A Relation between the Modified Wave Operators $W_J^{\pm}$ and $W_D^{\pm}$

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(Received August 28, 1986)

Abstract

A relation between the modified wave operator $W_J^{\pm}$ with a stationary modifier and $W_D^{\pm}$ with a time-dependent modifier and a relation between the corresponding scattering amplitudes $\mathcal{S}_J(\lambda, \omega, \omega')$ and $\mathcal{S}_D(\lambda, \omega, \omega')$ are obtained for long-range potentials. One related problem is proposed.

§ 0. Introduction

We consider the Schrödinger operators on $\mathcal{H}=L^2(\boldsymbol{R}^n)$, $n\geqq 2$:

$$(0.1)\qquad \begin{cases} H_0=-\dfrac{1}{2}\Delta=-\dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\partial^2/\partial x_j^2\,, \\ H=H_0+V\,. \end{cases}$$

The potential $V=V(x)$ is a real-valued C^∞ function on $\boldsymbol{R}^n$ such that for some constant $0<\varepsilon<1$ and for all multi-indices α

$$(0.2)\qquad |\partial_x^\alpha V(x)|\leqq C_\alpha\langle x\rangle^{-|\alpha|-\varepsilon}\,,$$

where $\langle x\rangle=(1+|x|^2)^{1/2}$ and $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $\alpha=(\alpha_1,\cdots,\alpha_n)$. Under this assumption, H_0 and H define self-adjoint operators on $\mathcal{H}$ with the domains $\mathcal{D}(H_0)=\mathcal{D}(H)=H^2(\boldsymbol{R}^n)$ (=the Sobolev space of order 2).

In [4] we have introduced the modified wave operators $W_J^{\pm}=s\text{-}\lim_{t\to\pm\infty} e^{itH}Je^{-itH_0}$ with a time-independent (stationary) modifier J for the long-range potentials satisfying (0.2), and proved that $W_J^{\pm}$ are complete. In a subsequent paper [5] we have proved that the corresponding scattering matrix $\mathcal{S}_J(\lambda)$ has a smooth integral kernel $\mathcal{S}_J(\lambda, \omega, \omega')$ when $\lambda>0$ and $\omega\neq\omega'$ (ω, $\omega'\in S^{n-1}$). Further, also in [5], we have solved the inverse problem for $V(x)$ satisfying (0.2) with $1/2<\varepsilon<1$ making use of $\mathcal{S}_J(\lambda, \omega, \omega')$ (see Th. 0.4 in [5]).

On the other hand, using a solution $W(\xi, t)$ of the Hamilton-Jacobi equation

$$\partial W/\partial t=|\xi|^2/2+V(\partial W/\partial\xi)\,, \tag{0.3}$$

and setting

$$X(\xi,\ t)=W(\xi,\ t)-t|\xi|^2/2\,, \tag{0.4}$$

we can construct the usual modified wave operators $W_D^{\pm}=W_D^{\pm}(X)=s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-iW(t)}=s\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-iX(t)}e^{-itH_0}$, where $e^{-iW(t)}$ and the time-dependent modifier $e^{-iX(t)}$ are defined by

$$\begin{cases} e^{-iW(t)}f(x)=(\mathcal{F}e^{-iW(\xi,t)}\mathcal{F}^{-1}f)(x)\,, \\ e^{-iX(t)}f(x)=(\mathcal{F}e^{-iX(\xi,t)}\mathcal{F}^{-1}f)(x)\,, \end{cases} \tag{0.5}$$

and $\mathcal{F}$ denotes the Fourier transformation:

$$\mathcal{F}f(\xi)=(2\pi)^{-n/2}\int e^{-i\xi\cdot x}f(x)dx\,. \tag{0.6}$$

It is known (see e.g. [3], [6], [11]) that $W_D^{\pm}(X)$ are complete. Further we can construct the usual S-matrix $\mathcal{S}_D(\lambda)=\mathcal{S}_D^X(\lambda)$ from $W_D^{\pm}(X)$. According to Agmon's announcement [1, Th. 2-(ii)], $\mathcal{S}_D^X(\lambda)$ has also a smooth integral kernel $\mathcal{S}_D^X(\lambda,\ \omega,\ \omega')$ when $\omega\neq\omega'$, but unfortunately the proof has not been published yet. In this paper, we shall prove

THEOREM 0.1. *Let $V(x)$ satisfy* (0.2) *for some constant* $0<\varepsilon<1$. *Then there exist real-valued C^∞ functions $\varphi_{\pm},{}_X(\xi)$ of $\xi\in\boldsymbol{R}^n-\{0\}$ such that*

$$\mathcal{S}_D^X(\lambda,\ \omega,\ \omega')=e^{i\varphi_{+,X}(\sqrt{2\lambda}\omega)}\mathcal{S}_J(\lambda,\ \omega,\ \omega')e^{-i\varphi_{-,X}(\sqrt{2\lambda}\omega')} \tag{0.7}$$

for $\lambda>0$ and $\omega\neq\omega'$.

We thus obtain the smoothness of $\mathcal{S}_D^X(\lambda,\ \omega,\ \omega')$ for $\omega\neq\omega'$ via (0.7) from that of $\mathcal{S}_J(\lambda,\ \omega,\ \omega')$. Moreover it follows from the proof of (0.7) (see Lemma 2.3 below) that there exists a solution $W_J(\xi,\ t)$ of the Hamilton-Jacobi equation (0.3) such that $W_D^{\pm}(X_J)=W_J^{\pm}$ hence $\mathcal{S}_D^{X_J}(\lambda,\ \omega,\ \omega')=\mathcal{S}_J(\lambda,\ \omega,\ \omega')$, where X_J is defined by (0.4) with $W=W_J$.

It is clear by (0.7) that the scattering cross sections $\sigma_J(\lambda,\ \omega,\ \omega')\equiv|\mathcal{S}_J(\lambda,\ \omega,\ \omega')|^2$ and $\sigma_D^X(\lambda,\ \omega,\ \omega')\equiv|\mathcal{S}_D^X(\lambda,\ \omega,\ \omega')|^2$ are identical. In this sence the physics is determined uniquely and independently of our choice of the modifiers J and $e^{-iX(t)}$. However the so-called "phase shift" is not uniquely determined. Thus there remains one problem:

(0.8) Which modifier $e^{-iX(t)}$ gives the "correct" phase shift?

This problem includes the one of finding out the "correct" phase shift for long-range potentials. From the mathematical point of view, the modifier $e^{-iX_J(t)}$ seems to give at least a convenient one, because the scattering amplitude

$\mathcal{S}_J(\lambda, \omega, \omega')$ constructed from $W_D^{\pm}(X_J)=W_J^{\pm}$ is suitable in treating the inverse problems for some long-range potentials as stated above.

The plan of the paper is as follows. In Section 1, we review some facts about the constructions of $W_J^{\pm}$ and $W_D^{\pm}=W_D^{\pm}(X)$ and of $\mathcal{S}_J(\lambda)$ and $\mathcal{S}_D(\lambda)=\mathcal{S}_D^X(\lambda)$. In Section 2, we prove the existence of the limits $U_{\pm}^X=s\text{-}\lim_{t\to\pm\infty} e^{iW(t)}Je^{-itH_0}$, which gives a relation between $W_J^{\pm}$ and $W_D^{\pm}(X)$. Theorem 0.1 will then be proved by using this relation and Theorem A.4 in the appendix. In the final appendix, we summarize from [7] some results of the method of stationary phase necessary for us.

§ 1. The modified wave operators $W_J^{\pm}$ and $W_D^{\pm}$, and the corresponding scattering amplitudes

We first consider $W_J^{\pm}$. We recall the following lemma from [4, Th. 2.5] or [5, Th. 2.1].

LEMMA 1.1. *Let* $-1<\sigma_0<\sigma_1<1$ *and* $d>0$. *Then there exist a constant* $R>2$ *and a real-valued* C^∞ *function* $\varphi(x, \xi)=\varphi_d(x, \xi)=\varphi_{\sigma_0, \sigma_1, d}(x, \xi)$ *satisfying the following properties:*

(i) *For* $|\xi|\geqq d/2$, $|x|\geqq R/2$ *and* $\cos(x, \xi)\equiv x\cdot\xi/|x||\xi|\in[-1, \sigma_0]\cup[\sigma_1, 1]$,

$$|\partial_x\varphi(x, \xi)|^2/2+V(x)=|\xi|^2/2\,. \tag{1.1}$$

(ii) *For any* α, β, *there is a constant* $C_{\alpha\beta}>0$ *such that for all* $(x, \xi)\in \boldsymbol{R}^{2n}$

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x, \xi)-x\cdot\xi)|\leqq C_{\alpha\beta}\langle x\rangle^{1-\epsilon-|\alpha|}\langle\xi\rangle^{-1}\,. \tag{1.2}$$

Further $\varphi(x, \xi)=x\cdot\xi$ *for* $|x|\leqq R/4$ *or* $|\xi|\leqq d/4$.

In the following, we shall use the solution $\varphi(x, \xi)=\varphi_{\sigma_0, \sigma_1, d}(x, \xi)$ of (1.1) defined by [4, (2.26)] or [5, (2.11)].

Given $d>0$, choose a real-valued C^∞ function $b(\xi)=b_d(\xi)$ of $\xi\in\boldsymbol{R}^n$ so that

$$b(\xi)=b_d(\xi)=1 \text{ for } |\xi|\geqq d \text{ and } =0 \text{ for } |\xi|\leqq d/2\,. \tag{1.3}$$

Letting $\varphi(x, \xi)=\varphi_d(x, \xi)$ where $-1<\sigma_0<\sigma_1<1$ are arbitrarily fixed, we define a Fourier integral operator $J(d)$ by

$$J(d)f(x)=(2\pi)^{-n}\ \mathrm{Os}\text{-}\iint e^{i(\varphi(x,\xi)-y\cdot\xi)}b(\xi)f(y)dyd\xi\,. \tag{1.4}$$

Here $\mathrm{Os}\text{-}\iint\cdots dyd\xi$ means the usual oscillatory integral (cf. e.g. [10]). $J(d)$ is known to define a bounded operator on $\mathcal{H}$(cf. e.g. [9, Sect. 4]). The modified wave operators $W_{J(d)}^{\pm}$ are defined by

$$W^{\pm}_{J(d)} = s\text{-}\lim_{t\to\pm\infty} e^{itH} J(d) e^{-itH_0}. \tag{1.5}$$

It is known ([4]) that $W^{\pm}_{J(d)}$ exist; define isometries on $E_{H_0}(\Gamma_d)\mathcal{H}$ where $\Gamma_d = [d^2/2, \infty)$ and $E_{H_0}(\Delta)$ is the spectral measure for H_0; verify $W^{\pm}_{J(d)} E_{H_0}(\Gamma_d)\mathcal{H} = E_H(\Gamma_d)\mathcal{H}^{ac}(H)$ (completeness) where $\mathcal{H}^{ac}(H)$ is the absolutely continuous subspace for H; and satisfy the intertwining property:

$$E_H(\Delta) W^{\pm}_{J(d)} = W^{\pm}_{J(d)} E_{H_0}(\Delta), \qquad \Delta \subset \boldsymbol{R}^1. \tag{1.6}$$

Therefore the S-operator $S_{J(d)}$ defined by

$$S_{J(d)} = (W^{+}_{J(d)})^* W^{-}_{J(d)} \tag{1.7}$$

is a unitary operator on $E_{H_0}(\Gamma_d)\mathcal{H}$ and commutes with H_0.

REMARK 1.2. $S_{J(d)}$ above is equal to $S(\Gamma_d)$ defined by (3.9) in [5], hence all the results concerning $S(\Gamma_d)$ in [5] also hold for our $S_{J(d)}$. This is seen by noting that $W^{\pm}_{J(d)}$ defined by (1.5) above are equal to $W^{\pm}_j(\Gamma_d)$ $(j=1, 2)$ defined by (3.8) of [5]. There $W^{\pm}_j(\Gamma_d)$ were defined by using a stationary modifier J_j (see [5, (3.1)]) with an amplitude function $a_j(x, \xi)$ satisfying a transport equation ([5, (2.12)]). Owing to the second estimate in (2.20) of [5] and applying the method of stationary phase ([2]), we can easily show that $J_j e^{-itH_0} f$ $(f \in E_{H_0}(\Gamma_d)\mathcal{H})$ asymptotically equals $J(d)e^{-itH_0}f$ as $t \longrightarrow \pm\infty$. From this follows $W^{\pm}_{J(d)} = W^{\pm}_j(\Gamma_d)$ hence $S_{J(d)} = S(\Gamma_d)$ on $E_{H_0}(\Gamma_d)\mathcal{H}$.

Next we consider the usual modified wave operators $W^{\pm}_D = W^{\pm}_D(X)$. For this purpose we record the following lemma (see [2, Th. 3.8] and [11, Prop. 2.7]).

LEMMA 1.3. *There exists a real-valued C^∞ function $W(\xi, t)$ satisfying the following properties:*

(i) *For any $d>0$ there is a constant $T>0$ such that for $|\xi| \geqq d$ and $|t| \geqq T$*

$$\partial_t W(\xi, t) = |\xi|^2/2 + V(\partial_\xi W(\xi, t)). \tag{1.8}$$

(ii) *For any $d>0$, $0<\varepsilon_0<\varepsilon$ and α, there is a constant $C_\alpha>0$ such that for $|\xi| \geqq d$ and $|t| \geqq 1$*

$$\begin{cases} |\partial_\xi^\alpha [W(\xi, t) - t|\xi|^2/2]| \leqq C_\alpha \langle t\rangle^{1-\varepsilon_0}, \\ |\partial_\xi^\alpha [V(\partial_\xi W(\xi, t))]| \leqq C_\alpha \langle t\rangle^{-\varepsilon_0}. \end{cases} \tag{1.9}$$

Given a $W(\xi, t)$ satisfying this lemma, we define the modified wave operators $W^{\pm}_D(X)$, $X(\xi, t) \equiv W(\xi, t) - t|\xi|^2/2$, by

$$\begin{aligned} W^{\pm}_D(X) &= s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-iX(t)} e^{-itH_0} \\ &= s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-iW(t)}, \end{aligned} \tag{1.10}$$

where $e^{-iW(t)}$ and $e^{-iX(t)}$ are defined by (0.5). It is also known (cf. e.g. [3], [6], [11]) that $W_D^{\pm}(X)$ exist; define isometries on $\mathcal{H}$; verify $W_D^{\pm}(X)\mathcal{H}=\mathcal{H}^{ac}(H)$ (completeness); and satisfy the intertwining property. Thus the S-operator S_D^X defined by

$$S_D^X=(W_D^+(X))^* W_D^-(X) \tag{1.11}$$

is a unitary operator on $\mathcal{H}$ and commutes with H_0.

We next define the scattering matrices $\mathcal{S}_{J(d)}(\lambda)$ and $\mathcal{S}_D^X(\lambda)$. Let $\mathcal{F}$ be the Fourier transformation defined by (0.6). We define the operators $\hat{S}_{J(d)}$ and $\hat{S}_D^X$ by

$$\begin{cases}\hat{S}_{J(d)}=\mathcal{F}S_{J(d)}\mathcal{F}^{-1}\,,\\ \hat{S}_D^X=\mathcal{F}S_D^X\mathcal{F}^{-1}\,.\end{cases} \tag{1.12}$$

Then $\hat{S}_{J(d)}$ is a unitary operator on $\hat{\mathcal{H}}_d\equiv\mathcal{F}E_{H_0}(\Gamma_d)=L^2(\boldsymbol{R}_d^n, d\xi)$ where $\boldsymbol{R}_d^n=\{\xi\in\boldsymbol{R}^n||\xi|\geqq d\}$, and $\hat{S}_D^X$ is a unitary operator on $\hat{\mathcal{H}}\equiv\mathcal{F}\mathcal{H}=L^2(\boldsymbol{R}^n, d\xi)$. Since $\hat{S}_{J(d)}$ and $\hat{S}_D^X$ both commute with $|\xi|^2/2$, they are decomposable with respect to $|\xi|^2/2$: $\hat{S}_{J(d)}=\int_{\lambda>d^2/2}^{\oplus}\mathcal{S}_{J(d)}(\lambda)d\lambda$ on $\hat{\mathcal{H}}_d$, and $\hat{S}_D^X=\int_{\lambda>0}^{\oplus}\mathcal{S}_D^X(\lambda)d\lambda$ on $\hat{\mathcal{H}}$. Namely $(\hat{S}_{J(d)}f)(\xi)=\mathcal{S}_{J(d)}(|\xi|^2/2)f(\xi)$ $(f\in\hat{\mathcal{H}}_d)$ for a.e. $\xi\in\boldsymbol{R}_d^n$, and $(\hat{S}_D^X g)(\xi)=\mathcal{S}_D^X(|\xi|^2/2)g(\xi)$ $(g\in\hat{\mathcal{H}})$ for a.e. $\xi\in\boldsymbol{R}^n$, where $\mathcal{S}_{J(d)}(\lambda)$ and $\mathcal{S}_D^X(\lambda)$ are unitary operators on $L^2(S^{n-1})$ defined for a.e. $\lambda>d^2/2$ and a.e. $\lambda>0$, respectively. $\mathcal{S}_{J(d)}(\lambda)$ and $\mathcal{S}_D^X(\lambda)$ are called scattering matrices or S-matrices.

It is known ([5] and Remark 1.2 above) that the integral kernel $\mathcal{S}_{J(d)}(\lambda, \omega, \omega')$ of $\mathcal{S}_{J(d)}(\lambda)$ exists for $\lambda>d^2/2$ and $\omega\neq\omega'\in S^{n-1}$ and is C^∞ in $(\lambda, \omega, \omega')$ if $\lambda>d^2/2$ and $\omega\neq\omega'$. From the representation formula ([5, Th. 3.3-(3.7)] of $\mathcal{S}_{J(d)}(\lambda)$ and the arguments in [5, Sect. 4], it follows that

$$\mathcal{S}_{J(d)}(\lambda, \omega, \omega')=\mathcal{S}_{J(d')}(\lambda, \omega, \omega') \tag{1.13}$$

for $\sqrt{2\lambda}>\max(d, d')$ and $\omega, \omega'\in S^{n-1}$ if $\omega\neq\omega'$ and $d, d'>0$. Since $d>0$ was arbitrary in the above, we can thus define $\mathcal{S}_J(\lambda, \omega, \omega')$ for all $\lambda>0$ and $\omega\neq\omega'$ by

$$\mathcal{S}_J(\lambda, \omega, \omega')=\mathcal{S}_{J(d)}(\lambda, \omega, \omega') \tag{1.14}$$

with choosing $d>0$ such that $\lambda>d^2/2$. $\mathcal{S}_J(\lambda, \omega, \omega')$ is called the scattering amplitude. The integral kernel $\mathcal{S}_D^X(\lambda, \omega, \omega')$ of $\mathcal{S}_D^X(\lambda)$ is also called the scattering amplitude, if it exists.

§ 2. The connecting operators $U_{\pm}^X$

Let $d>0$. We define on $E_{H_0}(\Gamma_d)\mathcal{H}$

$$U_{\pm}^X(d)=s\text{-}\lim_{t\to\pm\infty} e^{iW(t)}J(d)e^{-itH_0} \tag{2.1}$$

$$=s\text{-}\lim_{t\to\pm\infty}(e^{iW(t)}e^{-itH})(e^{itH}J(d)e^{-itH_0})$$

$$=(W_D^\pm(X))^* W_{J(d)}^\pm .$$

THEOREM 2.1. *For $d>0$, the limits $U_\pm^X(d)$ exist and define unitary operators on $E_{H_0}(\Gamma_d)\mathcal{H}$ commuting with H_0. Further*

$$W_{J(d)}^\pm = W_D^\pm(X)U_\pm^X(d) \tag{2.2}$$

and

$$S_{J(d)} = U_+^X(d)^* S_D^X U_-^X(d) . \tag{2.3}$$

Proof is clear by (2.1) and the completeness and intertwining property of $W_{J(d)}^\pm$ and $W_D^\pm(X)$.

We set

$$\hat{U}_\pm^X(d) = \mathcal{F} U_\pm^X(d)\mathcal{F}^{-1} \qquad \text{on } \hat{\mathcal{H}}_d = \mathcal{F}E_{H_0}(\Gamma_d)\mathcal{H} . \tag{2.4}$$

We shall prove

THEOREM 2.2. *Let $d>0$ and $u\in C_0^\infty(\boldsymbol{R}_d^n)$. Then there exist real-valued C^∞ functions $\varphi_{\pm,X}^d(\xi)$ of ξ, $|\xi|>d$, such that for $\xi\in\boldsymbol{R}_d^n$*

$$(\hat{U}_\pm^X(d)u)(\xi) = e^{i\varphi_{\pm,X}^d(\xi)}u(\xi) . \tag{2.5}$$

Further for another $d'>0$

$$\varphi_{\pm,X}^d(\xi) = \varphi_{\pm,X}^{d'}(\xi) \quad \textit{for } |\xi|>\max(d, d') . \tag{2.6}$$

From this theorem and density arguments it follows that for $f\in\hat{\mathcal{H}}_d$

$$(\hat{U}_\pm^X(d)f)(\xi) = e^{i\varphi_{\pm,X}^d(\xi)}f(\xi) \tag{2.7}$$

in $\hat{\mathcal{H}}_d$. This and (2.3) yield the existence of $\mathcal{S}_D^X(\lambda, \omega, \omega')$ and that

$$\mathcal{S}_{J(d)}(\lambda, \omega, \omega') = e^{-i\varphi_{+,X}^d(\sqrt{2\lambda}\omega)}\mathcal{S}_D^X(\lambda, \omega, \omega')e^{i\varphi_{-,X}^d(\sqrt{2\lambda}\omega')} \tag{2.8}$$

for $\lambda>d^2/2$ and $\omega\neq\omega'$. This with (1.14) and (2.6) proves Theorem 0.1.

Proof of Theorem 2.2. We consider $\hat{U}_+^X(d)$ only, since $\hat{U}_-^X(d)$ can be treated similarly. Let $d>0$ and $u\in C_0^\infty(\boldsymbol{R}_d^n)$ be fixed.

By (2.1), (2.4), and Theorem 2.1

$$\hat{U}_+^X(d)u = s\text{-}\lim_{t\to\infty}\hat{U}^X(d, t)u . \tag{2.9}$$

Here $\hat{U}^X(d, t)$ is defined by

$$(2.10)\qquad \begin{aligned}&(\hat{U}^X(d,\ t)u)(\xi)\\ &\equiv(\mathcal{F}e^{iW(t)}J(d)e^{-itH_0}\mathcal{F}^{-1}u)(\xi)\\ &=e^{iW(\xi,t)}(\mathcal{F}J(d)\mathcal{F}^{-1})(e^{-it|\eta|^2/2}u(\eta))(\xi)\\ &=e^{iW(\xi,t)}(2\pi)^{-n}\,\mathrm{Os}\text{-}\iint e^{i(-\xi\cdot x+\varphi(x,\eta)-t|\eta|^2/2)}\,u(\eta)dxd\eta\,,\end{aligned}$$

where we have used (0.6), (1.14), and $b(\eta)=1$ on supp $u\subset \boldsymbol{R}_d^n$.

Take compact sets K and K' of $\boldsymbol{R}^n-\{0\}$ such that supp $u\Subset K\Subset K'$, and choose a C^∞ function $a(\eta)$ such that

$$(2.11)\qquad a(\eta)=\begin{cases}1 & \text{on } K,\\ 0 & \text{outside } K'.\end{cases}$$

Then $a(\eta)u(\eta)\equiv u(\eta)$. Let χ be a rapidly decreasing function on $\boldsymbol{R}^n$ such that $\chi(0)=1$. Then by the definition of the oscillatory integral, we have from (2.10)

$$(2.12)\qquad \begin{aligned}&(2\pi)^n(\hat{U}^X(d,\ t)u)(\xi)\\ &=e^{iW(\xi,t)}\lim_{\varepsilon\downarrow 0}\iint e^{i(-\xi\cdot\theta+\varphi(\theta,x)-t|x|^2/2)}\,a(x)u(x)\chi(\varepsilon\theta)d\theta dx\end{aligned}$$

for $\xi\in\boldsymbol{R}_d^n$ and $t>0$.

In order to apply Theorem A.4 in the appendix, we set

$$(2.13)\qquad \begin{cases}\phi(\xi\,;\ x,\ \theta)=-\xi\cdot\theta+x\cdot\theta, & \psi(\xi\,;\ x)=|x|^2/2\,,\\ X(x,\ \theta)=-\varphi(\theta,\ x)+\theta\cdot x, & a(x,\ \theta)=a(x)\rho(\theta)\,,\end{cases}$$

where $\rho(\theta)$ is a C^∞ function such that $\rho(\theta)=1$ for $|\theta|\geqq 1$ and $=0$ for $|\theta|\leqq 1/2$. Then the integral on the right-hand side of (2.12) is equal to

$$(2.14)\qquad \langle A_{\xi,\varepsilon}\,,\ ue^{-it\psi(\xi;\cdot)}\rangle+I_\varepsilon(\xi,\ t)\,,$$

where $A_{\xi,\varepsilon}$, $\varepsilon>0$, is defined by (A.2) in the appendix, and

$$(2.15)\qquad \begin{aligned}&I_\varepsilon(\xi,\ t)\\ &=\iint e^{i(-\xi\cdot\theta+\varphi(\theta,x)-t|x|^2/2)}\,a(x)(1-\rho(\theta))u(x)\chi(\varepsilon\theta)d\theta dx\,.\end{aligned}$$

It is easy to check that the above ϕ, ψ, X and a satisfy the conditions (Cϕ), (Cψ), (CX) and (Ca) in the appendix with

$$(2.16)\quad \begin{cases} \Omega=K\times \boldsymbol{R}^n_d, \qquad \Omega'=K'\times \boldsymbol{R}^n_{d/2}, \\ \Gamma=V\times\{\theta\in\boldsymbol{R}^n \mid |\theta|>1/4\} \ (V \text{ being a bounded open neighborhood of } K' \text{ in } \boldsymbol{R}^n), \\ 1/2>\delta>0, \ \varepsilon\geqq\delta \ (\varepsilon \text{ being the constant in } (0.2)), \\ h'=1-\delta, \\ 1>\rho>1/2 \text{ with } h'\leqq 3\rho-2, \\ h_1=h_2=0. \end{cases}$$

(Here we have used the estimate (1.2) to show (CX).) Further, it is also easy to see that $\phi(\xi; x, \theta)$ and $\psi(\xi; x)$ satisfy the assumptions 1° and 2° of Proposition A. 1 in the appendix with $x_\infty(\xi)=\theta_\infty(\xi)=\xi$ and W being a compact set in Γ such that $\Omega'\times\Omega'\Subset W$.

Therefore we can apply Theorem A. 4 to the first term in (2.14) to see that for $\xi\in\Omega=K\cap\boldsymbol{R}^n_d$

$$(2.17)\quad \lim_{t\to\infty}\Big|\lim_{\varepsilon\downarrow 0}\langle A_{\xi,\varepsilon},\ ue^{-it\varphi}\rangle-(2\pi)^n e^{\pi i\sigma/4}e^{itf(t,\xi;\,x_c(t,\xi),\theta_c(t,\xi))}|\det J|^{-1/2}u^0_{t,\xi}(0)\Big|=0.$$

Here f and $(x_c, \theta_c)(t, \xi)$ are defined as in Proposition A. 1. Namely for $\xi\in\boldsymbol{R}^n_d$ and $t\gg 1$

$$(2.18)\quad f(t, \xi; x, \theta)=-\xi\cdot\theta-|x|^2/2+\varphi(t\theta, x)/t,$$

and $(x_c, \theta_c)(t, \xi)$ is a unique solution of

$$(2.19)\quad \begin{cases} \partial_x f(t, \xi; x_c, \theta_c)=-x_c+(\partial_x\varphi)(t\theta_c, x_c)/t=0, \\ \partial_\theta f(t, \xi; x_c, \theta_c)=-\xi-(\partial_\theta\varphi)(t\theta_c, x_c)=0 \end{cases}$$

such that for $t\gg 1$, $\xi\in\Omega'=K'\cap\boldsymbol{R}^n_{d/2}$, and all α

$$(2.20)\quad |\partial^\alpha_\xi[(x_c, \theta_c)(t, \xi)-(\xi, \xi)]|\leqq C_\alpha t^{-\delta}.$$

(See Proposition A. 2 and the estimate (1.2).) Further

$$(2.21)\quad J=J(t, \xi; x_c(t, \xi), \theta_c(t, \xi)),$$

where $J(t, \xi; x, \theta)$ is defined by (A. 7):

$$(2.22)\quad J(t, \xi; x, \theta)=\begin{pmatrix} t(\partial_\theta\partial_\theta\varphi)(t\theta, x) & (\partial_x\partial_\theta\varphi)(t\theta, x) \\ (\partial_\theta\partial_x\varphi)(t\theta, x) & -I \end{pmatrix},$$

σ is the signature of the real symmetric matrix J, and $u^0_{t,\xi}(y)$ is defined by (A. 15):

$$(2.23)\quad \begin{aligned} u^0_{t,\xi}(y)=&\,a(x+x_c(t, \xi))\rho(t(\theta+\theta_c(t, \xi)))u(x+x_c(t, \xi)) \\ &\times\tilde{\chi}(x, \theta)|_{(x,\theta)=\varphi_{t,\xi}(y)}|\det\partial_y\varphi_{t;\xi}(y)|, \end{aligned}$$

where $\varphi_{t,\xi}$ and $\tilde{\chi}$ are the C^∞ functions on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ determined by Morse lemma (Lemma A. 3) such that $\tilde{\chi}(x, \theta)=1$ near $(x, \theta)=0$, $\varphi_{t,\xi}(0)=0$, and $|\det\partial_y\varphi_{t,\xi}(0)|=1$. Thus for $\xi\in\Omega(\subset K)$ and $t\gg 1$

$$u^0_{t,\xi}(0)=u(x_c(t, \xi)), \tag{2.24}$$

which converges to $u(\xi)$ as $t\longrightarrow\infty$ by (2.20). Further by (1.2), (2.20), (2.21), and (2.22), we have for $\xi\in\Omega(\subset\boldsymbol{R}^n_d)$

$$\lim_{t\to\infty}|\det J|=1. \tag{2.25}$$

Therefore (2.17) implies

$$\lim_{t\to\infty}|\lim_{\varepsilon\downarrow 0}\langle A_{\xi,\varepsilon},\ ue^{-it\psi}\rangle-(2\pi)^n e^{-i\psi_d(\xi,t)}u(\xi)|=0 \tag{2.26}$$

for $\xi\in\Omega=K\cap\boldsymbol{R}^n_d$, where for $\xi\in\boldsymbol{R}^n_d$ and $t\gg 1$

$$\psi_d(\xi, t)=-\pi\sigma/4-tf(t, \xi\,;\, x_c(t, \xi),\ \theta_c(t, \xi)). \tag{2.27}$$

Since K was arbitrary as far as $\operatorname{supp}u\Subset K\Subset\boldsymbol{R}^n-\{0\}$, (2.26) holds for all $\xi\in\boldsymbol{R}^n_d$. Note also that, from our definition [4, (2.26)] or [5, (2.11)] of $\varphi(x, \xi)=\varphi_d(x, \xi)$, $\psi_d(\xi, t)$ defined by (2.27) satisfies for $d, d'>0$

$$\psi_d(\xi, t)=\psi_{d'}(\xi, t)\qquad\text{for } |\xi|>\max(d, d'). \tag{2.28}$$

To deal with the second term $I_\varepsilon(\xi, t)$ in (2.14), we integrate by parts with respect to x in the integral (2.15) using

$$L\equiv|x|^{-2}(ix\cdot\partial_x),\qquad t^{-1}Le^{-it|x|^2/2}=e^{-it|x|^2/2}. \tag{2.29}$$

Then denoting the transposed operator of L by ${}^tL$, we have for $\ell\geqq 1$, $t>0$, and $\xi\in\boldsymbol{R}^n$

$$\begin{aligned} &I_\varepsilon(\xi, t)\\ &=t^{-\ell}\iint e^{i(-\xi\cdot\theta-t|x|^2/2)}(1-\rho(\theta))\chi(\varepsilon\theta)({}^tL)^\ell(e^{i\varphi(\theta,x)}a(x)u(x))d\theta dx, \end{aligned} \tag{2.30}$$

where we have used $u(x)=0$ for $|x|<d$. Since the support of the integrand of (2.30) is compact in $\boldsymbol{R}^n\times\boldsymbol{R}^n$, the limit $I_0(\xi, t)=\lim_{\varepsilon\downarrow 0}I_\varepsilon(\xi, t)$ exists and satisfies $\lim_{t\to\infty}I_0(\xi, t)=0$ for $\xi\in\boldsymbol{R}^n$. Combining this with (2.26) by (2.12) and (2.14), we obtain for $\xi\in\boldsymbol{R}^n_d$

$$\lim_{t\to\infty}|(\hat{U}^X(d, t)u)(\xi)-e^{i(W(\xi,t)-\psi_d(\xi,t))}u(\xi)|=0. \tag{2.31}$$

Thus it now suffices to show that the limit

$$\varphi^d_{+,X}(\xi)=\lim_{t\to\infty}(W(\xi, t)-\psi_d(\xi, t)) \tag{2.32}$$

exists and defines a C^∞ function of ξ, $|\xi|>d$. In fact, (2.5) follows from (2.9), (2.10), (2.31), and (2.32), and (2.6) follows from (2.28) and (2.32).

For this purpose, we prepare the following

LEMMA 2.3. *Let $d>0$. Then there is a constant $T=T_d>0$ such that for $|\xi|\geqq d$ and $t\geqq T$, $\phi_d(\xi, t)$ satisfies the Hamilton-Jacobi equation*

$$\partial_t\phi_d(\xi, t)=|\xi|^2/2+V(\partial_\xi\phi_d(\xi, t)). \tag{2.33}$$

Further for any α, $|\alpha|\geqq 1$, and ε_0, $0<\varepsilon_0<\min(\varepsilon, 1/2)$

$$|\partial_\xi^\alpha[\phi_d(\xi, t)-t|\xi|^2/2]|\leqq C_{\alpha\varepsilon_0}\langle t\rangle^{1-\varepsilon_0}, \tag{2.34}$$

where $C_{\alpha\varepsilon_0}$ is a constant independent of $t\geqq T$ and $|\xi|\geqq d$.

Proof. Using (2.18), (2.19), and (2.27), we have

$$\begin{cases}\partial_t\phi_d(\xi, t)=-f(t, \xi; x_c, \theta_c)-t(\partial_t f)(t, \xi; x_c, \theta_c)=|x_c(t, \xi)|^2/2, \\ \partial_\xi\phi_d(\xi, t)=-t(\partial_\xi f)(t, \xi; x_c, \theta_c)=t\theta_c(t, \xi).\end{cases} \tag{2.35}$$

By (2.19)

$$|\xi|^2=|(\partial_\theta\varphi)(t\theta_c, x_c)|^2, \tag{2.36}$$

and by (2.20)

$$\begin{cases}|t\theta_c|\geqq R/2, \qquad |x_c|\geqq d/2, \\ \cos(t\theta_c, x_c)\geqq\sigma_1\end{cases} \tag{2.37}$$

for $|\xi|\geqq d$ and $t\geqq T$, if T is large enough, where R and σ_1 are the constants in Lemma 1.1. Therefore by (1.1) of Lemma 1.1

$$|(\partial_\theta\varphi)(t\theta_c, x_c)|^2/2+V(t\theta_c)=|x_c|^2/2 \tag{2.38}$$

for $t\geqq T$ and $|\xi|\geqq d$. Then (2.33) follows from (2.35), (2.36), and (2.38), and (2.34) follows from (2.20) and (2.35). The proof of the lemma is complete.

PROPOSITION 2.4. *The limit $\varphi^d_{+,X}(\xi)$ in (2.32) exists and defines a real-valued C^∞ function of $|\xi|>d$.*

Proof. We mimic the argument of Hörmander [2, pp. 86-87]. Let K_1 and K be compact sets in $\boldsymbol{R}^n$ such that $K_1\Subset K\Subset\boldsymbol{R}^n_d$. Since $W(\xi, t)$ and $\phi_d(\xi, t)$ are C^∞ and satisfy the same Hamilton-Jacobi equation (2.33), we have with $R(t, \xi)=W(\xi, t)-\phi_d(\xi, t)$

$$\partial_t\partial_\xi^\alpha R(t, \xi)=\partial_\xi^\alpha[\partial_t W(\xi, t)-\partial_t\phi_d(\xi, t)] \tag{2.39}$$

$$=\partial_\xi^\alpha[V(\partial_\xi W(\xi,\ t))-V(\partial_\xi\phi_d(\xi,\ t))]$$
$$=\partial_\xi^\alpha[\partial_\xi R(t,\ \xi)\cdot a(t,\ \xi)],$$

where

$$a(t,\ \xi)=\int_0^1\partial_x V(\partial_\xi\phi_d(\xi,\ t)+\theta\partial_\xi R(t,\ \xi))d\theta\,. \tag{2.40}$$

$a(t,\ \xi)$ satisfies for all α, $t\geqq T(\gg 1)$, and $\xi\in K$

$$|\partial_\xi^\alpha a(t,\ \xi)|\leqq C_\alpha\langle t\rangle^{-1-\epsilon} \tag{2.41}$$

for some constant C_α. This inequality follows from (0.2) and the inequality

$$|\partial_\xi^\alpha[\partial_\xi\phi_d(\xi,\ t)+\theta\partial_\xi R(t,\ \xi)-t\xi]|\leqq C_\alpha\langle t\rangle^{1-\epsilon_0}, \tag{2.42}$$

which is derived from (1.9) and (2.34). Let $\xi(t,\ \eta)$ be the solution of

$$\frac{d\xi}{dt}(t,\ \eta)=-a(t,\ \xi(t,\ \eta)),\qquad \xi(T,\ \eta)=\eta\in K_0, \tag{2.43}$$

where $K_0\Subset K$ is a compact set to be determined later. Then $\xi(t,\ \eta)\in K$ for $t\geqq T$ if T is large enough, and satisfies for any α and $t\geqq T$

$$|\partial_\eta^\alpha(\xi(t,\ \eta)-\eta)|\leqq C_\alpha\langle T\rangle^{-\epsilon}. \tag{2.44}$$

Therefore $K_0\ni\eta\longrightarrow\xi(t,\ \eta)\in K$ is a diffeomorphism for any fixed $t\geqq T$ if T is large enough, and has an inverse $K_1\ni\xi\longrightarrow\eta(t,\ \xi)\in K_0$ satisfying for any α and $t\geqq T$

$$|\partial_\xi^\alpha(\eta(t,\ \xi)-\xi)|\leqq C_\alpha\langle T\rangle^{-\epsilon}. \tag{2.45}$$

By (2.39) and (2.43), we have

$$\frac{d}{dt}[R(t,\ \xi(t,\ \eta))]=0, \tag{2.46}$$

hence for some function $\gamma(\eta)$ independent of t

$$R(t,\ \xi(t,\ \eta))=\gamma(\eta),\qquad \eta\in K_0,\ t\geqq T. \tag{2.47}$$

This implies that $\gamma(\eta)$ is C^∞ in $\eta\in K_0$ and $R(t,\ \xi)=\gamma(\eta(t,\ \xi))$. Thus by (2.45)

$$|\partial_\xi^\alpha R(t,\ \xi)|\leqq C_\alpha \tag{2.48}$$

for any α, $\xi\in K_1$, and $t\geqq T$. This with (2.39) and (2.41) implies that $\partial_t\partial_\xi^\alpha R(t,\ \xi)$ is integrable with respect to $t\geqq T$ uniformly in $\xi\in K_1$ for any α. This concludes the proof of the proposition.

The proof of Theorem 2.2 is complete.

Appendix

In this appendix, we summarize from [7] some facts about the asymptotic behavior of

$$\langle A_{\xi,\varepsilon},\ ue^{-it\psi(\xi;\cdot)}\rangle \qquad (u\in C_0^\infty(\boldsymbol{R}^n)) \tag{A. 1}$$

as $t\longrightarrow\infty$. Here the distribution $A_{\xi,\varepsilon}$ is defined by

$$\langle A_{\xi,\varepsilon},\ u\rangle=\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^N} e^{i(\phi(\xi;x,\theta)-X(x,\theta))}a(x,\ \theta)u(x)\chi(\varepsilon\theta)d\theta dx \tag{A. 2}$$

for $u\in C_0^\infty(\boldsymbol{R}^n)$, $\varepsilon>0$, and $\xi\in\Omega\Subset\boldsymbol{R}^m$ $(m\geqq 1)$, where χ is a rapidly decreasing function of $\theta\in\boldsymbol{R}^N$ $(N\geqq 1)$ with $\chi(0)=1$.

Let Γ be an open conic set in $\boldsymbol{R}^n\times(\boldsymbol{R}^N-\{0\})$ $(n\geqq 0)$. We assume the following conditions on ϕ and ψ:

(Cϕ) $\phi(\xi;\ x,\ \theta)$ is a real-valued C^∞ function defined on $\Omega'\times\Gamma$, where Ω' is a bounded open neighborhood of Ω, and satisfies for all $\xi\in\Omega'$

(a) $\phi(\xi;\ x,\ t\theta)=t\phi(\xi;\ x,\ \theta)$, $\quad t>0,\ (x,\ \theta)\in\Gamma$,

(b) $((\partial_x,\ \partial_\theta)\phi)(\xi;\ x,\ \theta)\neq 0$, $\quad (x,\ \theta)\in\Gamma$.

(Cψ) $\psi(\xi;\ x)$ is a real-valued C^∞ function on $\Omega'\times\boldsymbol{R}^n$ such that for all $\xi\in\Omega'$ and $x\in\operatorname{supp}u$, $\partial_x\psi(\xi;\ x)\neq 0$.

Let Π_x and Π_θ be the projections from $\boldsymbol{R}^n\times\boldsymbol{R}^N$ onto $\boldsymbol{R}^n$ and $\boldsymbol{R}^N$, respectively, and let real numbers ρ, δ, h_1, h_2, and h' be fixed as follows:

$$1>\rho>1/2>\delta>0, \qquad h_1,\ h_2\in\boldsymbol{R}^1, \qquad h'\leqq 3\rho-2. \tag{A. 3}$$

We assume the following conditions on X and a:

(CX) $X(x,\ \theta)$ is a real-valued C^∞ function on $\boldsymbol{R}^n\times\boldsymbol{R}^N$, and for any compact set L of $\Pi_x(\Gamma)$ and multi-indices α, β, there is a constant $C_{\alpha\beta}$ such that for any $(x,\ \theta)\in L\times\Pi_\theta(\Gamma)$

$$\begin{cases}|(\partial_\theta^\alpha\partial_x^\beta X)(x,\ \theta)|\leqq C_{\alpha\beta}\langle\theta\rangle^{1-\delta-|\alpha|}, & |\alpha|+|\beta|\leqq 2,\\ |(\partial_\theta^\alpha\partial_x^\beta X)(x,\ \theta)|\leqq C_{\alpha\beta}\langle\theta\rangle^{h'-\rho|\alpha|+(1-\rho)|\beta|}, & |\alpha|+|\beta|\geqq 3.\end{cases} \tag{A. 4}$$

(Ca) $a(x,\ \theta)$ is a C^∞ function on $\boldsymbol{R}^n\times\boldsymbol{R}^N$, and for any compact set L of $\boldsymbol{R}^n$ and multi-indices α, β, there is a constant $C_{\alpha\beta}$ such that for any $(x,\ \theta)\in L\times\boldsymbol{R}^N$

$$\begin{cases}|(\partial_\theta^\alpha\partial_x^\beta a)(x,\ \theta)|\leqq C_{\alpha\beta}\langle\theta\rangle^{h_1-|\alpha|}, & |\alpha|+|\beta|\leqq 1,\\ |(\partial_\theta^\alpha\partial_x^\beta a)(x,\ \theta)|\leqq C_{\alpha\beta}\langle\theta\rangle^{h_2-\rho|\alpha|+(1-\rho)|\beta|}, & |\alpha|+|\beta|\geqq 2.\end{cases} \tag{A. 5}$$

Further, for some compact set K of Γ, a satisfies $a(x,\ \theta)=0$ for $(x,\ \theta)\in\boldsymbol{R}^n\times\boldsymbol{R}^N-\{(x,\ t\theta)|t\geqq 1,\ (x,\ \theta)\in K\}$.

PROPOSITION A.1. *Suppose that there is a compact set W of Γ satisfying the following two conditions:*

1° *For any $\xi\in\Omega'$ there is a unique point $(x_\infty(\xi),\ \theta_\infty(\xi))$ in the interior of W such that*

$$\begin{cases}\partial_\theta\phi(\xi\,;\ x_\infty(\xi),\ \theta_\infty(\xi))=0,\\ \partial_x\phi(\xi\,;\ x_\infty(\xi),\ \theta_\infty(\xi))=\partial_x\psi(\xi;\ x_\infty(\xi)).\end{cases}\tag{A.6}$$

2° *For any $\xi\in\Omega'$ and $(x,\ \theta)\in W$,*

$$\det\begin{pmatrix}\partial_\theta\partial_\theta\phi & \partial_\theta\partial_x\phi\\ \partial_x\partial_\theta\phi & \partial_x\partial_x\phi-\partial_x\partial_x\psi\end{pmatrix}(\xi\,;\ x,\ \theta)\neq 0.$$

Then there exist a constant $T>1$ and a bounded open neighborhood U of Ω with $U\Subset\Omega'$ satisfying the following two conditions:

(i) *For any $t>T$, $\xi\in U$, and $(x,\ \theta)\in W$,*

$$J(t,\ \xi\,;\ x,\ \theta)=\begin{pmatrix}\partial_\theta\partial_\theta f & \partial_x\partial_\theta f\\ \partial_\theta\partial_x f & \partial_x\partial_x f\end{pmatrix}(t,\ \xi\,;\ x,\ \theta)\tag{A.7}$$

is a regular symmetric matrix, where

$$f(t,\ \xi\,;\ x,\ \theta)=\phi(\xi\,;\ x,\ \theta)-\psi(\xi\,;\ x)-X(x,\ t\theta)/t\,.\tag{A.8}$$

Further there is a constant C such that for $t>T$, $\xi\in U$, and $(x,\ \theta)\in W$

$$\left|J(t,\ \xi\,;\ x,\ \theta)-\begin{pmatrix}\partial_\theta\partial_\theta\phi & \partial_x\partial_\theta\phi\\ \partial_\theta\partial_x\phi & \partial_x\partial_x\phi-\partial_x\partial_x\psi\end{pmatrix}(\xi\,;\ x,\ \theta)\right|<Ct^{-\delta}\,.\tag{A.9}$$

(ii) *There exists a uniquely determined function $(x_c,\ \theta_c)(t,\ \xi)$: $(T,\ \infty)\times U\longrightarrow W$ such that*

(a) *for any $t>T$ and $\xi\in U$*

$$\begin{cases}\partial_x f(t,\ \xi\,;\ x_c,\ \theta_c)=0,\\ \partial_\theta f(t,\ \xi\,;\ x_c,\ \theta_c)=0,\end{cases}\tag{A.10}$$

(b) *$(x_c,\ \theta_c)$ is a C^∞ function on $(T,\ \infty)\times U$,*
(c) *$|(x_c,\ \theta_c)(t,\ \xi)-(x_\infty,\ \theta_\infty)(\xi)|\leqq Ct^{-\delta}$ for $(t,\ \xi)\in(T,\ \infty)\times U$, and*
(d) *$J(t,\ \xi\,;\ x_c,\ \theta_c)$ is a regular matrix for $(t,\ \xi)\in(T,\ \infty)\times U$.*

Proof is similar to that of Proposition 2.2 of [6].

Further if we assume the following condition on $X(x,\ \theta)$ in addition to (CX):

$$(A.11)\qquad |(\partial_\theta^\alpha \partial_x^\beta X)(x, \theta)| \leqq C_{\alpha\beta}\langle\theta\rangle^{1-\delta-|\alpha|} \text{ for all } \alpha, \beta,$$

then (ii)-(c) of Proposition A.1 is improved as follows.

Proposition A.2. *Let X satisfy* (CX) and (A.11). *Then $(x_c, \theta_c)(t, \xi)$ satisfies for all $(t, \xi)\in(T, \infty)\times U$ and α*

$$(A.12)\qquad |\partial_\xi^\alpha[(x_c, \theta_c)(t, \xi)-(x_\infty, \theta_\infty)(\xi)]| \leqq C_\alpha t^{-\delta},$$

where C_α is independent of (t, ξ).

Proof is again similar to that of Proposition 2.2 of [6].

In order to state our main theorem in this appendix, we prepare the following Morse lemma:

Lemma A.3. *There exists an open ball $B\subset \boldsymbol{R}^{n+N}$ with center 0 such that for any $(t, \xi)\in(T, \infty)\times\Omega$ there exist an open neighborhood $V_{t,\xi}$ of 0 in $\boldsymbol{R}^{n+N}$ and a C^∞ diffeomorphism $\varphi_{t,\xi}: V_{t,\xi}\longrightarrow B$ satisfying*

(i) $\varphi_{t,\xi}(0)=0, \quad (t, \xi)\in(T, \infty)\times\Omega$,

(ii) $f(t, \xi; \varphi_{t,\xi}(y)+(x_c, \theta_c)(t, \xi))=f(t, \xi; x_c, \theta_c)+\langle A(t, \xi)y, y\rangle/2, \quad y\in V_{t,\xi}$, $(t, \xi)\in(T, \infty)\times\Omega$, *where*

$$(A.13)\qquad A(t, \xi)=\begin{pmatrix}\partial_\theta\partial_\theta f & \partial_x\partial_\theta f\\ \partial_\theta\partial_x f & \partial_x\partial_x f\end{pmatrix}(t, \xi; x_c(t, \xi), \theta_c(t, \xi)), \text{ and}$$

(iii) $|\det \partial_y\varphi_{t,\xi}(0)|=1, \quad (t, \xi)\in(T, \infty)\times\Omega$.

For the proof, see [2, Appendix].

Theorem A.4. *Let the conditions* (Cϕ), (Cψ), (CX), (Ca), *and the conditions* 1° *and* 2° *of* Proposition A.1 *be satisfied. Then the following hold*:

(i) *For any $t>0$ and $\xi\in\Omega$, the limit*

$$(A.14)\qquad \lim_{\varepsilon\downarrow 0}\langle A_{\xi,\varepsilon}, ue^{-it\phi(\xi;\cdot)}\rangle$$

exists and defines a distribution $A_{\xi,0}$.

(ii) *Choose $\tilde\chi\in C_0^\infty(\boldsymbol{R}^{n+N})$ so that* $\operatorname{supp}\tilde\chi\subset B$ *and* $\tilde\chi(x, \theta)=1$ *near* $(x, \theta)=0$, *where B is the ball in* Lemma A.3. *Let $u_{t,\xi}^\varepsilon(y)$, $\varepsilon\geqq 0$, be defined by*

$$(A.15)\qquad \begin{aligned} u_{t,\xi}^\varepsilon(y)=&a(x+x_c(t, \xi), t(\theta+\theta_c(t, \xi)))u(x+x_c(t, \xi))\\ &\times\chi(\varepsilon t(\theta+\theta_c(t, \xi)))\tilde\chi(x, \theta)|_{(x,\theta)=\varphi_{t,\xi}(y)}|\det\partial_y\varphi_{t,\xi}(y)|.\end{aligned}$$

Then for any $\varepsilon\geqq 0$ and $\xi\in\Omega$

$$\text{(A.16)} \quad \lim_{t\to\infty} \left| \langle A_{\xi,\varepsilon},\ ue^{-it\phi(\xi;\cdot)}\rangle - (2\pi)^{(N+n)/2} e^{\pi i\sigma/4} e^{itf(t,\xi;x_c(t,\xi),\theta_c(t,\xi))} |\det J|^{-1/2} u_{t,\xi}^{\varepsilon}(0) \right| = 0.$$

Here $J=J(t, \xi; x_c(t, \xi), \theta_c(t, \xi))$ *and* σ *is the signature of* J.

This theorem is a special case of Theorem 1.2 in [7]. Note that our situation in the above corresponds to the case $\varepsilon_0=0$ in [7], so Theorem 1.2 in [7] is applicable to our case without any change (cf. [8]).

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