sigma)}X\longrightarrow {}_{d(\sigma')}X$. Now it is trivial that the identity

$$f_{q-1}\big||\sigma'|\times_{d(\sigma)}X=(f_{q-1}\big||\sigma'|\times_{d(\sigma')}X)\circ(1\times j)$$

holds.

First we assume that σ is degenerate. We write $\eta(\sigma')=\mathrm{Im}\,(\eta|\sigma')$. Let $\varphi:\ \sigma'\longrightarrow\eta(\sigma')$ be a morphism

$$\begin{tikzcd}
\sigma \arrow[r, "\eta"] & \nu(\sigma) \\
\sigma' \arrow[u] \arrow[ur, "\eta|\sigma"] \arrow[r, "\varphi"] & \eta(\sigma') \arrow[u]
\end{tikzcd}$$

is commutative.

Then the definition of f_σ implies

$$f_\sigma\big||\sigma'|\times_{d(\sigma)}X=(f_{q-1}\big||\eta(\sigma')|\times_{d(\sigma)}X)\circ(|\varphi|\times 1).$$

On the other hand Lemma 2.1 shows that there exists a degeneracy $\eta':\ \sigma'\longrightarrow\nu(\sigma')$ and a strong congruence $\alpha:\ \nu(\sigma')\longrightarrow\eta(\sigma')$ satisfying $\varphi=\alpha\circ\eta'$. From the assumption f_{q-1} is normalized, so we have

$$\begin{aligned}
f_{q-1}\big||\sigma'|\times_{d(\sigma)}X&=(f_{q-1}\big||\sigma'|\times_{d(\sigma')}X)\circ(1\times j)\\
&=(f_{q-1}\big||\nu(\sigma')|\times_{d(\sigma')}X)\circ(|\eta'|\times 1)\circ(1\times j)
\end{aligned}$$

$$
\begin{aligned}
&=(f_{q-1}\big||\eta(\sigma')|\times_{d(\sigma')}X)\circ(|\alpha|\times 1)\circ(|\eta'|\times 1)\circ(1\times j)\\
&=(f_{q-1}\big||\eta(\sigma')|\times_{d(\sigma')}X)\circ(|\varphi|\times 1)(1\times j)\\
&=(f_{q-1}\big||\eta(\sigma')|\times_{d(\sigma')}X)\circ(1\times j)\circ(|\varphi|\times 1)\\
&=(f_{q-1}\big||\eta(\sigma')|\times_{d(\sigma)}X)\circ(|\varphi|\times 1).
\end{aligned}
$$

Thus in this case f satisfies the reqired condition.

Next σ is assumed to be a nondegenerate q-simplex not belonging to N_q. We write $\alpha(\sigma')=\mathrm{Im}\,(\alpha|\sigma')$. Let $\alpha'\colon\ \sigma'\longrightarrow\alpha(\sigma')$ be a morphism such that the diagram

$$
\begin{tikzcd}
\sigma \arrow[r, "\alpha"] & \alpha(\sigma) \\
\sigma' \arrow[u] \arrow[ur, "\alpha|\sigma'"] \arrow[r, "\alpha'"] & \alpha(\sigma') \arrow[u]
\end{tikzcd}
$$

is commutative.

Analogously to the case of degenerate slmplex, we have

$$f_\sigma\big||\sigma'|\times_{d(\sigma)}X=(f'_q\big||\alpha(\sigma')|\times_{d(\sigma)}X)\circ(|\alpha'|\times 1).$$

Now we recall Lemma 2.3 which says that $\alpha'\colon\ \sigma'\longrightarrow\alpha(\sigma')$ gives a congruence. Since f_{q-1} is normalized, we obtain

$$
\begin{aligned}
f_{q-1}\big||\sigma'|\times_{d(\sigma)}X&=(f_{q-1}\big||\sigma'|\times_{d(\sigma')}X)\circ(1\times j)\\
&=(f_{q-1}\big||\alpha(\sigma')|\times_{d(\sigma')}X)\circ(|\alpha'|\times 1)\circ(1\times j)\\
&=(f_{q-1}\big||\alpha(\sigma')|\times_{d(\sigma')}X)\circ(1\times j)\circ(|\alpha'|\times 1|)\\
&=(f_{q-1}\big||\alpha(\sigma')|\times_{d(\sigma)}X)\circ(|\alpha'|\times 1).
\end{aligned}
$$

Consequently in both of these two cases, the compatibility condition

$$f_\sigma\big||\sigma'|\times_{d(\sigma)}X=f_{q-1}\big||\sigma'|\times_{d(\sigma)}X$$

is satisfied by f_σ for every fase σ' of σ.

§ 5. Functors from a simplicial complex to a category

In what follows, $\mathcal{A}$ is promised to denote a subcategory of $\mathcal{S}$ being the category of topological spaces.

Let $F_K\colon\ K\longrightarrow\boldsymbol{Z}$ be a filtration of a simplicial complex K. Let $\mathcal{A}$ be a category as above. Then a *contravariant functor* Y from K to $\mathcal{A}$ is a pair of functions denoted by Y: $K \longrightarrow \overset{\circ}{\mathcal{A}}$: One assigns to each simplex ρ in K an object $Y(\rho)$ in $\mathcal{A}$ and the other assigns to each inclusion i: $\rho \longrightarrow \rho'$ in K an inclusion $Y(i)$: $Y(\rho') \longrightarrow Y(\rho)$ in $\mathcal{A}$ satisfying the routine conditions $Y(1_\rho)=1_{Y(\rho)}$, $Y(i' \circ i)=Y(i) \circ Y(i')$ and the additioal ones:

i) There exists an object $Y(\emptyset)$ in $\mathcal{A}$ such that for each ρ in K there exists a unique inclusion $Y(\rho) \longrightarrow Y(\emptyset)$ in $\mathcal{A}$.

ii) If ρ, ρ' are two simplices contained in some simplex ρ'' in K, then we have $Y(\rho * \rho')=Y(\rho) \cap Y(\rho')$.

iii) If ρ, ρ' be two simplices in K satisfying $\rho \prec \rho'$ and $F_K(\rho)=F_K(\rho')$, then we have $Y(\rho)=Y(\rho')$.

Let F_K: $K \longrightarrow \boldsymbol{Z}$ be a filtration of an ordered simplicial complex. Let $\mathcal{A}$ be a category as above. Then *a covariant functor* Z from K to $\mathcal{A}$ is a pair of functions denoted by $Z : K \longrightarrow \mathcal{A}$: One assigns to each simplex σ in K an object $Z(\sigma)$ and the other assigns to each morphism φ: $\sigma \longrightarrow \sigma'$ in K an inclusion $Z(\varphi)$: $Z(\sigma) \longrightarrow Z(\sigma')$ satisfying the conditions $Z(1_\sigma)=1_{Z(\sigma)}$, $Z(\varphi' \circ \varphi)= Z(\varphi') \circ Z(\varphi)$ and the supplementary ones as follows:

i) There exists an object $Z(\omega)$ in $\mathcal{A}$ and for each simplex σ in K there exists a unique inclusion $Z(\sigma) \longrightarrow Z(\omega)$ in $\mathcal{A}$.

ii) If σ, σ' be two vertices of K satisfying $\sigma \leqq \sigma'$, and $F_K(\sigma)=F_K(\sigma')$, then we have $Z(\sigma)=Z(\sigma')$.

iii) If σ is a simplex of K with vertices $\sigma_0 < \cdots < \sigma_q$ then we have $Z(\sigma)=Z(\sigma_0)$.

Let F_K: $K \longrightarrow \boldsymbol{Z}$ be a filtration of a simplicial complex. Let Y: $K \longrightarrow \overset{\circ}{\mathcal{A}}$ be a contravariant functor from K to $\mathcal{A}$. Then we have a filtration $'F_K$: $'K \longrightarrow \boldsymbol{Z}$ of $'K$ derived from F_K. Let σ be a simplex of $'K$ with vertices $\hat{\rho}_0, \cdots, \hat{\rho}_q$ with $\rho_0 \prec \cdots \prec \rho_q$. $'F_K(\sigma)={}'F(\hat{\rho}_q)$ is known to equal $F(\rho_q)$. Now the functor $'Y$: $'K \longrightarrow \mathcal{A}$ *derived* from Y is defined as follows: Let σ be a simplex with vertices $\hat{\rho}_0, \cdots, \hat{\rho}_q$ such that $\rho_0 \prec \cdots \prec \rho_q$. We put $'Y(\sigma)={}'Y(\hat{\rho}_0)=Y(\rho_0)$. As easily seen, $'Y$ defines a covariant functor from $'K$ to $\mathcal{A}$.

LEMMA 5.1. *Let F_K be a filtration of an ordered simplicial complex K. Let Z be a covariant functor from K to $\mathcal{A}$. Then for each surjective morphism φ: $\sigma \longrightarrow \sigma'$, we have $Z(\sigma)=Z(\sigma')$. In particular we obtain the following*:

i) *For each degeneracy η: $\sigma \longrightarrow \nu(\sigma)$, we have $Z(\sigma)=Z(\nu(\sigma))$.*

ii) *For each congruence α: $\sigma \longrightarrow \alpha(\sigma)$, we have $Z(\sigma)=Z(\alpha(\sigma))$.*

Proof. Suppose given a elementary morphism φ: $\sigma \Longrightarrow \sigma'$. Then from the definition, there exists a simplex σ'' with vertices $\sigma''_0 < \cdots < \sigma''_r$ such that both of σ and σ' are faces of σ''. If σ is a simplex with vertices $\sigma_0 < \cdots < \sigma_p$ and σ' is one with vertices $\sigma'_0 < \cdots < \sigma'_q$. Then we can find increasing sequences $0 \leqq i_0 < \cdots < i_p \leqq r$ and $0 \leqq j_0 < \cdots < j_q \leqq r$ determined by the property

$$\begin{cases} \sigma''_{i_h}=\sigma_h & 0 \leqq h \leqq p \\ \sigma''_{j_k}=\sigma'_k & 0 \leqq k \leqq q . \end{cases}$$

Hence we can conclude that either $\sigma_0 \leqq \sigma_0'$ or $\sigma_0' \leqq \sigma_0$ holds.

On the other hand, the definition of a morphism implies $F(\sigma_0') \leqq F(\varphi(\sigma_0)) = F(\sigma_0)$ and the equality holds whenever $\sigma_0' = \varphi(\sigma_0)$. In particular, for any surjective morphism φ we obtain $F(\sigma_0') = F(\sigma_0)$.

Now applying the condition ii) in the definition of covariant functor shows $Z(\sigma_0) = Z(\sigma_0')$.

Finally the condition iii) assures $Z(\sigma) = Z(\sigma')$.

EXAMPLE $\boldsymbol{F}^n$: Objects, all euclidean subspaces of $\boldsymbol{F}^n$ passing through 0 with the usual metric; morphisms, all isometric affine maps of such preserving 0.

i) Given an object V in $\boldsymbol{F}^n$, we write V^* for the open set $V \setminus \{0\}$. Given a morphism v, we write v^* for the map induced from v.

$(\boldsymbol{F}^n)^*$: Objects, all V^*'s in one-to-one correspondence $V^* \longleftrightarrow V$ with the objects V's in $\boldsymbol{F}^n$; morphisms, all v^*'s in one-to-one correspondence $v^* \longleftrightarrow v$ with the morphisms v's in $\boldsymbol{F}^n$.

ii) For an object V in $\boldsymbol{F}^n$, SV stands for the unit sphere in V. For a morphism v, Sv stands for the map induced from v.

$S\boldsymbol{F}^n$: Objects, all SV's with V's in $\boldsymbol{F}^n$; morphisms, all Sv's with v's in $\boldsymbol{F}^n$.

iii) For an object V in $\boldsymbol{F}^n$, PV stands for the projective space associated with V. For a morphism v, Pv stands for the map induced from v.

$P\boldsymbol{F}^n$: Objects, all PV's with V's in $\boldsymbol{F}^n$; morphisms, all Pv's with v's in $\boldsymbol{F}^n$.

Let $\mathcal{V} = \{V_i | i \in I\}$ be a finite family of objects in $\boldsymbol{F}^n$. The filtration $F(\mathcal{V}) : \varDelta(I) \longrightarrow \boldsymbol{Z}$ of $\varDelta(I)$ is defined by $F(\mathcal{V})(\rho) = -\dim_F V_\rho$ with $V_\rho = \cap \{V_i | i \in \rho\}$. Now we define $Y(\mathcal{V}) : \varDelta(I) \longrightarrow \mathring{\boldsymbol{F}}^n$ to be a contravariant functor given by $Y(\mathcal{V})(\rho) = V_\rho$. In the same way, we define the contravariant functors $Y(\mathcal{V}^*) : \varDelta(I) \longrightarrow (\mathring{\boldsymbol{F}}^n)^*$, $Y(S\mathcal{V}) : \varDelta(I) \longrightarrow S\mathring{\boldsymbol{F}}^n$, $Y(P\mathcal{V}) : \varDelta(I) \longrightarrow P\mathring{\boldsymbol{F}}^n$ by $Y(\mathcal{V}^*)(\rho) = V_\rho^*$, $Y(S\mathcal{V})(\rho) = SV_\rho$, $Y(P\mathcal{V})(\rho) = PV_\rho$ respectively.

We know that the derived filtration $'F(\mathcal{V}) : {}'\varDelta(I) \longrightarrow \boldsymbol{Z}$ is defined as follows. Let σ be a simplex in $'\varDelta(I)$ of vertices $\hat{\rho}_0, \cdots, \hat{\rho}_q$ with $\rho_0 \prec \cdots \prec \rho_q$. The definition shows that $'F(\mathcal{V})(\sigma) = -\dim_F V_{\rho_q}$. Besides we get the covariant functor $'Y(\mathcal{V}) : {}'\varDelta(I) \longrightarrow \boldsymbol{F}^n$ derived from $Y(\mathcal{V})$ which is defined by the formula $'Y(\mathcal{V})(\sigma) = V_{\rho_0}$ with the above conventions. For the categories $(\boldsymbol{F}^n)^*$, $S\boldsymbol{F}^n$, $P\boldsymbol{F}^n$, we obtain the formulae $'Y(\mathcal{V}^*)(\sigma) = V_{\rho_0}^*$, $'Y(S\mathcal{V})(\sigma) = SV_{\rho_0}$, $'Y(P\mathcal{V})(\sigma) = PV_{\rho_0}$ correspondingly.

§ 6. Admissible maps from a half space

To begin with, we introduce some terminologies and hypotheses needed for later discussions.

Let $\mathcal{A}$ denote a subcategory of the category $\mathcal{S}$ of topological spaces.

Now a *filtration* F_X of an object X *in* $\mathcal{A}$ means a filtration F_X of a topological space X being an object in $\mathcal{A}$ such that for each l, ${}_lX$ is an object in $\mathcal{A}$ and for each pair of integers $l < l'$, the inclusion ${}_lX \longrightarrow {}_{l'}X$ is a morphism in $\mathcal{A}$.

The first we postulate is the following.

H_1) For each compact polyhedron B and each pair of objects X, Y in $\mathcal{A}$, the natural map

$$\mathcal{S}(B,\ \mathcal{S}(X,\ Y)) \longrightarrow \mathcal{S}(B\times X,\ Y)$$

induces a homeomorphism where $\mathcal{S}(S,\ T)$ denotes the space consisting of all continuous maps from the source S to the target T endowed with the compact open topology.

Let B be a compact polyhedron. Let X, Y be objects in $\mathcal{A}$. Then a map $f:\ B\times X \longrightarrow Y$ is called a *$\mathcal{A}$-map*, if f is transformed into $\mathcal{S}(B,\ \mathcal{A}(X,\ Y))$ through the correspondence given in the above hypothesis.

The next is the second hypothesis.

H_2) For each pair of integers l, l' such that $l<l'$, we have $\pi_{q-1}\mathcal{A}({}_lX,\ {}_{l'}X)=0$ if $q\leqq l'-l$.

Let F_K be a filtration of a simplicial complex K and let F be a filtration of an object X in $\mathcal{A}$. Let Y be a contravariant functor from K to $\mathcal{A}$.

The third hypothesis is the following.

H_3) For each simplex ρ in K, ${}_{d(\rho)}X$ and $Y(\rho)$ are equivalent in $\mathcal{A}$ where $d(\rho)$ denotes an integer $-F_K(\rho)$.

Here we introduce the notation $\tilde{Y}$ to denote the union $\cup\{Y(\rho)|\rho\in K\}$.

Suppose given a filtration F_K of an ordered simplicial complex. Then we have a filtration $|F_K|$ of $|K|$. Let F_X be a filtration of an object X in $\mathcal{A}$. Then $\tilde{K}$ means the half space of $|K|\times X$ with respect $|F_K|+F_X$. Suppose given a covariant functor Z from K to $\mathcal{A}$. Put $\tilde{Z}=\cup\{Z(\sigma)|\sigma\in K\}$.

We call a map $f:\ K\longrightarrow \tilde{Z}$ an *admissible map with carrier* Z, if for each σ in K there exists an $\mathcal{A}$-map

$$f_\sigma:\ \ |\sigma|\times{}_{d(\sigma)}X \longrightarrow Z(\sigma)$$

satisfying the condition

$$f\,\big|\,|\sigma|\times{}_{d(\sigma)}X=k_\sigma\circ f_\sigma$$

where $d(\sigma)$ denotes the integer $-F_K(\sigma)$ and $k_\sigma:\ Z(\sigma)\longrightarrow \tilde{Z}$ shows the inclusion.

In what follows, we specialize down to the case of the derived complex. With the above conventions, we know the existence of the filtration $'F_K$ of the complex $'K$. Now $|'F_K|$ denotes an induced filtration of a polyhedron $|K|$. Let F_X be a filtration of an object X in $\mathcal{A}$. Then we obtain a half space $'\tilde{K}$ of $|'K|\times X$ with respect to $|'F_K|+F_X$. Further we know that Y derives a covariant functor $'Y$ from $'K$ to $\mathcal{A}$. Put $'\tilde{Y}=\cup\{'Y(\sigma)|\sigma\in{}'K\}$. Clealy we have $'\tilde{Y}=\tilde{Y}$.

EXAMPLE Let $\boldsymbol{F}^n$ be a category of all euclidean subspaces of $\boldsymbol{F}^n$ passing through 0 and all isometric affine maps preserving 0. Let $\mathcal{V}=\{V_i|i\in I\}$ be a finite family of objects in $\boldsymbol{F}^n$. $F(\mathcal{V})$ is a filtration of $\Delta(I)$ transforming ρ into $-d(\rho)$ with $d(\rho)=\dim V_\rho$ $F_{\boldsymbol{F}^n}$ is a filtration of an object $\boldsymbol{F}^n$ in $\boldsymbol{F}^n$ defined by putting ${}_l\boldsymbol{F}^n=\boldsymbol{F}^l$. Now $Y(\mathcal{V})$ is a contravariant functor from $\Delta(I)$ to $\boldsymbol{F}^n$ assigning ρ into V_ρ.

First we can easily verify that $\boldsymbol{F}^n$ fulfils H_1).

Next we prove that taking $\boldsymbol{F}^n$ for $\mathcal{A}$, $\boldsymbol{F}^l$ for ${}_lX$ the hypothesis H_2) is satisfied. Actually, $\mathcal{A}({}_lX, {}_{l'}X)=U_{l'}/U_{l'-l}$ where U_l denotes $O(l)$, $U(l)$, $Sp(l)$ correspondingly whenever $\boldsymbol{F}$ is $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$. Then H_2) is wellknown.

In order to verify that H_3) is satisfied, we only need to remark the following. For each simplex ρ in K, we have $d(\rho)=\dim_{\boldsymbol{F}} V_\rho$ and $Y(\mathcal{V})(\rho)=V_\rho$, and hence ${}_{d(\rho)}X$ and $Y(\mathcal{V})(\rho)$ are equivalent to $\boldsymbol{F}^{d(\rho)}$.

The circumstances are the the same also when $\mathcal{A}$ is $(\boldsymbol{F}^n)^*$, $S\boldsymbol{F}^n$, $P\boldsymbol{F}^n$, F_X is $F_{(\boldsymbol{F}^n)^*}$, $F_{S\boldsymbol{F}^n}$, $F_{P\boldsymbol{F}^n}$ and Y is $Y(\mathcal{V}^*)$, $Y(S\mathcal{V})$, $Y(P\mathcal{V})$ respectively.

Now it is obvious that we have $\tilde{Y}(\mathcal{V}^*)=\cup\{V_i^*\subset(\boldsymbol{F})^*|i\in I\}$ $\tilde{Y}(S\mathcal{V})=\cup\{SV_i\subset S\boldsymbol{F}^n|i\in I\}$, $\tilde{Y}(P\mathcal{V})=\cup\{PV_i\subset P\boldsymbol{F}_n|i\in I\}$.

The first result is that an admissible map has an extention property.

PROPOSITION 6.1. *Let σ be a nondegenerate simplex in $'K$. Let $\widetilde{\partial\sigma}$ be the restriction of the half space $'\tilde{K}$ on $\partial\sigma$. Assume that there is given an admissible map f: $\widetilde{\partial\sigma}\longrightarrow\tilde{Y}$ with carrier $'Y$. Then there exists an $\mathcal{A}$-map*

$$f_\sigma:\ |\sigma|\times{}_{'d(\sigma)}X\longrightarrow{}'Y(\sigma)$$

with $'d(\sigma)=-'F_K(\sigma)$ satisfying the condition

$$k_\sigma\circ f_\sigma|\widetilde{\partial\sigma}=f$$

where k_σ: $'Y(\sigma)\longrightarrow\tilde{Y}$ denotes the inclusion.

Proof. Let σ' be a face of σ. Then we have $'d(\sigma')\geqq{}'d(\sigma)$, hence there exists an inclusion j: ${}_{'d(\sigma)}X\longrightarrow{}_{'d(\sigma')}X$ in terms of which we obtain a trivial identity

$$f||\sigma'|\times{}_{'d(\sigma)}X=(f||\sigma'|\times{}_{'d(\sigma')}X)\circ(1\times j).$$

Since f is admissible with carrier $'Y$, the definition shows the existence of an $\mathcal{A}$-map

$$f_{\sigma'}:\ |\sigma'|\times{}_{'d(\sigma')}X\longrightarrow{}'Y(\sigma')$$

satisfying the condition

$$f||\sigma'|\times{}_{'d(\sigma')}X=k_{\sigma'}\circ f_{\sigma'}$$

where $k_{\sigma'}$: $'Y(\sigma')\longrightarrow\tilde{Y}$ denotes the inclusion.

Let k: $'Y(\sigma')\longrightarrow{}'Y(\sigma)$ be the inclusion. Here we note that the identity $k_{\sigma'}=k_\sigma\circ k$ holds.

Now we define a map

$$f_{\partial\sigma}:\ |\partial\sigma|\times{}_{'d(\sigma)}X\longrightarrow{}'Y(\sigma)$$

by putting

$$f_{\partial\sigma}\big||\sigma'|\times{}_{'d(\sigma)}X=k\circ f_{\sigma'}\circ(1\times j).$$

From the hypothesis H_1), $f_{\partial\sigma}$ defines a map in $\mathcal{S}(|\partial\sigma|,\ \mathcal{A}({}_{'d(\sigma)}X,\ 'Y(\sigma)))$ and hence represents an element of $\pi_{q-1}\mathcal{A}({}_{'d(\sigma)}X,\ 'Y(\sigma))$ with $q=\dim\sigma$.

Let σ be a simplex with vertices $\hat{\rho}_0,\cdots,\ \hat{\rho}_q$ such that $\rho_0<\cdots<\rho_q$. Then we have $'d(\sigma)=d(\rho_q)$ and $'Y(\sigma)=Y(\rho_0)$ or equivalently we have $\mathcal{A}({}_{'d(\sigma)}X,\ 'Y(\sigma))=\mathcal{A}({}_{d(\rho_q)}X,\ Y(\rho_0))$.

Now the hypothesis H_3) yields a homeomorphism $\mathcal{A}({}_{'d(\sigma)}X,\ 'Y(\sigma))\approx\mathcal{A}({}_{d(\rho_q)}X,{}_{d(\rho_0)}X)$.

On the other hand, the assumption says that σ is nondegenerate and hence for each i with $0\leqq i\leqq q-1$ we have $F_K(\rho_i)<F(\rho_{i+1})$ and so $d(\rho_i)>d(\rho_{i+1})$. This implies the inequality $q\leqq d(\rho_0)-d(\rho_q)$.

These results and the hypothesis H_2) together assert $\pi_{q-1}\mathcal{A}({}_{'d(\sigma)}X,\ Y(\sigma))=0$, in other words we can form a map

$$f_\sigma:\ |\sigma|\times{}_{'d(\sigma)}X\longrightarrow{}'Y(\sigma)$$

satisfying the condition

$$f_\sigma\big||\partial\sigma|\times{}_{'d(\sigma)}X=f_{\partial\sigma}.$$

PROPOSITION 6.2. *For any simplex ρ in K, we choose a fixed equivalence $f(\rho)$: ${}_{d(\rho)}X\longrightarrow Y(\rho)$ in $\mathcal{A}$ whose existence is assured in the hypothesis* H_3).

Then there exists a normalized admissible map f: $'\tilde{K}\longrightarrow\tilde{Y}$ with carrier $'Y$ such that for each vertex $\sigma=\hat{\rho}$ in $'K^0$ with ρ in K, the diagram

$$\begin{tikzcd} |\sigma|\times{}_{'d(\sigma)}X \arrow[r, "f_\sigma"] \arrow[d] & 'Y(\sigma) \arrow[d, equal] \\ {}_{d(\rho)}X \arrow[r, "f(\rho)"] & Y(\rho) \end{tikzcd}$$

gets commutative where the left vertical arrow denotes the natural identification obtained by the projection.

Proof. We proceed by induction on the dimension q of σ.

If $q=0$, for each vertex σ we only need to define a map.

$$f_\sigma:\ |\sigma|\times{}_{'d(\sigma)}X\longrightarrow{}'Y(\sigma)$$

so as to render the above diagram commutative.

If $1 \leqq q$, we assume that a normalized admissible map f_{q-1}: $('\tilde{K})^{q-1} \longrightarrow \tilde{Y}$ with carrier $'Y$ satisfying the required condition. Let N_q be a complete set of representatives taken by one's in each congruence class of all nondegenerate q-simplices in $'K$.

Take a nondegenerate q-simplex σ in N_q. Clearly the restriction $f_{q-1}|\widetilde{\partial\sigma}$: $\widetilde{\partial\sigma} \longrightarrow \tilde{Y}$ is a normalized admissible map with carrier $'Y$. Hence from Proposition 6.1. there exists a $\mathcal{A}$-map

$$f_\sigma:\ |\sigma| \times_{'d(\sigma)} X \longrightarrow {}'Y(\sigma)$$

such that $k_\sigma \circ f_\sigma|\widetilde{\partial\sigma} = f_{q-1}|\widetilde{\partial\sigma}$ where k_σ: $'Y(\sigma) \longrightarrow \tilde{Y}$ denotes the inclusion. Gathering these maps, we obtain a normalized admissible map

$$f'_q:\ ('\tilde{K})^{q-1} \cup \tilde{N}_q \longrightarrow \tilde{Y}$$

with carrier $'Y$ such that $f'_q|('\tilde{K})^{q-1} = f_{q-1}$.

The argument in Proposition 4.1. allows us to define a normalized map f_q: $\tilde{K}^q \longrightarrow \tilde{Y}$ such that $f_q|('\tilde{K})^{q-1} \cup \tilde{N}_q = f'_q$. It remains to prove that this map is admissible and has the carrier $'Y$.

Since f'_q is an admissible map with carrier $'Y$ for each σ in $('K)^{q-1} \cup N_q$ there exists $\mathcal{A}$-map

$$f_\sigma:\ |\sigma| \times_{'d(\sigma)} X \longrightarrow {}'Y(\sigma)$$

such that $f'_q||\sigma| \times_{'d(\sigma)} X = k_\sigma \circ f_\sigma$ where k_σ is as above.

First we assume that σ is a degenerate q-simplex. Let η: $\sigma \longrightarrow \nu(\sigma)$ be a degeneracy with $\nu(\sigma)$ in K^{q-1}. The map η is a surjective morphism, so from the definition of $'d$ we have $'d(\nu(\sigma)) = {}'d(\sigma)$ and applying Lemma 5.1. we deduce $'Y(\nu(\sigma)) = {}'Y(\sigma)$. Hence we can define a map

$$f_\sigma:\ |\sigma| \times_{'d(\sigma)} X \longrightarrow {}'Y(\sigma)$$

by putting $f_\sigma = f_{\nu(\sigma)} \circ (\nu \times 1)$. Now it is obvious that f_σ is $\mathcal{A}$-map.

Next we consider a nondegenerate q-simplex σ not belonging to N_q. Let α: $\sigma \longrightarrow \alpha(\sigma)$ be a unique congruence such that $\alpha(\sigma)$ belongs to N_q. Again the map α is a surjective morphism, hence we obtain $'d(\alpha(\sigma)) = {}'d(\sigma)$ and $'Y(\alpha(\sigma)) = {}'Y(\sigma)$. We define a map

$$f_\sigma:\ |\sigma| \times_{'d(\sigma)} X \longrightarrow {}'Y(\sigma)$$

by putting $f_\sigma = f_{\alpha(\sigma)} \circ (\alpha \times 1)$. It is also an $\mathcal{A}$-map.

§ 7. Simple homotopy equivalence

The whole of assumptions adopted in the previous section will equally stand throughout this section too.

PROPOSITION 7.1. *Let f: $'\tilde{K} \longrightarrow \tilde{Y}$ be a normalized admissible map with carrier $'Y$ such that for each vertex $\sigma=\hat{\rho}$ in $'K$ with ρ in K, the diagram*

$$\begin{tikzcd} |\sigma| \times {}_{'d(\sigma)}X \arrow[r, "f_\sigma"] \arrow[d] & 'Y(\sigma) \arrow[d, equal] \\ {}_{d(\rho)}X \arrow[r, "f(\rho)"] & Y(\rho) \end{tikzcd}$$

becomes commutative where the vertical arrow is the projection, the top horizontal arrow denotes an $\mathcal{A}$-map and the bottom horizontal arrow means a prescribed equivalence in $\mathcal{A}$.

Then f gives rise to a simple homohopy equivalence.

Proof. First we remark that the proof can be reduced to the case $K=\Delta(I)$ with $I=K^0$.

If it is not the case, we take the standard simplex $\Delta(K^0)$ instead and extend the filtration F_K on $\Delta(K^0)$ by defining $F_K(\rho)=-\infty$ for ρ in $\Delta(K^0)$ not belonging to K. Further we agree to put ${}_{-\infty}X=\phi$. Finally we set $Y(\rho)=\phi$ for ρ not contained in K.

Now a map f: $'\tilde{K} \longrightarrow \tilde{Y}$ can be regarded as a map f: $'\Delta(\tilde{K}^0) \longrightarrow Y$ which has the required property if we adopt the convention that $f(\rho)$: ${}_{d(\rho)}X \longrightarrow Y(\rho)$ means nonsense for ρ not belonging to K.

Let us introduce some notations requisite for later discussions.

i) For a simplex ρ in $\Delta(I)$, we write $\Delta(\rho)$ for the simplicial closure of ρ and i_ρ: $\Delta(\rho) \longrightarrow \Delta(I)$ for the inclusion. Now i_ρ induces

$$F|\rho=F\circ i_\rho: \ \Delta(\rho) \longrightarrow Z \quad \text{defined by} \quad (F|\rho)(\rho')=F(\rho')$$

$$Y|\rho=Y\circ i_\rho: \ \Delta(\rho) \longrightarrow \mathring{\mathcal{A}} \quad \text{defined by} \quad (Y|\rho)(\rho')=Y(\rho').$$

Here we write $d|\rho=d\circ i_\rho$.

Clearly i_ρ derives the inclusion $'i_\rho$: $'\Delta(\rho) \longrightarrow {}'\Delta(I)$. Let σ' be a simplex in $'\Delta(\rho)$ with vertices $\hat{\rho}_0', \cdots, \hat{\rho}_q'$ such that $\rho_0' \prec \cdots \prec \rho_0'$. Then we have the identities

$$'(F|\rho)(\sigma')=(F|\rho)(\rho_q')=F(\rho_q')=('F\circ 'i_\rho)(\sigma')$$
$$'(Y|\rho)(\sigma')=(Y|\rho)(\rho_0')=Y(\rho_0')=('Y\circ 'i_\rho)(\sigma').$$

Moreover we use the notation $'(d|\rho)='d\circ 'i_\rho$.

ii) For a simplex ρ in $\Delta(I)$, $\lambda(\rho)$ denotes the link of ρ in $\Delta(I)$, more precisely $\lambda(\rho)$ is the simplex in $\Delta(I)$ satisfying the conditions $\lambda(\rho)\cap\rho=\phi$, $\lambda(\rho)*\rho=\Delta(I)$. As above $\Delta(\lambda(\rho))$ means the simplicial closure of $\lambda(\rho)$. We define $*_\rho$: $\Delta(\lambda(\rho)) \longrightarrow \Delta(I)$ to be the map obtained by joining with ρ. In other words, it is given by $*_\rho(\rho')=\rho'*\rho$ for ρ' in $\Delta(\lambda(\rho))$. This map provides us with the filtration and the contravariant functor

$$F*\rho=F\circ *_\rho:\ \Delta(\lambda(\rho))\longrightarrow Z \qquad \text{defined by} \qquad (F*\rho)(\rho')=F(\rho'*\rho)$$

$$Y*\rho=Y\circ *_\rho:\ \Delta(\lambda(\rho))\longrightarrow \mathring{\mathcal{A}} \qquad \text{defined by} \qquad (Y*\rho)(\rho')=Y(\rho'*\rho).$$

We adopt the convention here that write $d*\rho=d\circ(*_\rho)$.

Let $\hat{*}_\rho$: $'\Delta(\lambda(\rho))\longrightarrow{}'\Delta(I)$ be the simplicial map given as follows: Let σ' be a simplex in $'\Delta(\lambda(\rho))$ with vertices $\hat{\rho}'_0, \cdots, \hat{\rho}'_q$ such that $\rho'_0<\cdots<\rho'_q$. Then we define $(\hat{*}_\rho)(\sigma')$ to be the simplex with vertices $\widehat{(\rho'_0*\rho)}, \cdots, \widehat{(\rho'_q*\rho)}$.

Now we can deduce the following:

$$'(F*\rho)(\sigma')=(F*\rho)(\rho'_q)=F(\rho'_q*\rho)=('F\circ\hat{*}_\rho)(\sigma')$$

$$'(Y*\rho)(\sigma')=(Y*\rho)(\rho'_0)=Y(\rho'_0*\rho)=('Y\circ\hat{*}_\rho)(\sigma').$$

We put $'(d*\rho)={}'d\circ\hat{*}_\rho$.

Now we are getting down to the proof, assuming $K=\Delta(I)$.

If $\dim\Delta(I)=0$, the proof is trivial.

If $\dim\Delta(I)\geqq 1$, we assume that the proposition is proved whenever $K=\Delta(J)$ with $J\subsetneqq I$.

Suppose given a simplex ρ in $\Delta(I)$ whose dimension is less than $\dim\Delta(I)$. Then by the inductive hypothesis we can assume that the following assertions are true.

i) Let f': $\widetilde{'\Delta(\rho)}\longrightarrow\widetilde{Y|\rho}$ be a normalized admissible map with carrier $'(Y|\rho)$ such that for each vertex $\sigma'=\hat{\rho}'$ in $'\Delta(\rho)$ with ρ' in $\Delta(\rho)$ the diagram

$$\begin{tikzcd}
{|\sigma'|\times_{'(d|\rho)(\sigma')}X} \arrow[r, "f'_{\sigma'}"] \arrow[d] & {'(Y|\rho)(\sigma')} \arrow[d, equal] \\
{{}_{(d|\rho)(\rho')}X} \arrow[r, "f'(\rho')"] & {(Y|\rho)(\rho')}
\end{tikzcd}$$

becomes commutative where the vertical arrow and the horizontal arrows are as in Proposition 6.2. Then f' induces a simple homotopy equivalence.

Let f: $\widetilde{'\Delta(I)}\longrightarrow\tilde{Y}$ be a map given in the proposition. Then f gives rise to a map f': $\widetilde{'\Delta(\rho)}\longrightarrow\widetilde{Y|\rho}$ satisfying the property $f\circ\widetilde{'i_\rho}=k'\circ f'$ where k': $\widetilde{Y|\rho}\longrightarrow\tilde{Y}$ denotes the inclusion. f' inherits the properties from f which make f' a normalized admissible map with carrier $'(Y|\rho)$. Hence f' becomes a simple homotopy equivalence.

ii) Let f'': $\widetilde{\Delta(\lambda(\rho))}\longrightarrow\widetilde{Y*\rho}$ be a normalized admissible map with carrier $'(Y*\rho)$ such that for each vertex $\sigma'=\hat{\rho}'$ in $'\Delta(\lambda(\rho))$ with ρ' in $\Delta(\lambda(\rho))$ the diagram

$$\begin{tikzcd}
{|\sigma'| \times {}_{'(d*\rho)(\sigma')}X} \arrow[r, "f''_{\sigma'}"] \arrow[d] & {'(Y*\rho)(\sigma')} \arrow[d, equal] \\
{}_{(d*\rho)(\rho')}X \arrow[r, "f''(\rho')"] & (Y*\rho)(\rho')
\end{tikzcd}$$

becomes commutative where the vertical arrow and the horizontal arrows are as in Proposition 6.2.. Then f'' gives rise to a simple homotopy equivalence.

Let $f\colon \widetilde{'\Delta(I)} \longrightarrow \tilde{Y}$ be a map given in the proposition. Now f determines $f''\colon \widetilde{'\Delta(\lambda(\rho))} \longrightarrow \widetilde{Y*\rho}$ satisfying the property $f \circ \tilde{\ast}_\rho = k'' \circ f''$ where $k''\colon \widetilde{Y*\rho} \longrightarrow \tilde{Y}$ denotes the inclusion. f'' is shown to be a normalized admissible map with carrier $'(Y*\rho)$. Hence also in this case f'' is a simple homotopy equivalence.

We come to the final step in proving that f is a simple homotopy equivalence.

We assume that ρ is a vertex in $\Delta(I)$ and hence $\lambda(\rho)$ is a proper face of $\Delta(I)$ with the highest dimension.

Let us denote by N_ρ and $N_{\lambda(\rho)}$ the regular neighbourhoods of $\Delta(\rho)$ and $\Delta(\lambda\rho)$ respectively. So we have the collapsings

$$N_\rho \searrow {'\Delta(\rho)}, \qquad N_{\lambda(\rho)} \searrow {'\Delta(\lambda(\rho))}\,.$$

Further from the definition follows

$$N_\rho \cap N_{\lambda(\rho)} = \ast_\rho\, {'\Delta(\lambda(\rho))}, \qquad N_\rho \cup N_{\lambda(\rho)} = {'\Delta(I)}\,.$$

It is easily checked that $\tilde{N}_\rho$ and $\tilde{N}_{\lambda(\rho)}$ are the regular neighbourhood of $\widetilde{'\Delta(\rho)}$ and $\widetilde{'\Delta(\lambda(\rho))}$ respectively. Thus we have the collapsings

$$\tilde{N}_\rho \searrow \widetilde{'\Delta(\rho)}, \qquad \tilde{N}_{\lambda(\rho)} \searrow \widetilde{'\Delta(\lambda(\rho))}.$$

Now we have

$$\tilde{N}_\rho \cap \tilde{N}_{\lambda(\rho)} = \ast\, \widetilde{'\Delta(\lambda(\rho))}, \qquad \tilde{N}_\rho \cup \tilde{N}_{\lambda(\rho)} = \widetilde{'\Delta(I)}.$$

Here we note the observation in i) which yields the existence of simple homotopy equivalences

$$f'_\rho\colon \widetilde{'\Delta(\rho)} \longrightarrow \widetilde{Y|\rho}, \qquad f'_{\lambda(\rho)}\colon \widetilde{'\Delta(\lambda(\rho))} \longrightarrow \widetilde{Y|\lambda(\rho)}$$

satisfying the conditions $f \circ \widetilde{'i_\rho} = k_\rho \circ f'_\rho$, $f \circ \widetilde{'i_{\lambda(\rho)}} = k_{\lambda(\rho)} \circ f'_{\lambda(\rho)}$ respectively.

Further the arguments in ii) can be applied to show the existence of a simple homotopy equivalence

$$f'' : \ {}'\widetilde{\Delta(\lambda(\rho))} \longrightarrow \widetilde{Y*\rho}$$

satisfying the conditions $f \circ \tilde{\tilde{*}}_\rho = k'' \circ f''$.

On the other hand, we can easily verify

$$\widetilde{Y|\rho} \cap \widetilde{Y|\lambda(\rho)} = \widetilde{Y*\rho}, \qquad \widetilde{Y|\rho} \cup \widetilde{Y|\lambda(\rho)} = \tilde{Y}.$$

Finally we consider the following commutative diagram

$$\begin{tikzcd}
\widetilde{\Delta(\lambda(\rho))} \arrow[rr] \arrow[dr] \arrow[dd, "f''"'] & & \widetilde{N}_{\lambda(\rho)} \arrow[dr] \arrow[dd, "f'_{\lambda(\rho)}"] & \\
& \widetilde{N}_\rho \arrow[rr] \arrow[dd, "f'_\rho"] & & {}'\widetilde{\Delta(I)} \arrow[dd, "f"] \\
\widetilde{Y*\rho} \arrow[rr] \arrow[dr] & & \widetilde{Y|\lambda(\rho)} \arrow[dr] & \\
& \widetilde{Y|\rho} \arrow[rr] & & \tilde{Y}
\end{tikzcd}$$

where all the vertical arrows except the right extreme f give rise to simple homotopy equivalences.

Now using the sum formula of the Whitehead torsion, we can conclude that f induces a simple homotopy equivalence.

References

[1] Brieskorn, E., Sur les groupes de tresses. (d'aprés V.I. Arnold) *Séminaire Bourbaki* 24 *année*, 1971/2, *Springer Lecture Notes* No. **317**, (1973).

[2] Deligne, P., Les immeubles des groupes de tresse généralisés, *Invent. Math.*, **17**, 273–302 (1972).

[3] Hattori, A. and Kimura, T., On the Euler integral representations of hypergeometric functions in several variables, *J. Math. Soc. Japan.*, **26**, 1–16 (1974).

[4] Nakamura, T., On configurations, *Reports* (*Kōkyūroku*) *of Research Inst. Math. Sci. Kyoto Univ.* (1976) (in Japanese).

[5] Nakamura, T., A note on the $K(\pi, 1)$ property of the orbit space of the unitary reflection group $G(m, l, n)$, *Scientific Papers College Arts Sci. Univ. Tokyo.*, **33**, 1–6 (1983).

[6] Orlik, P. and Solomon, L., Combinatorics and topology of complements of hyperplanes, *Invent. Math.*, **56**, 167–189 (1980).

[7] Siebenmann, L. C., Infinite simple homotopy types, *Indag. Math.*, **32**, 479–495 (1970).