A Pushing up Theorem for Groups of Characteristic 2 Type

By Kensaku GOMI

Department of Mathematics, College of Arts and Sciences,
University of Tokyo

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Abstract

Let G be a finite group with $C_G(O_2(G)) \leq O_2(G)$ and S a Sylow 2-subgroup of G. Assume that S is contained in a unique maximal subgroup of G and that no nonidentity characteristic subgroup of S is normal in G. Then it will be shown that G is essentially equal to $LM\mathrm{wr}T$, where $L = \mathrm{SL}_2(2^m)$ or Σ_{2^e+1}, M is the natural $GF(2)L$-module, and T is a 2-group.

§ 1. Introduction

A finite group F is said to be of characteristic 2 type if it has even order and every 2-local subgroup G of F satisfies the condition $C_G(O_2(G)) \leq O_2(G)$. The program to classify the finite simple groups has been completed and in particular it has been shown that the nonabelian simple groups of characteristic 2 type are the simple Lie type groups of characteristic 2 plus certain isolated simple groups. In the course of the investigation to complete the classification and the subsequent investigation to revise the classification, several interesting questions have been raised. Among them is the question below. Note that if G is a group and S is its subgroup, then G has the largest normal subgroup contained in S. We shall denote it by $O_S(G)$, because it is equal to $O_p(G)$ if S is a Sylow p-subgroup of G.

QUESTION. Suppose F is a group and G_i ($i=1, 2, \ldots, n$) are finite subgroups of F which have a common 2-subgroup S such that $O_S(G_i) \neq 1$ for each i. Can we find a subgroup H of F which contains all the G_i and satisfies $O_S(H) \neq 1$?

Of course the answer is sometimes "yes" and at other times "no", and we wish to describe the structure of the G_i when the answer is "no". Several people have already investigated various special cases and their results have reasonably been called "pushing up" theorems. The proceedings of the Durham Conference [1] and of the Rutgers Conference [2] contain articles discussing certain pushing up theorems.

This paper is another contribution to the above question and concerned with the special case where $n=2$, S is a Sylow 2-subgroup of G_1, and S is normal in G_2. In this case, if some nonidentity characteristic subgroup C of S is normal in G_1, then we can push G_1 and G_2 up to a larger group $H=N_F(C)$. Thus we are led to the following:

PROBLEM. Suppose G is a finite group with $O_2(G)\neq 1$ and S is a Sylow 2-subgroup of G. Describe the structure of G when no nonidentity characteristic subgroup of S is normal in G.

In practice, we may place more restrictions on G in this problem. First, we may assume that $C_G(O_2(G))\leq O_2(G)$ because we have groups of characteristic 2 type in mind. Second, we may often assume that G is "small" in the sense that we make clear in the following:

DEFINITION. If a Sylow 2-subgroup S of a finite group G is contained in a unique maximal subgroup of G, then we say that G is 2-irreducible or that G is S-irreducible. In other words, G is S-irreducible if $S\neq G$ and G is not generated by any family of proper subgroups containing S.

In this paper, we restrict G to range over 2-irreducible groups in the above problem. Thus, we study the following situation:

HYPOTHESIS I. *The group G is a 2-irreducible group with $C_G(O_2(G))\leq O_2(G)$ and, for $S\in \mathrm{Syl}_2(G)$, no nonidentity characteristic subgroup of S is normal in G.*

In this case, Aschbacher's so-called (local) $C(G, T)$-theorem [3] has already shown that G is a product of S and blocks in $\mathscr{X}$, so our objective is to obtain more precise information. It should be noted that Aschbacher [4] and the author [5, 6] have improved the $C(G, T)$-theorem and that Baumann et al. considered Hypothesis I with the additional condition that $G/K\cong \mathrm{SL}_2(2^m)$, $m\geqq 1$, for some normal subgroup K of G (consult [1], [2]).

Hypothesis I actually contains too much restrictions on G. For the most part of our analysis, we shall only need the fact that none of certain three pre-assigned nonidentity characteristic subgroups of S is normal in G. Two of the characteristic subgroups are known ones, i.e., $\Omega_1(Z(S))$ and $Q(K(S))$ (consult [6] for the definition of the latter). The third one will be defined in this paper and denoted $R(S)$.

§ 2. The main result

The groups G of Hypothesis I will turn out to have a relatively simple structure, but we will describe it after rather lengthy definitions and remarks.

First, let P_n be the $GF(2)$-space of all n-dimensional row vectors with coefficients in $GF(2)$, $n \geqq 3$. The symmetric group Σ_n of degree n may be regarded as a group of linear transformations of P_n, i.e., we make each $\sigma \in \Sigma_n$ act on P_n according to the rule

$$(x_1, \dots, x_n)\sigma = (x_{1_\sigma{-1}}, \dots, x_{n_\sigma{-1}}).$$

Thus, we obtain a right $GF(2)\Sigma_n$-module P_n, which we call the natural permutation module for Σ_n over $GF(2)$. The module P_n has two distinct Σ_n-invariant subspaces T_n and U_n; T_n is the hyperplane defined by the equation

$$x_1 + x_2 + \cdots + x_n = 0,$$

and U_n is the line defined by the equations

$$x_1 = x_2 = \cdots = x_n.$$

By definition, the natural (or standard) $GF(2)\Sigma_n$-module is the quotient module

$$M_n = T_n + U_n / U_n.$$

(When we regard M_n as a module for the alternating group A_n, we call M_n the natural $GF(2)A_n$-module.) We denote by Δ_n the semidirect product of M_n by Σ_n:

$$\Delta_n = \Sigma_n M_n.$$

Let $\hat{\Sigma}_n$ be a representation group of Σ_n. Then an epimorphism $\hat{\Sigma}_n \longrightarrow \Sigma_n$ induces an action of $\hat{\Sigma}_n$ on M_n, and we can define the semidirect product

$$\Gamma_n = \hat{\Sigma}_n M_n.$$

This definition in general depends on the choice of $\hat{\Sigma}_n$, but a knowledge of the Schur multiplier of Σ_n shows that $\Gamma_3 \cong \Delta_3 \cong \Sigma_4$. The above definitions show that Γ_n is a central extension of Δ_n. In section 3, we will more precisely prove that if n is odd and $n \geqq 3$ then a central extension H of Δ_n is a central factor group of Γ_n defined by some $\hat{\Sigma}_n$ if and only if $[H, H] \geq Z(H)$ and the preimage of M_n in H is an elementary abelian 2-group.

Second, we can define the action of $SL_2(2^m)$ on the $GF(2^m)$-space V_m of 3-

dimensional row vectors with coefficients in $GF(2^m)$ as follows:

$$(x, y, z)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+cy, bx+dy, \sqrt{ab}x+\sqrt{cd}y+z).$$

Further, as there is an epimorphism $SL_2(5) \longrightarrow SL_2(4)$, the action of $SL_2(4)$ on V_2 induces an action of $SL_2(5)$ on V_2. Thus, we can construct the semidirect product

$$R_m = \begin{cases} SL_2(2^m)V_m & \text{if } m \geqq 3, \\ SL_2(5)V_2 & \text{if } m=2. \end{cases}$$

The group R_m may be characterized as follows. The matrix group $SL_2(2^m)$ acts, by right multiplication, on the set N_m of all 2-dimensional row vectors with coefficients in $GF(2^m)$, and thus N_m becomes the so-called natural (or standard) $GF(2)SL_2(2^m)$-module (of course, N_m is also a $GF(2^m)SL_2(2^m)$-module). By definition, the quadratic group over $GF(2^m)$ is the semidirect product

$$Q_m = SL_2(2^m)N_m .$$

It is easy to see that R_m is a central extension of Q_m. In section 4, we will more precisely prove that if $m \geqq 2$ then R_m is a representation group of Q_m. Note that if $m \geqq 2$ then Q_m is perfect and so a representation group of Q_m is nothing but a universal perfect central extension of Q_m.

Finally, for finite groups G, $J(G)$ is the subgroup generated by the set $\mathcal{A}(G)$ of all elementary abelian 2-subgroups of maximal order.

Now, we can state our main result.

THEOREM. *Suppose G is a finite group satisfying Hypothesis* I *and S is a Sylow 2-subgroup of G. Then $G=SJ(G)$ and one of the following holds:*

(1) *$J(G)$ is a central product, $J(G)=TUH_1H_2 \dots H_k$, of 2-groups T, U and the S-conjugates H_i, $1 \leqq i \leqq k$, of a subgroup H, where H is a central factor group of $\Gamma_{2^\ell+1}$, $\ell \geqq 1$, and T, U satisfy the following conditions:*

(a) *$U^2 \leq Z(H_1H_2 \dots H_k)$ and $\Omega_1(U) \leq T$;*
(b) *U is abelian of exponent at most 4;*
(c) *T is the direct product of copies of D_8 and copies of Z_2.*

(2) *$J(G)$ is a central product, $J(G)=TH_1H_2\dots H_k$, of a 2-group T and the S-conjugates H_i, $1\leqq i\leqq k$, of a subgroup H, where H is a central factor group of R_m, $m\geqq 2$.*

In this theorem, if some additional conditions on G force $Z(H)=1$, then the structure of $J(G)$ is easily determined. First, if $Z(H)=1$ in (1), then $U=\Omega_1(U)\leq T$ by (a) and so $J(G)$ is the direct product of k copies of $\Delta_{2^\ell+1}$, copies of D_8, and copies of Z_2 by (c). If $\ell=1$ in (1), then we necessarily have $Z(H)=1$ because $\Gamma_3\cong\Delta_3\cong\Sigma_4$. Therefore, we obtain the following:

COROLLARY. *Suppose G is a solvable finite group satisfying Hypothesis* I *and S is a Sylow 2-subgroup of G. Then $G=SJ(G)$ and $J(G)$ is the direct product of one or more copies of Σ_4, copies of D_8, and copies of Z_2.*

If $Z(H)=1$ in (2), then the Krull-Remak-Schmidt theorem yields that $J(G)$ is the direct product of k copies of Q_m, copies of a Sylow 2-subgroup of Q_m, and copies of Z_2.

The proof of the theorem begins at section 5 and ends at section 8. In section 5, we study $GF(2)$-representations of 2-irreducible groups. Recall that if G is a finite group and V is a faithful $GF(2)G$-module, then $\mathcal{P}(G, V)$ denotes the set of all nonidentity elementary abelian 2-subgroups A of G such that $|A||C_V(A)|\geqq|B||C_V(B)|$ for all subgroups B of A. We are interested in the following situation:

HYPOTHESIS II. *The group G is a 2-irreducible group with $O_2(G)=1$ and V is a faithful right $GF(2)G$-module with $\mathcal{P}(G, V)$ nonempty and $C_V(S)\nleq C_V(G)$ for $S\in\mathrm{Syl}_2(G)$.*

Under this hypothesis, we will show that the structures of both G and V are highly restricted. In this analysis, work of Aschbacher is very helpful. In particular, we will use Theorem 2 of [7], which implies that we appeal to the classification of the finite simple groups. However, since we aim to apply our theorem to the revision of the classification, we will first suppose that the classification has not yet been completed, and prove all our results in sections 5-8 for K-groups, i.e., finite groups all of whose simple sections are "known" simple groups. To be more precise, we will assume Hypothesis 0 given in section 5, which every K-group satisfies by Aschbacher's theorem. Then we obtain our theorem by appealing to the classification and Aschbacher's theorem.

We conclude this section with a remark on our terminology and notation (we have already defined some of them). Suppose G is a finite group and V is a right $GF(2)G$-module. Then V^* denotes the dual (or contragredient) right $GF(2)G$-module, and for subspaces W of V, $W^\perp$ denotes the annihilator (or orthogonal space) of W in V^*. Also, we define $V(G)=[V, G]/C_{[V,G]}(G)$. Suppose H is a finite group and there exists a monomorphism $\rho: G\longrightarrow H$. Let U be a right $GF(2)H$-module. The $GF(2)G$-module V is said to be induced by U

through ρ if there exists an isomorphism $\sigma: V \longrightarrow U$ of $GF(2)$-spaces such that $(vg)^{\sigma}=v^{\sigma}g^{\rho}$ for each $v \in V$ and $g \in G$.

§ 3. Symmetric groups

In this section, G is the symmetric group on the set $\Omega=\{1, 2, \ldots, n\}$, L is the alternating group on Ω, R is a Sylow 2-subgroup of G, and $S=R \cap L$. We will record here various facts concerning G, L, and their natural module M_n.

3.1. *Let E be an elementary abelian 2-subgroup of G of maximal order. Then the following hold:*

(1) $|E|=2^{[n/2]}$, *where* [ ] *is the Gaussian symbol;*

(2) *The length of each E-orbit on Ω is at most* 4, *and E has at most one fixed point;*

(3) $C_G(E)=E$.

Proof. First, assume that E is transitive on Ω and $n \geqq 2$. Then E is regular on Ω, so $|E|=n$ and $C_G(E)=E$. Since G contains an elementary abelian subgroup of order $2^{n/2}$ generated by transpositions, we have $n \geqq 2^{n/2}$. Therefore, $n=2^{n/2}=4$ or 2, and (1), (2), (3) hold. Next, let $O_1, O_2, \ldots, O_r$ be the E-orbits on Ω and assume $r>1$. Let H be the setwise stabilizer of $O_1, \ldots, O_r$, and G_i be the pointwise stabilizer of $\Omega-O_i$. Then $E \leq H=G_1 \times \cdots \times G_r$ and so the maximality of $|E|$ yields that E is the direct product of the projections E_i of E on G_i, and that every elementary abelian 2-subgroup of G_i has order at most $|E_i|$. Since G_i is the symmetric group on O_i and E_i is transitive on O_i, the previous discussion shows that

(i) $|E_i|=2^{[|O_i|/2]}$,

(ii) $|O_i|=4$, 2, or 1, and

(iii) $C_{G_i}(E_i)=E_i$

for each i. If $O_j=\{p_j\}$ and $O_k=\{p_k\}$, $j \neq k$, then the transposition (p_j, p_k) and E generate an elementary abelian 2-group larger than E, which is a contradiction. Therefore, $|O_i|=1$ for at most one i. Consequently, we have $|E|=2^{[n/2]}$. If $|O_j| \neq 1$, then O_j is the unique E_j-orbit on Ω of length greater than 1. Therefore, $C_G(E)$ leaves each O_i invariant, and $C_G(E)=C_H(E)=C_{G_1}(E_1) \ldots C_{G_r}(E_r)=E_1 \ldots E_r =E$. This completes the proof.

3.2. *If n is a power of* 2, *then $Z(S)$ is contained in a regular elementary abelian subgroup of G.*

Proof. We may assume $n \geqq 4$. The R contains a regular elementary abelian subgroup E. Since $n/2$ is even, we have $E \leq S$ and hence $Z(S) \leq C_G(E)=E$.

3.3. *Assume $n \equiv 0$ or* 1 (*mod* 4) *and $n \geqq 4$. Then we have $C_G(S)=Z(S)$.*

Proof. The assumption implies that S contains an elementary abelian subgroup E of order $2^{[n/2]}$. Since $C_G(E)=E$ by 3.1, we have $C_G(S)\leq S$, and hence $C_G(S)=Z(S)$.

3.4. *Assume that n is odd and $n\geqq 3$. Let T be a set of $(n-1)/2$ disjoint transpositions in G. Then G has an element g such that $G=\langle T\cup T^g\rangle$.*

Proof. We may assume that $n=2m+1$ and T consists of the transpositions $(2i-1, 2i)$, $1\leqq i\leqq m$. Let g denote the cycle $(1, 3, 5, \dots, 2m-1, 2m+1)$. Then $T\cup T^g$ consists of the transpositions $(i, i+1)$, $1\leqq i\leqq 2m$, and hence $G=\langle T\cup T^g\rangle$.

3.5. *Assume that n is odd and $n\geqq 3$. Let E be a subgroup of G and r the number of the E-orbits on Ω. Then* $\dim M_n/C_{M_n}(E)=n-r$. *In particular, if t is the product of $(n-1)/2$ disjoint transpositions, then* $\dim M_n/C_{M_n}(t)=(n-1)/2$.

Proof. We use the notation of section 2. Take the basis of the natural permutation module P_n consisting of the unit vectors $e_i=(0, \dots, 0, 1, 0, \dots, 0)$, where 1 lies in the i-th position, $1\leqq i\leqq n$. If $O_1, O_2, \dots, O_r$ are the E-orbits on Ω, then the r vectors $f_j=\sum e_i$ $(i\in O_j)$ form a basis of $C_{P_n}(E)$. Now $U_n\leq C_{P_n}(E)$ and since n is odd, we have $P_n=T_n\oplus U_n$. Therefore, $\dim T_n/C_{T_n}(E)=n-r$ and $M_n\cong T_n$. This completes the proof.

3.6. *Assume that n is odd and $n\geqq 3$. Let T be a group generated by $(n-1)/2$ disjoint transpositions in G. Then the semidirect product TM_n is isomorphic to the direct product of $(n-1)/2$ copies of D_8.*

Proof. We use the notation of 3.5. Let $n=2m+1$ and assume, without loss of generality, that T is generated by the transpositions $t_i=(2i-1, 2i)$, $1\leqq i\leqq m$. Identify M_n with P_n/U_n and, in the semidirect product TM_n, define $T_i=\langle t_i, e_{2i-1}+U_n\rangle$, $1\leqq i\leqq m$. Then we have $T_i\cong D_8$ and $TM_n=T_1\times\cdots\times T_m$.

3.7. *If n is odd and $n\geqq 3$, then* $H^1(L, M_n)\cong H^1(G, M_n)=0$.

Proof. We consider only the case $n\geqq 5$, but when suitably modified, the following argument works in the case $n=3$ as well. Since n is odd, L is generated by the two cycles $\alpha=(1, 2, 3)$ and $\beta=(3, 4, \dots, n)$. Suppose V is a $GF(2)L$-module which contains M_n as a $GF(2)L$-submodule and satisfies $[V, L]\leq M_n$. Since α has odd order, we have $V=M_n+C_V(\alpha)$, and hence $\operatorname{codim} C_V(\alpha)=2$ by 3.5. Similarly, we have $\operatorname{codim} C_V(\beta)=n-3$. Hence, $\operatorname{codim} C_V(L)\leqq n-1=\dim M_n$ and, since $C_{M_n}(L)=0$, we conclude that $V=M_n\oplus C_V(L)$. This implies that $H^1(L, M_n)=0$. Suppose U is a $GF(2)G$-module which contains M_n as a $GF(2)G$-submodule and satisfies $[U, G]\leq M_n$. Then $U=M_n\oplus C_{M_n}(L)$ by the above, and since $C_{M_n}(L)$ is G-invariant, we have $C_{M_n}(L)=C_{M_n}(G)$. Therefore, $H^1(G, M_n)=0$.

3.8. *M_n $(n\geqq 3)$ is self-dual, i.e., $M_n^*\cong M_n$ as $GF(2)G$-modules.*

Proof. Let P_n be the natural permutation module for G over $GF(2)$. Since permutation matrices are orthogonal, it follows that $P_n^* \cong P_n$. Hence $M_n^* \cong P_n(G)^* \cong P_n^*(G) \cong P_n(G) \cong M_n$.

3.9. *If n is odd and $n \geqq 3$, then every extension of M_n by Σ_n splits.*

Proof. See [8].

3.10. *Let H be a finite group and K a normal subgroup with $C_H(K)=1$. Assume that $K/O_2(K) \cong A_n$, n odd $\geqq 3$, that $O_2(K)$ is elementary abelian, and that, when considered a $GF(2)(K/O_2(K))$-module, $O_2(K)$ is induced by M_n through an isomorphism $K/O_2(K) \longrightarrow A_n$. Then the following hold*:

(1) *$H/O_2(K) \cong A_n$ or Σ_n;*

(2) *When considered a $GF(2)(H/O_2(K))$-module, $O_2(K)$ is induced by M_n through a monomorphism $H/O_2(K) \longrightarrow \Sigma_n$;*

(3) *Let E be an elementary abelian 2-subgroup of H. Then $|E| \leqq 2^{n-1}$. If $|E|=2^{n-1}$ and $E \neq O_2(K)$, then $H/O_2(K) \cong \Sigma_n$ and $H=EK$.*

Proof. If $n \neq 3$, define $C=C_H(K/O_2(K))$, while if $n=3$, define $C=C_H(O_2(K))$. Then C always centralizes both $K/O_2(K)$ and $O_2(K)$, because $C_K(O_2(K))=O_2(K)$ and, if $n \neq 3$ and $K_i/O_2(K)$ $(i=1, 2, \ldots, n)$ are the distinct subgroups of $K/O_2(K)$ of index n, then $C_{O_2(K)}(K_i)$'s have order 2 and generate $O_2(K)$. Since $C_H(K)=1$, it follows that C is a 2-group. If $O_2(K) < B \leq C$ with $|B:O_2(K)|=2$, then B is elementary abelian and then $C_B(K) \neq 1$ by 3.7, a contradiction. Thus, $C=O_2(K)$ and hence (1) holds. (2) now follows from 3.4 of [9].

In order to prove (3), let E be an elementary abelian 2-subgroup of H of maximal order, and assume $E \neq O_2(K)$. Then we have $|E| \geqq |O_2(K)|=2^{n-1}$ and hence

$$\begin{aligned}|EO_2(K)/O_2(K)| &= |E:E \cap O_2(K)| \\ &\geqq |O_2(K):E \cap O_2(K)| \\ &\geqq |O_2(K):C_{O_2(K)}(E)|.\end{aligned}$$

3.1 of [9] now shows that $H/O_2(K) \cong \Sigma_n$, that $EO_2(K)/O_2(K)$ is generated by transpositions in $H/O_2(K)$, and that

$$|EO_2(K)/O_2(K)| = |O_2(K):C_{O_2(K)}(E)|.$$

Hence (3) follows.

3.11. *Let H be a central extension of $\Delta_n = \Sigma_n M_n$, n odd $\geqq 3$. Assume that*

$H' \geq Z(H)$ and that the preimage of M_n in H is an elementary abelian 2-group. Then H is isomorphic to a central factor group of Γ_n.

Proof. First, $H/Z(H) \cong \Delta_n$ because $Z(\Delta_n)=1$. Let V and K be the preimages of M_n and Σ_n in H, respectively. Regard V as a $GF(2)(H/V)$-module and consider the dual module V^*. Since $V/Z(H)$ is a natural module for $H/V \cong \Sigma_n$, $Z(H)^{\perp}$ is the dual of a natural module, and so $Z(H)^{\perp}$ is a natural module by 3.8. Also, since $Z(H)=C_V(H/V)$, we have $Z(H)^{\perp}=[V^*, H/V]$. Thus, 3.7 yields

$$\begin{aligned} V^* &= Z(H)^{\perp} + C_{V^*}(H/V) \\ &= Z(H)^{\perp} + [V, H/V]^{\perp} \\ &= (Z(H) \cap [V, H/V])^{\perp}. \end{aligned}$$

This implies that $Z(H) \cap [V, H]=1$ and hence $V=Z(H) \times [V, H]$. Therefore, H is a semidirect product of $[V, H]$ by K, and $[V, H]$ is a natural module for $K/Z(H) \cong \Sigma_n$. Now, since $H'=K'[V, H]$, we have $K' \geq Z(H)$ and so, if $Z(H) \neq 1$, K is a representation group of Σ_n. This completes the proof.

§ 4. Special linear groups $SL_2(2^m)$

In this section, $L=SL_2(2^m)$, $m \geqq 1$, $\Gamma=\mathrm{Aut}\ GF(2^m)$, and $G=\Gamma L$ is the natural semidirect product. Embed L and Γ into G, take $R \in \mathrm{Syl}_2(G)$, and define $S=R \cap L$. We will record here various facts concerning G, L, and the natural module N_m for L. Note that $G \cong \mathrm{Aut}\, L$.

4.1. *Let E be an elementary abelian 2-subgroup of G of maximal order. Then* $|E|=2^m$ and $C_G(E)=E$.

Proof. If $m=1$ or 2, then $G \cong \Sigma_3$ or Σ_5 and the assertion is proved in 3.1. Therefore, assume $m>2$. We may assume that S consists of the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \in GF(2^m)$, and $R=\Phi S$, where $\Phi \in \mathrm{Syl}_2(\Gamma)$. Suppose there is an involution t in $R-S$. Then Φ contains a unique involution ϕ and $t \in \phi C_S(\phi)$ because S is elementary abelian. Since $|C_S(\phi)|=|S:C_S(\phi)|$, we conclude that t is conjugate to ϕ in R. Thus, if $E \not\leq L$, then we may assume $\phi \in E$. But then $E \leq \langle \phi \rangle C_L(\phi)$ and $|E| \leqq 2^{m/2}2 < 2^m=|S|$, a contradiction. Therefore, $E \leq L$ and $|E|=|S|=2^m$. In proving $C_G(E)=E$, we may assume $E=S$. We have $N_G(S)=\Gamma N_L(S)$ and $N_L(S)=HS$, where H is the group of matrices $\begin{pmatrix} h^{-1} & 0 \\ 0 & h \end{pmatrix}$, $h \in GF(2^m)^{\times}$. Hence $C_L(S)=S$ and so $[C_G(S), N_L(S)] \leq S$. Also, since Γ acts faithfully on H, we have $C_G(N_L(S)/S) = N_L(S) \leq L$. Therefore, $C_G(S)=C_L(S)=S$.

4.2. *L has an element g such that* $L=\langle S, S^g\rangle$.

Proof. If S consists of the matrices $\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$, $a\in GF(2^m)$, then $g=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ meets our requirement.

4.3. (1) *If* $1\neq x\in S$, *then* $|C_{N_m}(x)|=2^m$.
(2) $|[N_m, S]|=2^m$.
(3) *L transitively acts on* $N_m-\{(0, 0)\}$.

Proof. We may assume that S consists of the matrices $\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$, $a\in GF(2^m)$. Then both $C_{N_m}(x)$ $(1\neq x\in S)$ and $[N_m, S]$ consist of the vectors $(0, y)$, $y\in GF(2^m)$. Thus, (1) and (2) hold. The stabilizer in L of the vector $(0, 1)\in N_m$ is equal to S and hence (3) follows.

4.4. *If* $m\geqq 2$, *then* $|H^1(L, N_m)|=2^m$ (*consult* 3.7 *for the case* $m=1$).

Proof. Suppose V is a $GF(2^m)L$-module which contains N_m as a $GF(2^m)L$-submodule and satisfies $[V, L]\leq N_m$. If t is an involution of L, then $[V, t]\leq C_{N_m}(t)$ and so $|V:C_V(t)|=|[V, t]|\leqq 2^m$ by 4.3. Since L is generated by the three involutions $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, and $\begin{pmatrix}0 & h^{-1}\\ h & 0\end{pmatrix}$, where h is a generator of $GF(2^m)^\times$, we conclude that $|V:C_V(L)|\leq 2^{3m}$. Now, $C_{N_m}(L)=0$ by 4.3, so $|V:N_m+C_V(L)|$ is bounded from above by 2^m. It remains to prove that this upper bound is attained. Let U be the $GF(2^m)$-space of 3-dimensional row vectors with coefficients in $GF(2^m)$, and define the action of L on U by

$$(x, y, z)\begin{pmatrix}a & b\\ c & d\end{pmatrix}=(x, \sqrt{ac}x+ay+cz, \sqrt{bd}x+by+dz).$$

We can verify that this action of L on U makes U a right $GF(2^m)L$-module (U is the dual of the $GF(2^m)L$-module V_m defined in section 2). It is clear that the vectors $(0, y, z)$ of U form a $GF(2^m)L$-submodule, W, isomorphic to N_m and that $[U, L]\leq W$. Furthermore, since $m\geqq 2$, we can easily deduce that $C_U(L)=0$. Thus, $|U: W+C_U(L)|=2^m$ and the proof is complete.

4.5. *If V is an irreducible GF(2)L-module with* $|V|=2^{2m}$ *and* $|V:C_V(S)|\leqq 2^m$, *then* $V\cong N_m$.

Proof. This is a special case of a much more general result, e.g., 2.1 of [9]. A very elementary proof was given in (1K) of [10].

4.6. N_m *is a self-dual GF(2)L-module.*

Proof. By 4.3, we can apply 4.5 to $V=N_m^*$ and conclude that $N_m^*\cong N_m$.

REMARK. N_m is self-dual also as a $GF(2^m)L$-module.

4.7. *If $m\geqq 2$, then R_m is a representation group of Q_m.*

Proof. Since $m\geqq 2$, we have $[L, L]=L$. We have remarked in the proof of 4.4 that $C_{V_m^*}(L)=0$. Hence $V_m=[V_m, L]$ and R_m is perfect. Since R_m is a central extension of Q_m, it therefore suffices to prove that the order of an arbitrary perfect central extension H of Q_m does not exceed that of R_m. To this end, take an epimorphism $\pi: H\longrightarrow Q_m$ with $\operatorname{Ker}\pi\leq Z(H)$ and define $Z=\operatorname{Ker}\pi$ (in fact, $Z=Z(H)$). Since Sylow p-subgroups of Q_m are cyclic for all odd primes p, Z is a 2-group, and so is the preimage V of N_m in H. Furthermore, if X is an arbitrary maximal subgroup of Z, then since $|V/Z|=2^{2m}>4$ and H acts transitively on $V/Z-\{1\}$ by 4.3, we have that V/X is elementary abelian. Hence V is elementary and we can regard V as a $GF(2)(H/V)$-module. Denote by V^* its dual module. Then since V/Z is a natural module for $H/V\cong \mathrm{SL}_2(2^m)$ and $C_V(H/V)=Z$ by 4.3, $Z^\perp$ is also a natural module by 4.6 and $[V^*, H/V]=Z^\perp$. Thus, $|V^*: Z^\perp+C_{V^*}(H/V)|\leqq 2^m$ by 4.4. Also, since $H/[V, H]$ is a perfect central extension of $\mathrm{SL}_2(2^m)$, a knowledge of the Schur multiplier of $\mathrm{SL}_2(2^m)$ shows that $|C_{V^*}(H/V)|=|V:[V, H]|\leqq 2$ with equality only when $m=2$. Therefore, $|V|=|V^*|\leqq 2^{3m+1}$ with equality only when $m=2$. This shows that $|H|\leqq|R_m|$, as required.

§ 5. $GF(2)$-representations of 2-irreducible groups

The following hypothesis was stated in section 2.

HYPOTHESIS II. *The group G is a 2-irreducible group with $O_2(G)=1$ and V is a faithful right $GF(2)G$-module with $\mathcal{P}(G, V)$ nonempty and $C_V(S)\nleq C_V(G)$ for $S\in \mathrm{Syl}_2(G)$.*

In this section, we will give a description of the structure of the pair G, V of this hypothesis under an additional hypothesis satisfied by all K-groups. In order to state it, we define Q to be the set of all quadruples (H, W, A, K) of a finite group H, a faithful $GF(2)H$-module W, a nonidentity elementary abelian 2-subgroup A of H such that $|A|\geqq|W: C_W(A)|$, and a quasisimple normal subgroup K of H such that $C_H(K)=Z(K)=O(K)$ and $H=AK$. The additional hypothesis is the following:

HYPOTHESIS 0. *If $(H, W, A, K)\in Q$ and $K\cong L/M$ for some subgroup L of the group G and a normal subgroup M of L, then either $K\in$ Chev$(2)-\{(\mathrm{P})\mathrm{SU}_3(2^m), \mathrm{Sz}(2^{2m-1})\mid m\geqq 2\}$ or $K\cong A_n$, $n\geqq 7$.*

Here, Chev (2) denotes the set of all (isomorphism classes of) quasisimple groups K with $O_2(K)=1$ and $K/Z(K)$ isomorphic to a simple group of Lie type and characteristic 2 (we consider the groups A_6, $PSU_3(3)$, and ${}^2F_4(2)'$ to be of Lie type and characteristic 2). We note that all K-groups satisfy Hypothesis 0 [7, 11].

Under Hypotheses 0 and II, let $\mathcal{P}^*(G, V)$ be the set of all minimal elements of $\mathcal{P}(G, V)$ under the partial order $\leqq_{(V)}$ defined as follows: $A\leqq_{(V)}B$ if and only if $A\leq B$ and $|A||C_V(A)|=|B||C_V(B)|$. Furthermore, let $\mathcal{P}_0^*(G, V)$ be the set of all elements of $\mathcal{P}^*(G, V)$ contained in $O_{2',2}(G)$, and let $\mathcal{P}_1^*(G, V)$ be the complementary set of $\mathcal{P}_0^*(G, V)$ in $\mathcal{P}^*(G, V)$. We can now describe the structure of G.

5.1. *Define a subgroup N of G by*

$$N=\begin{cases}\langle\mathcal{P}_0^*(G, V)\rangle & \textit{if } \mathcal{P}_0^*(G, V) \textit{ is nonempty,}\\ E(\langle\mathcal{P}_1^*(G, V)\rangle) & \textit{if } \mathcal{P}_0^*(G, V) \textit{ is empty.}\end{cases}$$

Then $G=SN$, $C_G(N)=1$, and N is the direct product of one or more subgroups $L_1, L_2, \dots, L_k$ with $L_1^G=\{L_1, L_2, \dots, L_k\}$ and $L_1\cong SL_2(2^m)$ $(m\geqq 1)$ or $L_1\cong A_{2m+1}$ $(m=2^r\geqq 2)$.

Proof. By 1.11 and 1.14 of [9], N is a central product of one or more subgroups $L_1, L_2, \dots, L_k$ and either $L_i\cong SL_2(2)$ for each i or L_i is quasisimple for each i. Of course, if $L_i\cong SL_2(2)$, then N is the direct product of the L_i and hence $L_i=C_N(O^{2'}(C_N(L_i')))$. The components L_i are, therefore, uniquely determined by the Krull-Remak-Schmidt theorem and, consequently, G permutes the L_i by conjugation because $N\lhd G$.

If $H\lhd G$, then $G=N_G(S\cap H)SH$ by a Frattini argument and so, since $O_2(G)=1$, the 2-irreducibility of G shows that either $H\leq O(G)$ or $SH=G$. Hence

$$G=SN.$$

Similarly, we have $G=S\langle L_1^G\rangle$ and hence

$$L_1^G=L_1^S=\{L_1, L_2, \dots, L_k\}.$$

Since $G=SN$ and $O_2(G)=1$, we have

$$C_G(N)=Z(N)\leq O(N).$$

Therefore, we may assume from now on that L_1 is quasisimple.

Since $\langle\mathscr{P}(G, V)\rangle\nleq C_G(L_1)$ by the above remarks, 4.1 of [9] shows that there exists an element (H, W, A, K) of Q such that K is a homomorphic image of L_1. The structure of K is restricted by Hypothesis 0, so a knowledge of the Schur multipliers of the relevant simple groups (e.g., [12]) and 4.2 of [9] show that either $L_1\cong A_n$, $n\geqq 7$, or $L_1\in \mathrm{Chev}(2)-\{(\mathrm{P})\mathrm{SU}_3(2^m), \mathrm{Sz}(2^{2m-1})\mid m\geqq 2\}$. Consequently, G is 2-isolated only if $L_1\cong \mathrm{SL}_2(2^m)$, $m\geqq 2$.

We interrupt the proof with a lemma which gives a criterion for the 2-irreducibility. For finite groups G, we define $\mathscr{H}_{0,2,G}$ to be the set of all nonidentity 2-subgroups T of G such that $N_G(T)/T$ is 2-isolated. For $T\in\mathscr{H}_{0,2,G}$, we define $N_G^*(T)$ to be the subgroup of $N_G(T)$ containing T such that $N_G^*(T)/T$ is the unique minimal subnormal subgroup of $N_G(T)/T$ of even order.

LEMMA. *Let G be a finite group which is neither 2-isolated nor 2-closed. Assume that, for each $T\in\mathscr{H}_{0,2,G}$, $N_G^*(T)$ is contained in a proper subgroup of G of odd index. Then G is not 2-irreducible.*

Proof. Suppose G is 2-irreducible. Take $T\in\mathrm{Syl}_2(G)$ and let M be the unique maximal subgroup of G containing T. Then the conjugates of M control Sylow 2-intersections in G by 1.7 of [5] and so G is generated by $N_G(T)$ and the conjugates of M containing T by 1.5 of [5]. However, this implies that $G=M$, a contradiction.

Let us return to the proof of 5.1. Since G is 2-irreducible, the above lemma and the main theorems of [13] show that one of the following holds (see also [14]):

(1) $L_1\cong A_{2^\ell+1}$, $\ell\geqq 2$;

(2) $L_1\cong \mathrm{SL}_2(2^m)$, $m\geqq 2$;

(3) $L_1\cong(\mathrm{P})\mathrm{SL}_3(2^m)$, $\mathrm{Sp}_4(2^m)'$, or $\hat{A}_6$ and some element of $N_S(L_1)$ interchanges, by conjugation, the two nontrivial parabolic subgroups of L_1 containing $S\cap L_1$. Here, $\hat{A}_6$ is the perfect central extension of A_6 by Z_3 and by "parabolic subgroups," we mean subgroups containing a Sylow 2-normalizer.

Now, since $C_V(S)\nleq C_V(G)$ and $G=S\langle L_1^S\rangle$, we have $[C_V(S), L_1]\neq 0$. Also, since $C_G(N)\leq O(N)$, we have $[L_1, \langle\mathscr{P}^*(S, V)\rangle]\neq 1$. Therefore, case (3) does not occur by 4.7 of [9], and L_1 has the desired structure. In particular, $Z(L_1)=1$ so $C_G(N)=1$ and $N=L_1\times\cdots\times L_k$. The proof of 5.1 is complete.

The letters used in 5.1 will retain their meaning for the remainder of this section. Thus, m is the integer such that $L_1\cong\mathrm{SL}_2(2^m)$ or A_{2m+1}, and k is the number of the G-conjugates of L_1.

5.2. *If* $1\neq X\lhd G$, *then* $O^2(G)\leq X$.

Proof. Since $G=SN$ and $N\lhd G$, we have $O^2(G)\leq N$ and in particular $O^2(X)$ normalizes each L_i. Also, since $O_2(G)=1$ and $C_G(N)=1$, we have $[N, O^2(X)]\neq 1$. Thus, $1\neq[L_i, O^2(X)]\lhd L_i$ and hence $O^2(L_i)\leq[L_i, O^2(X)]\leq X$ for some i. Since the L_j, $1\leqq j\leqq k$, are all conjugate and $O^2(G)=O^2(L_1)\ldots O^2(L_k)$, we conclude that $O^2(G)\leq X$.

We will next describe the action of G on V. To this end, we define

$$W=[V, N],$$

$$W_i=[V, L_i], \qquad 1\leqq i\leqq k,$$

and denote by $\sim$ the natural homomorphism $W\longrightarrow W/C_W(N)$.

5.3. (1) W *is* G-*invariant and* $W\neq 0$.

(2) *If* $g\in G$ *and* $L_i^g=L_j$, *then* $W_ig=W_j$.

(3) $W=W_1+W_2+\cdots+W_k$.

(4) $W_i=[W_i, O^2(L_i)], \quad 1\leqq i\leqq k$.

(5) $W=[W, O^2(N)]$.

(6) $C_G(W)=1$.

Proof. (1), (2), and (3) immediately follow from the definition. In proving (4), we may assume $L_i\cong SL_2(2)$, in which case $|W_i|=4$ by 1.11 of [9] and (4) clearly holds. Finally, (5) is a consequence of (4), and (6) follows from 5.2 and (5).

5.4. (1) *When considered a module for* L_1, $\tilde{W}_1$ *is either induced by the natural* $GF(2)A_{2m+1}$-*module through an isomorphism* $L_1\longrightarrow A_{2m+1}$, *or induced by the natural* $GF(2)SL_2(2^m)$-*module through an isomorphism* $L_1\longrightarrow SL_2(2^m)$. (*We shall call the former case* A_{2m+1}-*case and the latter case* $SL_2(2^m)$-*case*.)

(2) $[W_i, L_j]=0 \quad$ *if* $i\neq j$.

(3) $\tilde{W}=\tilde{W}_1\oplus\tilde{W}_2\oplus\cdots\oplus\tilde{W}_k$.

(4) $\langle\mathcal{P}(G, V)\rangle$ *normalizes each* L_i *and, in* $SL_2(2^m)$-*case*, $\langle\mathcal{P}(G, V)\rangle=N$.

Proof. Set $L=L_1$. *If* $L\cong SL_2(2)$, the assertions follow from 1.11 and 1.13 of [9]. So we assume $L\ncong SL_2(2)$. Define $K=N_S(L)L$, $R=O_2(K)$, and let bars denote images in K/R. Further, define $U=C_V(R)$ and recall that $U(L)=[U, L]/C_{[U,L]}(L)$. If $C_L(U(L))\neq 1$, then $[U, L]=0$ and so $[V, L]=0$ by Thompson's $A\times B$-lemma, a contradiction. Therefore, $C_L(U(L))=1$ and the definition of K shows that

$$C_K(U(L))=R.$$

Since $Z(L)=1$, we also have

$$C_K(L)=R.$$

4.2 of [9] shows that the former part of (4) holds and in particular

$$\langle \mathcal{P}(S, V)\rangle \leq N_S(L) \leq K.$$

Also, since $C_G(N)=1$ and $N=\langle L^S\rangle$ by 5.1, we have

$$\langle \mathcal{P}^*(S, V)\rangle \nleq C_G(L).$$

Thus, we can take $A\in\mathcal{P}(S, V)$ so that $[L, A]\neq 1$ and, using the definition of $\mathcal{P}(S, V)$, we can deduce as follows:

$$\begin{aligned} |U(L):C_{U(L)}(A)| &\leqq |[U, L]:C_{[U,L]}(A)| \\ &=|[U, L]+C_V(A):C_V(A)| \\ &\leqq |C_V(R\cap A):C_V(A)| \\ &\leqq |A:R\cap A| \\ &=|\bar{A}|. \end{aligned}$$

Hence $(\overline{AL}, U(L), \bar{A}, \bar{L})\in Q$, and so one of the following holds by 2.1, 3.1, 3.4, and 3.8 of [9]:

(i) $\overline{AL}\cong SL_2(2^m)$ and $U(L)$ is induced by the natural $GF(2)SL_2(2^m)$-module;

(ii) $\overline{AL}\cong\Sigma_{2m+1}$ and $U(L)$ is induced by the natural $GF(2)\Sigma_{2m+1}$-module.

Furthermore, $|U(L):C_{U(L)}(A)|=|\bar{A}|$ in either case by 2.1 and 3.1 of [9]. Therefore, $C_V(R\cap A)=[U, L]+C_V(A)$, and if $R\cap A\neq 1$, then $R\cap A\in\mathcal{P}(G, V)$ and $R\cap A<_{(V)}A$. Suppose we have taken A from $\mathcal{P}^*(S, V)$ (this is possible). Then we must have $R\cap A=1$ and hence $V=[U, L]+C_V(A)$. This implies that $A\leq C_K(V/[U, L])$, so $L=[L, A]\leq C_K(V/[U, L])$. Thus, $[V, L]=[U, L]$ and $V(L)$

$=U(L)$. We conclude that $V(L)$ is the natural module for $L\cong SL_2(2^m)$ or for $L\cong A_{2m+1}$ according as (i) or (ii) holds. Also, since $[V, L]\leq U=C_V(R)$ and $R=C_S(L)$, we have $[[V, L], S\cap L_j]=0$ for all $j\geqq 2$. Thus, we have $[[V, L], L_j]=0$ for all $j\geqq 2$, proving (2). Consequently, we have $C_{[V,L]}(L)=C_{[V,L]}(N)$, so $V(L)\cong\widetilde{W}_1$ and (1) holds. (3) now follows from (1), (2), and 5.3. In case (i), A induces a group of inner automorphisms on L. Therefore, in $SL_2(2^m)$-case, $\langle\mathcal{P}(S, V)\rangle$ induces a group of inner automorphisms on L and hence on N as well, which implies that $\langle\mathcal{P}(S, V)\rangle\leq N$ because $C_G(N)=1$. Since $N\leq\langle\mathcal{P}(G, V)\rangle$ by 5.2, we have proved the latter part of (4).

5.5. *In A_{2m+1}-case and* $SL_2(2)$*-case*, $C_W(N)=0$.

Proof. In $SL_2(2)$-case, this follows from 1.11 of [9]. In A_{2m+1}-case, 3.7 and 3.8 imply that $C_{W_i}(L_i)=0$ because $W_i=[W_i, L_i]$ and $W_i/C_{W_i}(L_i)$ is a natural module by 5.4. Hence, by (3) of 5.3 and (2) of 5.4, we have $C_W(N)=0$.

5.6. *If $1\neq x\in Z(S)$, then x normalizes each L_i and $|\widetilde{W}:C_{\widetilde{W}}(x)|=2^{mk}$.*

Proof. Since $Z(S)$ centralizes each $S\cap L_i$, $Z(S)$ normalizes each L_i and leaves each W_i invariant. By 3.3 and 4.1, $Z(S)$ induces inner automorphisms on each L_i, and hence on N as well. Since $C_G(N)=1$, this shows that $Z(S)\leq Z(S\cap N)$. Thus, if $x\in Z(S)$, then $x=x_1\ldots x_k$ with $x_i\in Z(S\cap L_i)$, and if furthermore $x\neq 1$, then each $x_i\neq 1$ because S transitively permutes the $Z(S\cap L_i)$, $1\leqq i\leqq k$. Since $\widetilde{W}_i$ is a natural module for L_i, 3.2, 3.5, and 4.3 show that $|\widetilde{W}_i:C_{\widetilde{W}_i}(x_i)|=2^m$. We conclude by (2) and (3) of 5.4 that $|\widetilde{W}:C_{\widetilde{W}}(x)|=2^{mk}$.

5.7. *Assume that S has an elementary abelian normal subgroup E of order at least 2^{mk} such that $[W, E, E]=0$. Then the following hold:*

(1) *E normalizes each L_i;*
(2) $|E|=2^{mk}$;
(3) $C_G(E)=E$.

Proof. Let $1\neq x\in E\cap Z(S)$. Then $[W, E]\leq C_W(E)\leq C_W(x)$ and so E centralizes $\widetilde{W}/C_{\widetilde{W}}(x)$. Suppose $y\in E$ and $L_i^y=L_j$, $i\neq j$. Then $\widetilde{W}_i+C_{\widetilde{W}}(x)=(\widetilde{W}_i+C_{\widetilde{W}}(x))y=\widetilde{W}_j+C_{\widetilde{W}}(x)$ and hence $[\widetilde{W}_i, x]=[\widetilde{W}_i+C_{\widetilde{W}}(x), x]=[\widetilde{W}_j+C_{\widetilde{W}}(x), x]=[\widetilde{W}_j, x]$. Thus, $[\widetilde{W}_i, x]\leq\widetilde{W}_i\cap\widetilde{W}_j$ by 5.6. But then $[\widetilde{W}_i, x]=0$ by 5.4 and, since $\widetilde{W}=\sum\widetilde{W}_i s$ $(s\in S)$ by 5.1 and 5.3, we conclude that $[\widetilde{W}, x]=0$, contrary to 5.6. Therefore, E normalizes each L_i. Now define

$$H=\bigcap_{i=1}^{k}N_G(L_i),$$

$$C_i=C_H(L_{i+1}L_{i+2}\ldots L_k),\ i=0, 1, \ldots, k-1,$$

$$C_k=H,$$

$$H_i=C_H(L_1\ldots L_{i-1}L_{i+1}\ldots L_k),\ i=1, 2, \ldots, k,$$

and note that $C_0=C_G(N)=1$ and $H_1=C_1$. This definition shows that C_i/C_{i-1} is isomorphic to a subgroup of $\mathrm{Aut}\, L_i$ containing $\mathrm{Inn}\, L_i$. So $|E\cap C_i : E\cap C_{i-1}|\leqq 2^m$ for each i by 3.1 and 4.1, which yields that $|E|\leqq 2^{mk}$. Therefore, $|E|=2^{mk}$ and then we have $|E\cap C_i : E\cap C_{i-1}|=2^m$ for each i. In particular, $|E\cap H_1|=|E\cap C_1|=2^m$ and so

$$|E\cap H_i|=2^m, \qquad 1\leqq i\leqq k,$$

because S transitively permutes the H_i by 5.1. Now, $L_i\lhd H_i$ and $C_{H_i}(L_i)=C_G(N)=1$, so a knowledge of the structure of $\mathrm{Aut}\, L_i$ shows that H_i/L_i is cyclic and, if $m=1$ or $L_i\cong \mathrm{SL}_2(2)$, then $H_i=L_i$. Thus, $E\cap L_i\neq 1$ and hence $C_G(E)=C_H(E)$. Also, since $(E\cap C_i)C_{i-1}/C_{i-1}$ is self-centralizing in C_i/C_{i-1} by 3.1 and 4.1, we have $C_{C_i}(E)\leq EC_{i-1}$ for each i and hence $C_H(E)=E$. Therefore, $C_G(E)=E$.

5.8. *Under the hypothesis of* 5.7, *if furthermore* $|W : C_W(E)|=2^{mk}$, *then* $\langle \mathcal{P}(S, V)\rangle\leq E$ *and* G *has an element* g *such that* $N\leq\langle E, E^g\rangle$. *Furthermore, the following hold:*

(1) *In* $\mathrm{SL}_2(2^m)$*-case,* $E=S\cap N$;

(2) *In* A_{2m+1}*-case and* $\mathrm{SL}_2(2)$*-case, if we set* $H=\bigcap N_G(L_i)$ $(1\leqq i\leqq k)$ *and* $H_i=C_H(L_1\ldots L_{i-1}L_{i+1}\ldots L_k)$ *for* $i=1, 2, \ldots, k$, *then* $H=H_1\times\cdots\times H_k$, $E=(E\cap H_1)\times\cdots\times(E\cap H_k)$, $H_i\cong\Sigma_{2m+1}$, *and* $E\cap H_i$ *is generated by* m *disjoint transpositions in* H_i *(note that* $m=1$ *in* $\mathrm{SL}_2(2)$*-case). Furthermore,* $W=W_1\oplus\cdots\oplus W_k$, $[W_i, H_j]=0$ *if* $i\neq j$, *and when considered a* $GF(2)H_i$*-module,* W_i *is induced by the natural* $GF(2)\Sigma_{2m+1}$*-module.*

Proof. $\mathrm{SL}_2(2^m)$*-case*: Recall from 5.7 that E normalizes each L_i and define $K_i=EL_i$. Also, define $R_i=O_2(K_i)$ and let bars denote images in K_i/R_i. Then K_i leaves W_i invariant, and

$$C_{K_i}(\widetilde{W}_i)=C_{K_i}(L_i)=R_i.$$

Since $|W : C_W(E)|=2^{mk}$, we have $|\widetilde{W} : C_{\widetilde{W}}(E)|\leqq 2^{mk}$ and so $|\widetilde{W}_i : C_{\widetilde{W}_i}(E)|\leqq 2^m$ by 5.4 because S transitively permutes the W_i by 5.1. Also, $E\cap H_i\cap R_i\leq C_G(N)=1$ and, as was shown in the proof of 5.7, $|E\cap H_i|=2^m$. Hence

$$|\bar{E}|\geqq 2^m\geqq|\widetilde{W}_i : C_{\widetilde{W}_i}(E)|.$$

If $m\geqq 2$, then the above remarks show that $(\bar{K}_i, \widetilde{W}_i, \bar{E}, \bar{L}_i)\in Q$ and so $\bar{K}_i=\bar{L}_i$

by 2.1 of [9]. Thus, E induces a group of inner automorphisms on each L_i in $\mathrm{SL}_2(2^m)$-case, $m \geqq 2$. The same is true of $\mathrm{SL}_2(2)$-case as well because $\mathrm{Out}\,\mathrm{SL}_2(2) = 1$. Hence, $E \leq S \cap N$ and, since orders coinside, we conclude that (1) holds. The former part of 5.8 in $\mathrm{SL}_2(2^m)$-case now follows from 5.4 and 4.2.

A_{2m+1}-*case and* $\mathrm{SL}_2(2)$-*case*: As was remarked in 5.7, H_i is isomorphic to a subgroup of $\mathrm{Aut}\,L_i$ containing $\mathrm{Inn}\,L_i$. Also, $W = W_1 \oplus \cdots \oplus W_k$ and W_i is a natural module for $L_i \cong A_{2m+1}$ or for $L_i \cong \mathrm{SL}_2(2)$ by 5.4 and 5.5. Therefore, when considered a $GF(2)H_i$-module, W_i is induced by the natural $GF(2)\Sigma_{2m+1}$-module through a monomorphism $H_i \longrightarrow \Sigma_{2m+1}$ by 3.4 of [9] (note that $m=1$ in $\mathrm{SL}_2(2)$-case). Furthermore, since $|W : C_W(E)| = 2^{mk}$, we have $|W_i : C_{W_i}(E)| = 2^m$ as in $\mathrm{SL}_2(2^m)$-case, and hence $|E \cap H_i| = 2^m \geqq |W_i : C_{W_i}(E \cap H_i)|$. Thus, 3.1 of [9] shows that $H_i \cong \Sigma_{2m+1}$ and that $E \cap H_i$ is generated by m disjoint transpositions in H_i. Now, since $C_G(N) = 1$, we have $\langle H_1, \ldots, H_k \rangle = H_1 \times \cdots \times H_k$ and, consequently, $|H| \geqq |\Sigma_{2m+1}|^k$. On the other hand, $|H| = \Pi_{i=1}^k |C_i : C_{i-1}| \leqq |\Sigma_{2m+1}|^k$, where C_i is the same as in the proof of 5.7. Therefore, $H = H_1 \times \cdots \times H_k$. Similarly, we have $E = (E \cap H_1) \times \cdots \times (E \cap H_k)$ because $|E \cap H_i| = 2^m$ and $|E| = 2^{mk}$ by 5.7. If $i \neq j$, then $[W_i, L_j] = 0$ by 5.4, so $|H_j / C_{H_j}(W_i)| \leqq 2$ and $C_{W_i}(H_j) \neq 0$. Since L_i is irreducible on W_i, we conclude that $C_{W_i}(H_j) = W_i$, i.e., $[W_i, H_j] = 0$. We have proved (2). 3.4 now shows that there is an element $g \in G$ such that $N \leq \langle E, E^g \rangle$. In order to prove $\langle \mathcal{P}(S, V) \rangle \leq E$, we let bars denote images in $H / C_H(W_i)$. Then by (2), $\bar{H} \cong H_i \cong \Sigma_{2m+1}$. Furthermore, $\overline{E \cap H_i}$ is generated by the m distinct transpositions in $\overline{S \cap H} \in \mathrm{Syl}_2(\bar{H})$. Suppose $A \in \mathcal{P}(S, V)$. Then $A \leq S \cap H$ by 5.4 and, if $\bar{A} \neq 1$, then $\bar{A} \in \mathcal{P}(\overline{S \cap H}, W_i)$ by 1.2 of [9]. Therefore, $\bar{A} \leq \overline{E \cap H_i}$ for each i by 3.1 of [9], and hence it follows that $A \leq E$.

§ 6. Pushing up and $GF(2)$-representations

In this section, we will consider the following situation:

HYPOTHESIS III. *The group G is a 2-irreducible group with $C_G(O_2(G)) \leq O_2(G)$ and, for $S \in \mathrm{Syl}_2(G)$, $\Omega_1(Z(S)) \not\leq Z(G)$ and $Q(K(S))$ is not normal in G.*

Under this hypothesis, we define $Q = O_2(G)$, $V = \Omega_1(Z(Q))$, and $C = C_G(V)$. Also, we denote by $\bar{\ }$ the natural homomorphism $G \longrightarrow G/C$, and regard V as a faithful right $GF(2)\bar{G}$-module. The main purpose of this section is to show that the pair $\bar{G}$, V satisfies Hypothesis II of section 5 (see 6.6 below).

6.1. $G \neq SC$.

Proof. Since $C_G(Q) \leq Q$, we have $\Omega_1(Z(S)) \leq V$ and so, if $G = SC$, then $\Omega_1(Z(S)) \leq Z(G)$, contrary to Hypothesis III.

6.2. *If $X \lhd G$, then either $S \cap X \lhd G$ or $G = SX$.*

Proof. By a Frattini argument, $G = N_G(S \cap X)SX$ and so $G = N_G(S \cap X)$ or

$G=SX$ by the 2-irreducibility of G.

6.3. $J(S)\not\leq C$.

Proof. 6.1 and 6.2 show that $C_S(V)=Q$. Hence if $J(S)\leq C$, then $J(S)=J(Q)$, $V\leq\Omega_1(Z(J(Q))=\Omega_1(Z(J(S)))$, and $K(S)=C_S(\Omega_1(Z(J(S))))\leq C_S(V)=Q$. However, this shows that $K(S)=K(Q)$ and so $Q(K(S))\lhd G$, contrary to Hypothesis III.

6.4. $G=SJ(G)$.

Proof. If $G\neq SJ(G)$, then $S\cap J(G)\leq Q\leq C$ by 6.2 and so $J(S)=J(S\cap J(G))\leq C$, contrary to 6.3.

6.5. *If* $A\in\mathcal{A}(S)$ *and* $A\not\leq C$, *then* $\bar{A}\in\mathcal{P}(\bar{S}, V)$.

Proof. If $C\cap A\leq B\leq A$, then the maximality of $|A|$ shows that $C_V(A)=V\cap A=C_V(B)\cap B$ and $|A|\geqq|BC_V(B)|$. Hence $|A:B|\geqq|C_V(B):C_V(A)|$, which implies that $\bar{A}\in\mathcal{P}(\bar{S}, V)$.

6.6. *When* V *is regarded as a* $GF(2)\bar{G}$*-module, the pair* $\bar{G}$, V *satisfies Hypothesis* II.

Proof. Since $\bar{G}\neq\bar{S}$ by 6.1, $\bar{G}$ is 2-irreducible. Let $X/C=O_2(\bar{G})$. Then $X=(S\cap X)C$. Hence $SX=SC\neq G$ by 6.1 and so $S\cap X\leq Q\leq C$ by 6.2. This implies that $O_2(\bar{G})=1$. When considered a $GF(2)\bar{G}$-module, V is clearly faithful and $\mathcal{P}(\bar{G}, V)$ is not empty by 6.3 and 6.5. Finally, $C_V(\bar{S})=\Omega_1(Z(S))\not\leq C_V(\bar{G})$ by Hypothesis III.

Now assume that the group G satisfies Hypothesis 0 in addition to Hypothesis III. Then $\bar{G}$ also satisfies Hypothesis 0 and so, by 6.6, we can apply the results of section 5 to the pair $\bar{G}$, V. We can also apply Theorem H of [6] to G. Thus, the following two results can be proved.

6.7. *Under Hypothesis* 0, *we have* $O^2(G)\leq C_G(Q/V)$.

Proof. Since $G\neq\langle C_G(\Omega_1(Z(S))), N_G(Q(K(S)))\rangle$ by Hypothesis III, Theorem H of [6] shows that G has an Aschbacher block B. The definition of Aschbacher blocks adopted in [6] or [5] in particular shows that $1\neq B=O^2(B)\leq O^2(C_G(Q/V))$. Since $O^2(C_G(Q/V/1))=1$ by Hypothesis III, we must have $B\not\leq C$. Thus, $C_G(Q/V)\not\leq C$ and so $O^2(G)\leq C_G(Q/V)C$ by 5.2. Hence $G=SC_G(Q/V)\cdot SC$ and, since $G\neq SC$ by 6.1, we have $G=SC_G(Q/V)$ by the 2-irreducility. Thus $O^2(G)\leq C_G(Q/V)$.

6.8. *Under Hypothesis* 0, *we have* $C=Q$.

Proof. Since $O^2(C_G(Q/V/1))=1$, $C_C(Q/V)\leq Q$ and so $|G:C_G(Q/V)|$ is divided by $|C:Q|$. Now, $|G:C_G(Q/V)|$ is a power of 2 by 6.7, while $Q\in\mathrm{Syl}_2(C)$ as was remarked in the proof of 6.3. Therefore, $C=Q$.

§ 7. A characteristic subgroup for pushing up

In this section, S is a 2-group. Here, we will define a characteristic subgroup $R(S)$ of S mentioned in the introduction, and prove one of the key results. Let $\mathscr{B}(S)$ be the collection of the finite groups which satisfy Hypothesis III of section 6 together with Hypothesis 0 and contain S as a Sylow 2-subgroup. If $\mathscr{B}(S)$ is nonempty, we define

$$R(S)=\bigcap O_2(G) \qquad (G\in\mathscr{B}(S)).$$

If $\mathscr{B}(S)$ is empty, we define

$$R(S)=S.$$

Thus, we have defined a subgroup $R(S)$ of S.

7.1. *$R(S)$ is a characteristic subgroup of S containing $Z(S)$.*

Proof. That $R(S)$ is characteristic follows from remarks in section 1 of [6]. If $G\in\mathscr{B}(S)$, then $Z(S)\leq C_G(O_2(G))\leq O_2(G)$. Therefore, $Z(S)\leq R(S)$.

7.2. *If $G\in\mathscr{B}(S)$ and $R(S)$ is not normal in G, then there exists an element X of $\mathscr{B}(S)$ such that $[O_2(G), O^2(G)]\nleq O_2(X)$.*

Proof. If $[O_2(G), O^2(G)]\leq O_2(X)$ for all $X\in\mathscr{B}(S)$, then $O_2(G)\cap O_2(X)\lhd SO^2(G)=G$ for all $X\in\mathscr{B}(S)$, hence $R(S)\lhd G$.

§ 8. Pushing up for 2-irreducible groups

In this section, we will conclude the proof of the theorem stated in section 2, but as was noted in section 1, we will more generally consider a 2-irreducible group G with $C_G(O_2(G))\leq O_2(G)$ such that, for $S\in \mathrm{Syl}_2(G)$, none of $\Omega_1(Z(S))$, $Q(K(S))$, and $R(S)$ is normal in G. We will also assume that G satisfies Hypothesis 0. Thus, G is a member of $\mathscr{B}(S)$ and so by 7.2 we are led to the following situation:

HYPOTHESIS IV. *S is a 2-group, $G\in\mathscr{B}(S)$, and there exists an element $X\in\mathscr{B}(S)$ such that $[O_2(G), O^2(G)]\nleq O_2(X)$.*

Under this hypothesis, we define $Q=Q_2(G)$, $V=\Omega_1(Z(Q))$, and denote by $-$ the natural homomorphism $G\longrightarrow G/Q$. Since $G\in\mathscr{B}(S)$, G satisfies Hypotheses 0 and III, and so we can apply the results of section 6 to G. Thus, $C_G(V)=Q$ by 6.8, and we can regard V as a faithful $GF(2)\bar{G}$-module. The pair $\bar{G}$, V satisfies Hypothesis II by 6.6, and $\bar{G}$ satisfies Hypothesis 0, so we can apply the results of section 5 to the pair $\bar{G}$, V. Thus, if we define

$$N/Q=\begin{cases}\langle\mathscr{P}_0^*(\bar{G},\ V)\rangle & \text{if } \mathscr{P}_0^*(\bar{G},\ V) \text{ is nonempty,}\\ E(\langle\mathscr{P}_1^*(\bar{G},\ V)\rangle) & \text{if } \mathscr{P}_0^*(\bar{G},\ V) \text{ is empty,}\end{cases}$$

then, by 5.1, N is a normal subgroup of G, $\bar{N}$ is the direct product of one or more subgroups

$$\bar{L}_i=L_i/Q, \qquad i=1,\ 2,\ \dots,\ k,$$

and either $\bar{L}_i\cong\mathrm{SL}_2(2^m)$, $m\geqq 1$, for each i or $\bar{L}_i\cong A_{2m+1}$, $m=2^r\geqq 2$, for each i. Furthermore, if we define

$$W=[V,\ N],$$

$$W_i=[V,\ L_i],\ 1\leqq i\leqq k,$$

then, according to 5.4, there are two possibilities for the action of $\bar{L}_i$ on $W_i+C_W(N)/C_W(N)$, i.e., $\mathrm{SL}_2(2^m)$-case and A_{2m+1}-case. Since $X\in\mathscr{B}(S)$, the above remarks apply also to X and, in particular, we can define the subgroups Q_X, N_X, W_X of X analogous to the subgroups Q, N, W of G, respectively. Thus, $Q_X=O_2(X)$ and so $[Q,\ O^2(G)]\nleqq Q_X$ by Hypothesis IV. We will use this condition only in the proof of 8.3 below.

8.1. $O^2(G)\leq C_G(Q/W)$.

Proof. Since $O^2(G)\leq N$ by 5.1, the definition of W shows $O^2(G)\leq C_G(V/W)$. The remark now follows from 6.7.

8.2. $C_S(W_X)=Q_X$.

Proof. This follows from (6) of 5.3 applied to X/Q_X and W_X.

8.3. $W\nleqq Q_X$.

Proof. If $W\leq Q_X$, then $[Q,\ O^2(G)]\leq Q_X$ by 8.1, contrary to Hypothesis IV.

8.4. $W_X \not\leq Q$.

Proof. If $W_X \leq Q$, then since $W \leq Z(Q)$, we have $W \leq Q_X$ by 8.2, contrary to 8.3.

We will not use the condition $[Q, O^2(G)] \not\leq Q_X$ any more, and results analogous to 8.1 and 8.2 also hold for X and G, respectively, because their proofs depend only on the fact that G, $X \in \mathscr{B}(S)$. Therefore, 8.3 and 8.4 permit us to exploit the symmetry between G and X in the remainder of this section.

8.5. (1) *W_X normalizes each L_i.*
(2) $|Q : Q \cap Q_X| = |W : W \cap Q_X| = 2^{mk} = |W_X : W_X \cap Q|$.
(3) *G has an element g such that $N \leq \langle W_X, W_X^g \rangle Q$.*
(4) $J(S) \leq W_X Q$.
(5) *If G is in* $\mathrm{SL}_2(2^m)$*-case, then* $W_X Q = S \cap N$.
(6) *If G is in A_{2m+1}-case or* $\mathrm{SL}_2(2)$*-case, then* $|W| = |W_X| = 2^{2mk}$, $W_X \cap Q = W \cap Q_X = [W, W_X]$, *and $W_X W$ is the direct product of mk copies of D_8.*

Proof. Since

$$1 \neq \bar{W}_X \lhd \bar{S}$$

by 8.4, we can take an element x of $W_X - Q$ so that $xQ \in Z(S/Q)$. For this element, we have

$$C_W(x) \geqq C_W(W_X) \geqq W \cap Q_W$$

and hence

$$|W : W \cap Q_X| \geqq |W : C_W(W_X)| \geqq |W : C_W(x)| \geqq 2^{mk}$$

by 5.6. By the symmetry between G and X, we have

$$|W_X : W_X \cap Q| \geqq 2^{m_X k_X},$$

where m_X and k_X have the same meaning for X as m and k have for G. Furthermore, since W_X is abelian normal in S, we have

$$[W,\ W_X,\ W_X]=1.$$

If $m_X k_X \geqq mk$, then $|W_X : W_X \cap Q| \geqq 2^{m_X k_X} \geqq 2^{mk}$, so $|W_X : W_X \cap Q| = 2^{m_X k_X} = 2^{mk}$ by (2) of 5.7. If $mk \geqq m_X k_X$, then by symmetry $|W : W \cap Q_X| = 2^{mk} = 2^{m_X k_X}$. Hence we conclude that

$$mk = m_X k_X$$

and

$$|W_X : W_X \cap Q| = |W : W \cap Q_X| = 2^{mk}.$$

Thus, (1) holds by 5.7. It also follows that

$$|W : C_W(W_X)| = |W : C_W(x)| = 2^{mk},$$

and hence

$$C_W(x) = C_W(W_X) = W \cap Q_X.$$

Since $\bar{Q}_X \leq C_{\bar{G}}(\bar{W}_X)$, 5.7 also shows that $Q_X \leq W_X Q$. By symmetry, $Q \leq W Q_X$ and so $|Q : Q \cap Q_X| = |W : W \cap Q_X|$. This proves (2). Since $|W : C_W(W_X)| = 2^{mk}$, (3), (4), and (5) hold by 5.8, because $C_G(V) = Q$ by 6.8 and $\overline{J(S)} \leq \langle \mathcal{P}(\bar{S},\ V) \rangle$ by 6.5.

In order to prove (6), suppose G is in A_{2m+1}-case or $SL_2(2)$-case. Then

$$C_W(N) = 1$$

by 5.5 and so

$$|W| = 2^{2mk}$$

by 5.4. Hence $|C_W(x)| = |W : C_W(x)| = 2^{mk}$ and $W \cap Q_X = C_W(x) = [W,\ x] \leq [W,\ W_X] \leq W \cap W_X \leq W \cap Q_X$. We conclude that

$$W\cap Q_X=[W,\ W_X]$$

and consequently

$$|[W,\ W_X]|=2^{mk}.$$

Now, we can take an element h of X so that $N_X\leq\langle W,\ W^h\rangle Q_X$ by the symmetry between G and X. By 5.3, $W_X=[W_X,\ N_X]\leq[W_X,\ W]\cdot[W_X,\ W^h]$, which yields $|W_X|\leqq 2^{2mk}$. Therefore,

$$|W_X|=2^{2mk} \quad \text{and} \quad C_{W_X}(N_X)=1$$

by 5.4. So far in this paragraph, we have used only the fact that $C_W(N)=1$, and we have shown that $C_{W_X}(N_X)=1$. Thus we can conclude, by symmetry, that

$$W_X\cap Q=[W_X,\ W]=W\cap Q_X.$$

Finally, take a complement Y to $W_X\cap Q$ in W_X. Then WW_X is a semidirect product of W by Y and the action of Y on W is described by 5.8. We conclude by 3.6 that WW_X is the direct product of mk copies of D_8.

8.6. *Let g be an element of G as in* (3) *of* 8.5, *and define* $P=Q\cap Q_X\cap Q_X^g$. *Then the following hold:*

(1) $[P,\ W_X]=1$;
(2) $Q=WP$ *and* $W\cap P=C_W(N)$;
(3) $P=C_Q(O^2(N))$.

Proof. Since $W\leq Z(Q)$ and $W_X\leq Z(Q_X)$, the definition of P and (3) of 8.5 show that $[P,\ W_X]=1$ and $W\cap P\leq C_W(N)$. Hence $|W:C_W(N)|\leqq|W:W\cap P|=|WP:P|\leqq|Q:P|\leqq|Q:Q\cap Q_X|^2$. Now, $|W:C_W(N)|=2^{mk}$ by 5.4, while $|Q:Q\cap Q_X|^2\leqq 2^{mk}$ by (2) of 8.5. Hence (2) follows. (3) of 8.5 also shows that $N=C_N(P)Q$, so $P\lhd N$ and then $O^2(N)\leq C_N(P)$. Since $C_W(O^2(N))=C_W(N)$ by 5.4, (2) shows that $C_Q(O^2(N))=P$.

8.7. *Define* $M=O^2(N)$ *and* $K_i=O^2(L_i)$, $1\leqq i\leqq k$. *Then the following hold:*

(1) $W=[Q,\ M]\leq Q\cap M$ *and* $W_i=[Q,\ K_i]\leq Q\cap K_i$;

(2) *If* $\bar{L}_i \cong SL_2(2^m)$ $(m \geqq 2)$ *or* $\bar{L}_i \cong A_{2m+1}$, *then* K_i *is perfect*, K_i/W_i *is quasisimple with* $Z(K_i/W_i) = Q \cap K_i/W_i$, $K_i/Q \cap K_i \cong \bar{L}_i$, *and* $W_i/C_{W_i}(N)$ *is a natural module for* $K_i/Q \cap K_i \cong SL_2(2^m)$ *or* A_{2m+1}*;*

(3) *If* $\bar{L}_i \cong SL_2(2)$, *then* $K_i/W_i \cong A_3$, $Q \cap K_i = W_i$, *and* W_i *is a natural module for* $K_i/W_i \cong A_3$*;*

(4) M *is a central product of* $K_1, \ldots, K_k$.

Proof. First, (1) follows from 6.7 and 5.3. Then (2) and (3) follow from 5.4 (and 5.5 for $SL_2(2)$-case). Suppose i, $j \in \{1, 2, \ldots, k\}$ and $i \neq j$. Then $[K_i, K_j] \leq Q \cap K_i$ by 5.1 and so $[K_i, K_j] \leq W_i$ by (1)-(3). Since $[W_i, K_j] = 1$ by 5.4, we conclude that $[K_i, K_j] = 1$. Now, since $O^2(\bar{N}) = O^2(\bar{L}_1) \ldots O^2(\bar{L}_k)$ by 5.1, we have $K_1 \ldots K_k \leq M \leq K_1 \ldots K_k Q$ and so $M = K_1 \ldots K_k$.

8.8. *In* $SL_2(2^m)$*-case* $(m \geqq 2)$, *the following hold:*

(1) N *is a central product of* M *and* P*;*

(2) $M \leq J(G) \leq N$*;*

(3) K_i *is a central factor group of* R_m.

Proof. Since $m \geqq 2$, we have $N = MQ$ and so $N = MP$ by (2) of 8.6 and (1) of 8.7. This is a central product by (3) of 8.6. We have $M \leq J(G)$ by 6.4, and $J(G) \leq N$ by 6.5 and (4) of 5.4. Now, since W_X is elementary abelian, (5) of 8.5 shows that Q has a complement in $S \cap L_i$ and so Q/P has a complement in $S \cap L_i/P$ (note that $P \lhd G$ by (3) of 8.6). Since Q/P is abelian by (2) of 8.6, Gaschütz' theorem shows that Q/P has a complement X_i/P in L_i/P. Let $J_i = O^2(X_i)$. Then $J_i \leq K_i$ and X_i is a central product of J_i and P by (3) of 8.6. Hence J_i is a perfect central extension of $SL_2(2^m)$ and $Z(J_i) = P \cap J_i = Q \cap J_i$. Furthermore, $L_i = J_i Q$ and so $K_i = J_i(Q \cap K_i)$. We now argue by (2) of 8.7 that K_i is a perfect central extension of the quadratic group Q_m: Since K_i/W_i is quasisimple with $Z(K_i/W_i) = Q \cap K_i/W_i$, we have $K_i = J_i W_i$ and so $Q \cap K_i = (Q \cap J_i) \cdot W_i = (P \cap J_i) W_i$. Thus, $K_i/C_{W_i}(N)(P \cap J_i) \cong Q_m$. Since $C_{W_i}(N)(P \cap J_i) \leq Z(K_i)$ by (3) of 8.6, we conclude that K_i is a perfect central extension of Q_m. Therefore, K_i is a central factor group of R_m by 4.7.

8.9. *In* A_{2m+1}*-case and* $SL_2(2)$*-case, the following hold:*

(1) *Let* $H = W_X M$. *Then* $J(G)$ *is a central product of* $J(P)$ *and* H*;*

(2) H *is a central product of* $P \cap H$ *and the* S*-conjugates* $H_1, \ldots, H_k$ *of* $H_1 = \langle w_1, K_1 \rangle$, *where* $w_1 \in W_X$ *and* H_1 *is a central factor group of* Γ_{2m+1}*;*

(3) $J(S) = J(P) \times W_X W$*;*

(4) *Let* $G_0 = \bigcap_{i=1}^{k} N_G(L_i)$. *Then* G_0 *is a central product of* P *and* H*;*

(5) *If no nonidentity characteristic subgroup of* $J(S)$ *is normal in* G, *then* $J(P)$ *is the direct product of copies of* D_8 *and copies of* Z_2.

Proof. By 8.6, $W_X Q$ is a central product of $W_X W$ and P. Since $C_W(N) = 1$

by 5.5, we have

$$Q=W\times P$$

by (2) of 8.6. Further, $W_X\cap Q\leq W$ by (6) of 8.5. Hence $W_X W\cap P=W_X W\cap Q\cap P=(W_X\cap Q)W\cap P=W\cap P=1$ and so

$$W_X Q=W_X W\times P.$$

Now, $J(S)=J(W_X Q)$ by (4) of 8.5, and $J(W_X W)=W_X W$ by (6) of 8.5. Therefore,

$$J(S)=W_X W\times J(P).$$

Since $M=O^2(G)\leq J(G)$ by 5.1 and 6.4, we have $J(G)=\langle J(S)^M\rangle=J(S)[J(S), M]=J(S)M=J(P)W_X WM$ and therefore

$$J(G)=J(P)H$$

by (1) of 8.7. This product is a central product by 8.6.

Recall that $P\lhd G$ by (3) of 8.6, and define π to be the natural homomorphism $G\longrightarrow G/P$. By 8.7 and 5.5, W_i is a natural module for $K_i/Q\cap K_i\cong A_{2m+1}$ (note that $m=1$ in $SL_2(2)$-case) and $|Q\cap K_i : W_i|\leqq 2$. Hence it follows that

$$Q\cap K_i \text{ is elementary abelian}$$

and then $Q\cap K_i=W_i\times C_{Q\cap K_i}(K_i)$ by 3.7. Since $C_{Q\cap K_i}(K_i)=C_{Q\cap K_i}(M)=P\cap K_i$ by (4) of 8.7 and (3) of 8.6, we conclude that

$$Q\cap K_i=W_i\times(P\cap K_i),$$

$$K_i^\pi/W_i^\pi\cong A_{2m+1}, \text{ and}$$

$$W_i^\pi \text{ is a natural module for } K_i^\pi/W_i^\pi.$$

In particular, $Z(K_i^\pi)=1$ and so

$$M^\pi=K_1^\pi\times\cdots\times K_k^\pi$$

by (4) of 8.7. Since $W^\pi\leq M^\pi$ by (1) of 8.7, we have $C_{G^\pi}(M^\pi)\leq C_{G^\pi}(W^\pi)$. Since $Q=W\times P$ and $C_G(W)=Q$ by (6) of 5.3, we have $C_{G^\pi}(W^\pi)=C_G(W)^\pi=Q^\pi=W^\pi$. Therefore,

$$C_{G^\pi}(M^\pi)=Z(M^\pi)=1.$$

Note that H normalizes each K_i by (1) of 8.5, and define

$$H_i^\pi=C_{H^\pi}(K_1^\pi\ldots K_{i-1}^\pi K_{i+1}^\pi\ldots K_k^\pi),\ 1\leqq i\leqq k,$$

$$C_i^\pi=C_{H^\pi}(K_{i+1}^\pi\ldots K_k^\pi),\ i=0,1,\ldots,k-1,$$

$$C_k^\pi=H^\pi.$$

Then $H_1^\pi=C_1^\pi$ and, since $H\lhd SM=G$, S transitively permutes the H_i^π $(1\leqq i\leqq k)$ by 5.1. Further, since $C_{G^\pi}(M^\pi)=1$, we have $C_0^\pi=1$ and

$$\langle H_1^\pi,\ldots,H_k^\pi\rangle=H_1^\pi\times\cdots\times H_k^\pi.$$

We can apply 3.10 to C_i^π/C_{i-1}^π because $K_i^\pi\lhd C_i^\pi$ and $C_{i-1}^\pi\cap K_i^\pi=Z(K_i^\pi)=1$. Thus, $|W_X^\pi\cap C_i^\pi:W_X^\pi\cap C_{i-1}^\pi|\leqq 2^{2m}$ by (3) of 3.10. This yields $|W_X^\pi|\leqq 2^{2mk}$, while since $W_X\cap P=1$, we have $|W_X^\pi|=2^{2mk}$ by (6) of 8.5. We conclude that $|W_X^\pi\cap C_i^\pi:W_X^\pi\cap C_{i-1}^\pi|=2^{2m}$ for each i and, in particular, $|W_X^\pi\cap H_1^\pi|=2^{2m}$. Therefore, $|W_X^\pi\cap H_i^\pi|=2^{2m}$ for each i and hence

$$W_X^\pi=(W_X^\pi\cap H_1^\pi)\times\cdots\times(W_X^\pi\cap H_k^\pi).$$

We can also apply 3.10 to H_i^π because $C_{H_i^\pi}(K_i^\pi)\leq C_{G^\pi}(M^\pi)=1$. If $W_X^\pi\cap H_i^\pi=W_i^\pi=O_2(K_i^\pi)$ for some i, then this holds for each i and so $W_X^\pi=O_2(M^\pi)=Q^\pi$, which is a contradiction because $W_X\nleq Q$ by 8.4. Therefore, $W_X^\pi\cap H_i^\pi\neq W_i^\pi$ for each i, and we conclude by 3.10 that

$$H_i^\pi/W_i^\pi \cong \Sigma_{2m+1},$$

W_i^π is a natural module for H_i^π/W_i^π, and

$$H_i^\pi=(W_X^\pi \cap H_i^\pi)K_i^\pi.$$

Furthermore,

$$H_i^\pi \cong \Delta_{2m+1}$$

by 3.9. Take an element $w_i \in W_X$ so that $H_i^\pi=\langle w_i^\pi\rangle K_i^\pi$ and define $H_i=\langle w_i\rangle K_i$, $1\leqq i\leqq k$. Then $H_i/P\cap H_i\cong \Delta_{2m+1}$ and, since

$$[H, P]=1$$

by 8.6, H_i is a central extension of Δ_{2m+1} with $Z(H_i)=P\cap H_i$. Further,

$$P\cap H_i=P\cap K_i$$

because $|H_i^\pi : K_i^\pi|=2=|H_i : K_i|$. Hence $Z(H_i)=Z(K_i)\leq H_i'$ by 8.7. Also, since $Q\cap K_i=W_i\times(P\cap K_i)=W_i\times(P\cap H_i)$, the preimage of M_{2m+1} in H_i is equal to $Q\cap K_i$, which is elementary abelian. Thus, we conclude by 3.11 that H_i is a central factor group of Γ_{2m+1}. We have that

$$H_i'=K_i.$$

In A_{2m+1}-case, this follows from (2) of 8.7. In $SL_2(2)$-case, this follows from the fact that $(H_i^\pi)'=K_i^\pi$, because $P\cap H_i=P\cap K_i=1$ by (3) of 8.7. Let $i, j\in\{1, 2, \dots, k\}$, $i\neq j$. Then $[H_i^\pi, H_j^\pi]=1$ and so $[H_i, H_j]\leq P$. Hence $[H_i, H_j, H_i]=1$ and the three-subgroup lemma yields that $[H_i', H_j]=1$. Thus, $[K_i, H_j]=1$ and since W_X is abelian, we conclude that $[H_i, H_j]=1$ whenever $i\neq j$. Since $H\leq G_0$ by (1) of 8.5, we have $|G_0^\pi|\geqq|H^\pi|\geqq|H_1^\pi \dots H_k^\pi|=|\Delta_{2m+1}|^k$. On the other hand, since $C_{\bar{G}}(\bar{N})=1$, the structure of $\operatorname{Aut}\bar{L}_i$ shows that $|\bar{G}_0|\leqq|\Sigma_{2m+1}|^k$ and hence $|G_0^\pi|\leqq|\Delta_{2m+1}|^k$. Hence $G_0=PH$ and $H=(P\cap H)H_1\dots H_k$, which are central products. Since we can choose the w_i so that $\{w_1, \dots, w_k\}=w_1^S$, we have proved (1)-(4).

Assume that no nonidentity characteristic subgroup of $J(S)$ is normal in G. Let $J(P)=D\times E$, where D is the direct product of copies of D_8 and E is the direct product of indecomposable groups which are not isomorphic to D_8. Then since $J(S)=J(P)\times W_xW$ and W_xW is the direct product of copies of D_8 by (6) of 8.5, Krull-Remak-Schmidt theorem shows that E' is a characteristic subgroup of $J(S)$. Since it is normal in $SM=G$ by (3) of 8.6, we conclude that $E'=1$. Hence $E^2=Z(J(S))^2$ and we similarly have $E^2=1$, proving (5).

This completes our analysis of the pair S, G satisfying Hypothesis IV. We can derive our theorem stated in section 2 from 8.8 and 8.9. If the group G satisfies Hypothesis I and $S\in \mathrm{Syl}_2(G)$, then $G=SJ(G)$ by 6.4 and the pair S, G satisfies Hypothesis IV. If G is in A_{2m+1}-case or $\mathrm{SL}_2(2)$-case, then 8.9 shows that condition (1) of our theorem is satisfied with $T=J(P)$ and $U=P\cap H$. If G is in $\mathrm{SL}_2(2^m)$-case ($m\geqq 2$), 8.8 shows that condition (2) of our theorem is satisfied.

We mentioned theorems of Baumann et al. on groups G satisfying Hypothesis I with $G/K\cong \mathrm{SL}_2(2^m)$ for some normal subgroup K of G. All these theorems can also be derived from the results of this paper because the following holds:

8.10. *Let G be a finite group satisfying Hypotheses* 0 *and* III. *If K is a proper normal subgroup of G with $O_2(G/K)=1$, then $K=O_2(G)$.*

Proof. Since $O_2(G/K)=1$, we have $O_2(G)\leq K$. If $O_2(G)<K$, then 5.2, 6.6, and 6.8 yield that $O^2(G)\leq K$, which is a contradiction.

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