An Imbedding Theorem for a Hilbert Space Appearing in Electromagnetics

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Abstract

Let Ω be a bounded domain in $\boldsymbol{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$. In electromagnetics, we consider a Hilbert space of vector-valued functions which, along with their rotations and divergences, are square summable in Ω and whose normal components on $\partial\Omega$ vanish. We will show that this space is continuously imbedded to $\{H^1(\Omega)\}^3$ when Ω is convex, where $H^1(\Omega)$ is the usual first order Sobolev space. In addition, we will derive an inequality for functions in this Hilbert space. To these aims, we adopt the techniques of Kadlec-Grisvard and the mixed formulation.

Key words: imbedding theorem, Hilbert space, electromagnetics, convex domain, mixed formulation

§ 1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$. In electromagnetics, we often use the Hilbert space of vector-valued functions which, along with their rotations and divergences, are square summable in Ω and whose normal components vanish on $\partial\Omega$, see Duvaut-Lions [4]. This space is used for example to describe magnetic fields. It is difficult to tell whether functions in this space have square summable first order derivatives. In other words, we want to clarify relations between this space and $\{H^1(\Omega)\}^3$, where $H^1(\Omega)$ is the usual first order Sobolev space. It is well-known that the answer is affirmative when $\partial\Omega$ is sufficiently smooth, see e.g. Duvaut-Lions [4]. Unfortunately, this result is not fully satisfactory for practical purposes, since we must often deal with domains with non-smooth boundaries such as polyhedra. Furthermore, the answer is in fact negative for some non-smooth domains, see e.g. Weck [10].

In this paper, we will show that the answer is affirmative and a special inequality holds when Ω is convex. To this end, we will utilize the techniques of Kadlec-Grisvard and the mixed formulation. That is, after Kadlec [8] and Grisvard [7], we approximate Ω from the outside by convex domains with smooth boundaries, for which the answer is affirmative. On the other hand, the mixed formulation is known as the Lagrange multiplier method in optimization theory and in numerical analysis of the finite element method. We will take advantage of the fact that this approach is well suited for dealing with certain constraint conditions, see e.g. Brezzi [2]. These techniques are essentially the same as those employed by the second author [9] to derive similar results for the Hilbert space of electric fields. Finally, it is to be noted that the same results are obtained by Girault-Raviart [6]. However, we publish this paper since our method of proof is different from theirs and in a sense more natural than theirs.

§ 2. Function spaces and preliminaries

Let Ω be a bounded domain in $\boldsymbol{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$. As usual, $L_2(\Omega)$ designates the Hilbert space of all real functions square summable in Ω. We will denote by $(\cdot,\cdot)$ the inner product of $L_2(\Omega)$ or $\{L_2(\Omega)\}^3$. Similarly, we will use $\|\cdot\|$ to denote the norm of $L_2(\Omega)$ or $\{L_2(\Omega)\}^3$ or $\{L_2(\Omega)\}^9$. We will also use notations such as $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$ when it is necessary to specify domains. $H^1(\Omega)$ is the usual first order Sobolev space, which consists of all real functions in $L_2(\Omega)$ with square summable first order derivatives in the distributional sense. Later, we will also consider $H^1(\boldsymbol{R}^3)$ in the usual sense. The norm of $H^1(\Omega)$ is denoted by $\|\cdot\|_{H^1(\Omega)}$ and is given by

$$\|q\|_{H^1(\Omega)}=(\|q\|^2+\|\mathrm{grad}\ q\|^2)^{1/2} \text{ for } q\in H^1(\Omega), \tag{1}$$

where *grad* denotes the gradient operator in the distributional sense. We will also use the notation $\mathrm{grad}\, v$ for any vector-valued function $v\in\{H^1(\Omega)\}^3$, which implies that $\mathrm{grad}\, v=\{\mathrm{grad}\, v^{(1)}, \mathrm{grad}\, v^{(2)}, \mathrm{grad}\, v^{(3)}\}\in\{L_2(\Omega)\}^9$ with $v^{(i)}$ $(i=1, 2, 3)$ being the components of v.

Let us introduce some real Hilbert spaces appearing in electromagnetics [4]:

$$H(\mathrm{rot},\ \Omega)=\{v\in\{L_2(\Omega)\}^3;\ \mathrm{rot}\, v\in\{L_2(\Omega)\}^3\}$$

equipped with the norm

$$\|v\|_{H(\mathrm{rot},\ \Omega)}=(\|v\|^2+\|\mathrm{rot}\, v\|^2)^{1/2}, \tag{2}$$

where *rot* implies the rotation operator in the distributional sense,

$$H(\mathrm{div},\ \Omega)=\{v\in\{L_2(\Omega)\}^3;\ \mathrm{div}\, v\in L_2(\Omega)\}$$

equipped with the norm

$$\|v\|_{H(\mathrm{div},\,\Omega)}=(\|v\|^2+\|\mathrm{div}\,v\|^2)^{1/2}, \tag{3}$$

where *div* implies the divergence operator in the distributional sense,

$$H_0(\mathrm{div},\,\Omega)=\{v\in H(\mathrm{div},\,\Omega);\ (\mathrm{div}\,v,\,q)=-(v,\,\mathrm{grad}\,q)\ \text{for any}\ q\in H^1(\Omega)\}$$

equipped with the norm of $H(\mathrm{div},\,\Omega)$, and

$$V(\Omega)=H(\mathrm{rot},\,\Omega)\cap H_0(\mathrm{div},\,\Omega)$$

equipped with the norm

$$\|v\|_{V(\Omega)}=(\|v\|^2+\|\mathrm{rot}\,v\|^2+\|\mathrm{div}\,v\|^2)^{1/2}\ \text{for}\ v\in V(\Omega). \tag{4}$$

The inner products of the above Hilbert spaces are standard ones associated with the norms, although we omit the explicit expressions. Note that the normal component of any function v in $H_0(\mathrm{div},\,\Omega)$ vanishes on $\partial\Omega$ in a weak sense, since the condition $(\mathrm{div}\,v,\,q)=-(v,\,\mathrm{grad}\,q)$ for all $q\in H^1(\Omega)$ is rewritten by the divergence theorem as follows, when v belongs to $\{H^1(\Omega)\}^3$ as well and $\partial\Omega$ is sufficiently smooth:

$$\int_{\partial\Omega}(v\cdot\nu)q\ d\gamma=0;\ \forall q\in H^1(\Omega). \tag{5}$$

Here $\cdot$ denotes the inner product of two vectors, ν is the unit outward normal vector on $\partial\Omega$, and $d\gamma$ is the infinitesimal element on $\partial\Omega$. For details of the present discussion, see Lemma 7.5.2 of Duvaut-Lions [4].

In what follows, let us list up some known results. The notations Ω, Ω^* and Ω_n ($n=1, 2, 3, \ldots$) below are all for bounded domains in $\boldsymbol{R}^3$ with Lipschitz continuous boundaries.

(i) For each $q\in H^1(\Omega)$, there exists an extension $q^\dagger$ to $\boldsymbol{R}^3$ such that $q^\dagger\in H^1(\boldsymbol{R}^3)$, see Theorem 11.12 of Agmon [1].

(ii) $\{H^1(\Omega)\}^3$ is dense in $H(\mathrm{rot},\,\Omega)$, see Lemma 7.4.1 of Duvaut-Lions [4].

(iii) Let Ω and Ω^* be two domains such that $\Omega\subset\Omega^*$. Let v be an arbitrary element of $H_0(\mathrm{div},\,\Omega)$, and v^* be the extension of v to Ω^* by zero outside Ω, that is,

$$v^*(\xi)=v(\xi)\ \text{for}\ \xi\in\Omega,\qquad v^*(\xi)=0\ \text{for}\ \xi\in\Omega^*\backslash\Omega. \tag{6}$$

Then v^* belongs to $H_0(\mathrm{div},\,\Omega^*)$ with $\|v^*\|_{H(\mathrm{div},\,\Omega^*)}=\|v\|_{H(\mathrm{div},\,\Omega)}$ and $\mathrm{div}\,v^*$ is equal to the extension of $\mathrm{div}\,v$ to Ω^* by zero outside Ω, see the proof of Lemma 7.4.3 of [4].

(iv) Let Ω be a domain with a boundary of C^∞-class. Then $\{H^1(\Omega)\}^3\cap V(\Omega)$ is dense in $V(\Omega)$, see Lemma 7.6.1 of [4].

(v) Let Ω be a convex domain with a boundary of C^∞-class. Then it holds for any $v \in \{H^1(\Omega)\}^3 \cap V(\Omega)$ that

$$\|\text{grad } v\|^2 \leqq \|\text{rot } v\|^2 + \|\text{div } v\|^2 . \tag{7}$$

This follows from Theorem 3.1.1.1 of Grisvard [7] together with the convexity of Ω and the vanishing of normal component of v on $\partial\Omega$.

(vi) Let Ω be a convex domain with a boundary of C^∞-class. Then, from (iv) and (v), $V(\Omega)$ is continuously imbedded to $\{H^1(\Omega)\}^3$ with

$$\|\text{grad } v\|^2 \leqq \|\text{rot } v\|^2 + \|\text{div } v\|^2 ; \ \forall v \in V(\Omega) . \tag{8}$$

(vii) Let Ω be a convex domain. Then there exists a sequence of domains $\{\Omega_n\}_{n=1}^\infty$ such that

(vii-1) Ω_n is convex and $\Omega_n \supset \Omega$ $(n=1, 2, 3, \ldots)$,
(vii-2) $\partial\Omega_n$ is of C^∞-class $(n=1, 2, 3, \ldots)$, and
(vii-3) $\sup_{\xi \in \Omega_n} \inf_{\eta \in \Omega} \text{dist}(\xi, \eta) \longrightarrow 0$ as $n \longrightarrow \infty$,

where dist$(\cdot, \cdot)$ is the Euclidean distance function for $\boldsymbol{R}^3$. As for these, see Lemma 3.2.1.1 of Grisvard [7]. Note here that any convex domain has a Lipschitz continuous boundary. Actually, the statement of Grisvard corresponding to (vii-2) is that $\partial\Omega_n$ is of C^2-class. However, it is not difficult to improve the original statement to the present form since its proof is based on Theorem 33 of Eggleston [5] and the standard mollifier techniques.

(viii) *Friedrichs' inequality* Let Ω be a domain which is star-shaped with respect to any point in an open ball of radius $\rho>0$ contained in Ω. Then each $q \in H^1(\Omega)$ with $(q, 1)=0$ (1=constant function with unit value) satisfies

$$\|q\| \leqq C(\rho, \delta)\|\text{grad } q\| , \tag{9}$$

where δ is the diameter of Ω, and $C(\rho, \delta)$ is a positive constant which is independent of q but can depend continuously on ρ and δ. For the present results, see Theorem 3.2 of Dupont-Scott [3].

§ 3. An imbedding theorem

We will show that $H(\text{rot}, \Omega) \cap H_0(\text{div}, \Omega)$ is continuously imbedded to $\{H^1(\Omega)\}^3$ when Ω is convex. To this end, we first present the following lemma, which is a simplified version of Theorem 1.1 of Brezzi [2] and is commonly used for the analysis of the mixed formulation. Hereafter, we will denote the range and the null space of a linear operator T by $R(T)$ and $N(T)$, respectively.

LEMMA. *Let X and Y be two real Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let B be a linear bounded operator from X into Y, and let*

B^* be the dual of B. We assume that $R(B^*)$ is closed in X, or, in other words, there exists a positive constant k such that

$$\|B^*y\|_X \geqq k\|y\|_Y \text{ for any } y \in R(B). \tag{10}$$

Let us consider the problem: given $\{f, g\} \in X \times R(B)$, find $\{x, y\} \in X \times R(B)$ such that

$$x + B^*y = f, \ Bx = g. \tag{11}$$

Then the solution $\{x, y\}$ of (11) exists uniquely in $X \times R(B)$ for each $\{f, g\} \in X \times R(B)$ and satisfies the estimation

$$\|x\|_X + \|y\|_Y \leqq M(k)(\|f\|_X + \|g\|_Y), \tag{12}$$

where $M(k)$ is a positive number dependent only on k.

REMARK 1. Since $R(B^*)$ is closed in X, $R(B)$ is closed in Y. If the solution $\{x, y\}$ of (11) is looked for in $X \times Y$, y is not unique if $N(B^*) \neq \{0\}$.

Now we can state our main results. The main idea for the proof is accredited to Grisvard [7]. We will refer to the results in the preceding section as (i), (ii) etc.

THEOREM. *Let Ω be a convex bounded domain in $\boldsymbol{R}^3$. Then $V(\Omega) = H(\text{rot}, \Omega) \cap H_0(\text{div}, \Omega)$ is continuously imbedded to $\{H^1(\Omega)\}^3$ with*

$$\|\text{grad } v\|^2 \leqq \|\text{rot } v\|^2 + \|\text{div } v\|^2 \text{ for any } v \in V(\Omega). \tag{13}$$

REMARK 2. Similar results are obtained in [9] for the Hilbert space of vector-valued functions which as well as their rotations and divergences are square summable in Ω and their tangential components vanish on $\partial\Omega$. However, an inequality corresponding to (13) is not given there. Such an inequality is now easy to derive if we adopt the techniques given in the proof below.

Proof. 1° For Ω, we consider $\{\Omega_n\}_{n=1}^{\infty}$ of (vii). For any $u \in V(\Omega)$ and $n=1, 2, 3, \ldots$, denote the extensions of u, $\text{rot } u$ and $\text{div } u$ to Ω_n by zero outside Ω as u_n^*, $(\text{rot } u)_n^*$ and $(\text{div } u)_n^*$, respectively. Then $(\text{rot } u)_n^* \in \{L_2(\Omega_n)\}^3$ and, by (iii), $u_n^* \in H_0(\text{div}, \Omega_n)$ with $\text{div } u_n^* = (\text{div } u)_n^*$. Moreover, we will use the notations:

$$X_n = H(\text{rot}, \Omega_n), \ Y_n = H^1(\Omega_n), \ Z_n = \{H^1(\Omega_n)\}^3.$$

2° For $n=1, 2, 3, \ldots$, let us find a pair $\{u_n, p_n\} \in X_n \times Y_n$ such that $(p_n, 1)_{\Omega_n} = 0$ and, for all $\{w_n, q_n\} \in X_n \times Y_n$,

$$\left.\begin{aligned} &(u_n, w_n)_{X_n} + (\text{grad } p_n, w_n)_{\Omega_n} = (u_n^*, w_n)_{\Omega_n} + ((\text{rot } u)_n^*, \text{rot } w_n)_{\Omega_n} \\ &(u_n, \text{grad } q_n)_{\Omega_n} = (u_n^*, \text{grad } q_n)_{\Omega_n} \end{aligned}\right], \tag{a}$$

where $(\cdot,\cdot)_{X_n}$ is the inner product of $X_n=H(\mathrm{rot},\Omega_n)$. This is a typical problem described by the mixed formulation. In fact, if we set $X=X_n$ and $Y=Y_n$, we can use the Riesz representation theorem to express (a) in the form of (11). Particularly, B^* in this case is an operator from Y_n into X_n, and is characterized by the relation

$$(B^*q_n, w_n)_{X_n}=(\mathrm{grad}\ q_n, w_n)_{\Omega_n} \text{ for all } \{w_n, q_n\}\in X_n\times Y_n.$$

From this relation, we can show that $B^*q_n=\mathrm{grad}\,q_n$ for any $q_n\in Y_n$ if we notice that $\mathrm{grad}\,q_n\in X_n$ with $\mathrm{rot}\,\mathrm{grad}\,q_n=0$ and hence $(\mathrm{grad}\,q_n, w_n)_{\Omega_n}=(\mathrm{grad}\,q_n, w_n)_{X_n}$ for any $w_n\in X_n$. Then we can also see that $\overline{R(B)}$, the closure of $R(B)$ in Y_n, is the totality of functions in $H^1(\Omega_n)$ whose integrals over Ω_n vanish, since $N(B^*)$ (=orthogonal complement of $\overline{R(B)}$) consists of all real constant functions defined in Ω_n.

3° To conclude the unique existence of the solution of (a) by the preceding lemma, let us first show the following inequality for all $q_n\in\overline{R(B)}$, that is, for all $q_n\in H^1(\Omega_n)$ with $(q_n, 1)_{\Omega_n}=0$,

$$\sup_{w_n\in X_n\setminus\{0\}}(\mathrm{grad}\ q_n, w_n)_{\Omega_n}/\|w_n\|_{X_n}\geqq\|\mathrm{grad}\ q_n\|_{\Omega_n}.$$

Due to (1), (vii), and (viii), this is a sufficient condition for (10) and the positive constant corresponding to k in (10) can be chosen to be independent of n. To prove the above inequality, let us consider $w_n=\mathrm{grad}\ q_n$. Then, as we have already seen, $w_n\in X_n$ with $\mathrm{rot}\,w_n=0$, and hence $\|w_n\|_{X_n}=\|\mathrm{grad}\,q_n\|_{\Omega_n}$. The desired inequality immedeately follows from this relation. It is also clear that the function corresponding to g in (11) actually belongs to $R(B)$, since $R(B)$ has been shown to be closed and $(u_n^*, \mathrm{grad}\,q_n)_{\Omega_n}=0$ for any $q_n\in N(B^*)$=orthogonal complement of $R(B)$.

4° Now we have shown the unique existence of $\{u_n, p_n\}$ for each $n\geqq 1$ with the estimation

$$\|u_n\|_{X_n}+\|p_n\|_{Y_n}\leqq C_1(\|u_n^*\|_{\Omega_n}^2+\|(\mathrm{rot}\ u)_n^*\|_{\Omega_n}^2+\|(\mathrm{div}\ u)_n^*\|_{\Omega_n}^2)^{1/2}$$
$$=C_1\|u\|_{V(\Omega)},$$

where C_1 is a positive number independent of u and n (a similar notation C_2 will be used later). Moreover, by considering the restriction q_n^0 of $q_n\in H^1(\Omega_n)$ to Ω, which belongs to $H^1(\Omega)$, we have from the second equation of (a) that

$$(u_n, \mathrm{grad}\,q_n)_{\Omega_n}=(u_n^*, \mathrm{grad}\ q_n)_{\Omega_n}=(u, \mathrm{grad}\ q_n^0)_\Omega$$
$$=-(\mathrm{div}\ u, q_n^0)_\Omega=-((\mathrm{div}\ u)_n^*, q_n)_{\Omega_n},$$

since $u\in H_0(\mathrm{div},\Omega)$. Thus $\mathrm{div}\,u_n=(\mathrm{div}\,u)_n^*\in L_2(\Omega_n)$ with $(u_n, \mathrm{grad}\,q_n)_{\Omega_n}=-(\mathrm{div}\,u_n, q_n)_{\Omega_n}$ for any $q_n\in H^1(\Omega_n)$. This implies that u_n belongs to $V(\Omega_n)$, and also to $Z_n=\{H^1(\Omega_n)\}^3$ by (vi), since each $\partial\Omega_n$ is of C^∞-class by (vii). To sum up, each $\{u_n, p_n\}$ belongs to $\{Z_n\cap V(\Omega_n)\}\times Y_n$ with $(p_n, 1)_{\Omega_n}=0$ and

$$||u_n||_{Z_n}+||p_n||_{Y_n}\leqq C_2||u||_{V(\Omega)}\,. \tag{b}$$

5° Let us denote the restrictions of u_n and p_n to Ω by u_n^0 and p_n^0, respectively. Then $u_n^0\in\{H^1(\Omega)\}^3$ and $p_n^0\in H^1(\Omega)$. Moreover, the norms of u_n^0 and p_n^0 do not exceed those of u_n and p_n, respectively. Then, from (b), $||u_n^0||_{\{H^1(\Omega)\}^3}$ and $||p_n^0||_{H^1(\Omega)}$ are uniformly bounded with respect to n. Thus we can choose a subsequence of $\{\{u_n^0,\ p_n^0\}\}_{n=1}^\infty$, again denoted by $\{\{u_n^0,\ p_n^0\}\}_{n=1}^\infty$ for simplicity, such that as $n\longrightarrow\infty$

$$u_n^0\longrightarrow u_0 \text{ weakly in } \{H^1(\Omega)\}^3\,,\ p_n^0\longrightarrow p_0 \text{ weakly in } H^1(\Omega)\,,$$

where $\{u_0,\ p_0\}$ is an element of $\{H^1(\Omega)\}^3\times H^1(\Omega)$ and satisfies

$$||u_0||_{\{H^1(\Omega)\}^3}+||p_0||_{H^1(\Omega)}\leqq C_2||u||_{V(\Omega)}\,. \tag{c}$$

6° For each $\{w,\ q\}\in\{H^1(\Omega)\}^3\times H^1(\Omega)$, there exists by (i) an extension $\{w^\dagger,\ q^\dagger\}$ to $\boldsymbol{R}^3$ that belongs to $\{H^1(\boldsymbol{R}^3)\}^3\times H^1(\boldsymbol{R}^3)$. Then the restriction $\{w_n^\dagger,\ q_n^\dagger\}$ to Ω_n belongs to $Z_n\times Y_n$ for any $n\geqq1$, and we find from (a) that

$$(u_n,\ w_n^\dagger)_{X_n}+(\text{grad}\ p_n,\ w_n^\dagger)_{\Omega_n}=(u,\ w)_{H(\text{rot},\ \Omega)}\,,$$

$$(u_n,\ \text{grad}\ q_n^\dagger)_{\Omega_n}=(u,\ \text{grad}\ q)_\Omega\,.$$

To see the behaviors of the left-hand sides of these equations as $n\longrightarrow\infty$, we first notice the identity

$$(u_n,\ w_n^\dagger)_{X_n}=(u_n^0,\ w)_{H(\text{rot},\ \Omega)}+\int_{\Omega_n\setminus\Omega}\{u_n\cdot w^\dagger+(\text{rot}\ u_n)\cdot(\text{rot}\ w^\dagger)\}d\xi\,,$$

where $d\xi$ is the infinitesimal element in $\boldsymbol{R}^3$. Since $||u_n||_{\Omega_n}$ and $||\text{rot}\ u_n||_{\Omega_n}$ are uniformly bounded with respect to n and the Lebesgue measure of $\Omega_n\setminus\Omega$ tends to 0 as $n\longrightarrow\infty$ by (vii), the last term of this identity converges to 0 as $n\longrightarrow\infty$. Thus, $(u_n,\ w_n^\dagger)_{X_n}\longrightarrow(u_0,\ w)_{H(\text{rot},\ \Omega)}$ as $n\longrightarrow\infty$. Similarly, we find that $(\text{grad}\ p_n,\ w_n^\dagger)_{\Omega_n}\longrightarrow(\text{grad}\ p_0,\ w)_\Omega$, $(u_n,\ \text{grad}\ q_n^\dagger)_{\Omega_n}\longrightarrow(u_0,\ \text{grad}\ q)_\Omega$, and $(p_n,\ 1)_{\Omega_n}=0\longrightarrow(p_0,\ 1)_\Omega$ as $n\longrightarrow\infty$. We have now shown that $\{u_0,\ p_0\}\in\{H^1(\Omega)\}^3\times H^1(\Omega)$ satisfies $(p_0,\ 1)_\Omega=0$ and

$$\left.\begin{aligned}&(u_0,\ w)_{H(\text{rot},\ \Omega)}+(\text{grad}\ p_0,\ w)_\Omega=(u,\ w)_{H(\text{rot},\ \Omega)}\\&(u_0,\ \text{grad}\ q)_\Omega=(u,\ \text{grad}\ q)_\Omega\end{aligned}\right]\,, \tag{d}$$

which actually holds for any $\{w,\ q\}\in H(\text{rot},\ \Omega)\times H^1(\Omega)$ since, by (ii), $\{H^1(\Omega)\}^3$ is dense in $H(\text{rot},\ \Omega)$. As $\{u,\ 0\}\in H(\text{rot},\ \Omega)\times H^1(\Omega)$ also satisfies (d), it holds that $u_0=u$ and $p_0=0$ by the uniqueness of the solution assured by applying the preceding lemma to (d) with $X=H(\text{rot},\ \Omega)$ and $Y=H^1(\Omega)$. Therefore, any $u\in V(\Omega)$ belongs to $\{H^1(\Omega)\}^3$ with $||u||_{\{H^1(\Omega)\}^3}\leqq C_2||u||_{V(\Omega)}$ from (c). This implies the continuous imbedding of $V(\Omega)$ to $\{H^1(\Omega)\}^3$.

7° Finally, let us derive (13). As we have seen in 4°, it holds that

$$\operatorname{div} u_n = (\operatorname{div} u)_n^* . \tag{e}$$

On the other hand, equating $\{w_n, q_n\}$ to $\{u_n, p_n\}$ in (a), we have

$$\|u_n\|_{X_n}^2 + (\operatorname{grad} p_n, u_n)_{\Omega_n} = \|u_n\|_{X_n}^2 + (\operatorname{grad} p_n, u_n^*)_{\Omega_n}$$
$$= (u_n^*, u_n)_{\Omega_n} + ((\operatorname{rot} u)_n^*, \operatorname{rot} u_n)_{\Omega_n},$$

from which follows

$$\|u_n\|_{X_n}^2 + (\operatorname{grad} p_n^0, u)_\Omega = (u, u_n^0)_{H(\mathrm{rot},\, \Omega)} .$$

Letting n tend to ∞ in the above, we have

$$\lim_{n\to\infty} \|u_n\|_{X_n} = \|u\|_{H(\mathrm{rot},\, \Omega)} . \tag{f}$$

It follows from (vi), (e), and (f) that

$$\|u\|_\Omega^2 + \|\operatorname{grad} u\|_\Omega^2 \leqq \liminf_{n\to\infty} (\|u_n^0\|_\Omega^2 + \|\operatorname{grad} u_n^0\|_\Omega^2)$$
$$\leqq \liminf_{n\to\infty} (\|u_n\|_{\Omega_n}^2 + \|\operatorname{grad} u_n\|_{\Omega_n}^2)$$
$$\leqq \lim_{n\to\infty} (\|u_n\|_{\Omega_n}^2 + \|\operatorname{rot} u_n\|_{\Omega_n}^2 + \|\operatorname{div} u_n\|_{\Omega_n}^2)$$
$$= \|u\|_\Omega^2 + \|\operatorname{rot} u\|_\Omega^2 + \|\operatorname{div} u\|_\Omega^2 .$$

This implies (13), and the proof is complete.

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