Bifurcation and Stability of Periodic Orbits of Nonlinear Evolution Equations

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Introduction

We consider the semilinear evolution equation in a real Banach space X with a real n-dimensional parameter λ of the form

$$\text{(E)} \qquad \frac{du}{dt}=L\,u+N(u,\ \lambda)\,, \qquad t>0,$$

with an initial value $u(0)=x_0$. Here L is the generator of an analytic semigroup and $N(x, \lambda)$ is a nonlinear operator of class C^2 with $N(0, 0)=0$ and $D_xN(0, 0)=0$ ($D_xN(0, 0)$ denotes the Fréchet derivative of $N(x, \lambda)$ with respect to x at $(x, \lambda)=(0, 0)$). We are interested in the case that the evolution equation (E) is "structually unstable" at $\lambda=0$. The simplest situation occurs in the case that L satisfies one of the following conditions:

(i) 0 is an eigenvalue of L with algebraic multiplicity one and

$$\sup_{\mu\in\sigma(L)\setminus\{0\}} \operatorname{Re}\mu<-\alpha \qquad (\alpha>0)\,.$$

(ii) $\pm i$ are eigenvalues of L with algebraic multiplicity one and

$$\sup_{\mu\in\sigma(L)\setminus\{i,-i\}} \operatorname{Re}\mu<-\alpha \qquad (\alpha>0)\,.$$

The case (i) is discussed in [17]. In the present paper we consider the case (ii). In this case, it is well known that non-equilibrium periodic orbits may bifurcate from $(x, \lambda)=(0, 0)$. (See, e.g., [6], [14], [27], [32].) Also, we may assume without loss of generality that $N(0, \lambda)=0$. Then $(0, \lambda)$ is a stationary solution of (E). (A pair (x, λ) is, by definition, a stationary solution of (E) if (x, λ) satisfies $Lx+N(x, \lambda)=0$.)

The purpose of this paper is to investigate the asymptotic behavior of a solution $u(t, x_0, \lambda)$ as $t\longrightarrow\infty$ with an initial value x_0 given near 0. In particular we are interested in the case that for some λ there exist at least two periodic orbits of (E). First we want to ask if $u(t, x_0, \lambda)$ converges to a periodic orbit as $t\longrightarrow\infty$. If it does, the following comes into question: to which periodic orbit does it converge as $t\longrightarrow\infty$? We shall answer these questions by classifying initial values x_0 in terms of the asymptotic behavior of a solution $u(t, x_0, \lambda)$ of (E) as $t\longrightarrow\infty$.

Here we interpret the problem which will be discussed throughout this paper. We only consider the case that a nonlinear operator N is small and a solution of (E) may blow up, hence we introduce the local version of the notion of an ω-limit set. Let U and V be some neighborhoods of 0 in X with $U\subset V$. Let $x_0\in U$. A *local ω-limit set* $\Omega_{U,V}(x_0, \lambda)$ of $u(t, x_0, \lambda)$ with respect to $\{U, V\}$ is defined as follows:

$$\Omega_{U,V}(x_0, \lambda)=\begin{cases}\bigcap_{s\geq 0}\overline{\{u(t, x_0, \lambda): t\in[s, \infty)\}} & \text{if } u(t, x_0, \lambda)\in V,\ t>0\\ \emptyset & \text{otherwise}\end{cases}$$

where the bar denotes the closure.

Now our problem can be formulated as follows.

Problem. For appropriate neighborhoods U and V with $U\subset V$, determine $\Omega_{U,V}(x_0, \lambda)$. Here U is independent of λ.

To begin with we state our results on a local ω-limit set.

Theorem 1. *There exists a positive number d and neighborhoods U, $U(\lambda)$ ($|\lambda|<d$) of 0 in X with $U\subset U(\lambda)$ such that a local ω-limit set $\Omega_{U,U(\lambda)}(x_0, \lambda)$ of a solution $u(t, x_0, \lambda)$ of (E) (with respect to $\{U, U(\lambda)\}$) consists of a single periodic orbit $\gamma(x_0, \lambda)$ (which may be an equilibrium point $\{0\}$) in $U(\lambda)$ if $\Omega_{U,U(\lambda)}(x_0, \lambda)$ is not empty. Furthermore, $u(t, x_0, \lambda)$ converges to the periodic orbit $\gamma(x_0, \lambda)$ as $t\longrightarrow\infty$.*

In order to solve our problem, we therefore have to determine whether $\Omega_{U,U(\lambda)}(x_0, \lambda)$ is empty or not, and if it is not empty, we have to determine the mapping $x_0\longrightarrow\gamma(x_0, \lambda)$. We shall show that these are completely determined by the behavior of the bifurcation function and the position of an initial value situated around the stable manifolds of periodic orbits of (E). In the next we state a special form of this result (see Theorem 3.14 for a more general form).

Theorem 2. *Let d, U, and $U(\lambda)$ ($|\lambda|<d$) be as in Theorem 1. Suppose that there exists a unique non-equilibrium periodic orbit $\gamma(\lambda)$ in $U(\lambda)$, and suppose that it is asymptotically stable. Then, for all $x_0\in U$, $\Omega_{U,U(\lambda)}(x_0, \lambda)$ is not empty. Furthermore if $x_0\in\mathcal{M}(0, \lambda)$, then $\gamma(x_0, \lambda)=\{0\}$, and if $x_0\in U\setminus\mathcal{M}(0, \lambda)$, then $\gamma(x_0, \lambda)=\gamma(\lambda)$. Here $\mathcal{M}(0, \lambda)$ is the stable manifold of a stationary solution $(0, \lambda)$ of codimension two.*

Further we can conclude that by the study of the asymptotic behavior we can completely determine the stability of periodic orbits of (E) near 0 (Th. 3.1).

As a direct consequence of the stability theorem (Th. 3.1), we obtain the following theorem.

THEOREM 3. (*Bifurcation thorem*) *Let* λ *be a real number. If a statinary solution* $(0, \lambda)$ *changes its stability at* $\lambda=0$, *then non-equilibrium periodic orbits bifurcate from* $(x, \lambda)=(0, 0)$.

In Section 1, we state assumptions made throughout the present paper. In Section 2, we define the bifurcation function using the center manifold theorem, and establish the relation between the critical eigenvalue of the linearized Poincaré map and the bifurcation function (Th. 2.11). In Section 3, we define the stable manifolds of periodic orbits (which are of condimension one) and the inside (outside) of the stable manifolds. Using them, in the end of Section 3 we shall establish our main results (Ths. 3.1, 3.2, 3.13, and 3.14).

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§ 1. Preliminaries

1.1. Let X be a real Banach space. Suppose that L is a linear operator in X and that $N(\cdot, \lambda)$ is a nonlinear operator in X with a real n-dimensional parameter λ. Throughout the present paper we postulate the following two hypotheses concerning L and N.

HYPOTHESIS 1. (i) L generates an analytic semigroup $\{e^{tL}\}_{t\geq 0}$ in X.

(ii) $\pm i$ are eigenvalues of L with algebraic multiplicity one and there exists a positive constant α such that the other part of the spectrum $\sigma'(L)$ of L satisfies

$$\sup_{\mu\in\sigma'(L)} \mathrm{Re}\ \mu < -\alpha .$$

In the above, L is regarded as a unique linear extension of L to the complexification X_c of X.

Throughout the present paper we fix β such that $0\leq\beta<1$. All the results are valid for any fixed β within this range. We denote by X_β the Banach space consisting of all elements in the domain of $(-L+1)^\beta$. The norm of X_β is the graph norm of $(-L+1)^\beta$.

HYPOTHESIS 2. The nonlinear operator N is a C^2-mapping of some neighborhood of $(0, 0)$ in $X_\beta\times \boldsymbol{R}^n$ into X such that $N(0, \lambda)=0$ and $D_x N(0, 0)=0$.

1.2. It follows from Hypothesis 1 (ii) that the kernel Ker $(L\text{-}i)$ of $L\text{-}i$ is spanned by some $\phi_c \in X_c$. Since i is an isolated eigenvalue with algebraic multiplicity one, the range $R(L\text{-}i)$ of $L\text{-}i$ is a closed subspace of codimension one in X_c, and ϕ_c is not contained in $R(L\text{-}i)$ (Kato [21, Chap. IV, Th. 5.28]). Hence there exists a continuous linear functional ϕ_c^* on X_c such that $\langle x, \phi_c^* \rangle = 0$ for $x \in R(L\text{-}i)$ and $\langle \phi_c, \phi_c^* \rangle \neq 0$. We may assume that $\langle \phi_c, \phi_c^* \rangle = 2$.

Since we treat an evolution equation (E) in a real Banach space X, we set

$$\phi_c = \phi_1 + i\phi_2, \qquad \phi_j \in X, \qquad j=1,\ 2,$$

$$\phi_c^* | X = \phi_1^* - i\phi_2^*,$$

where ϕ_j^*, $j=1, 2$, is a continuous linear functional on X. Then we have

$$L\phi_1 = -\phi_2, \qquad L\phi_2 = \phi_1,$$

$$\langle Lx, \phi_j^* \rangle = (-1)^{3-j} \langle x, \phi_{3-j}^* \rangle, \qquad j=1,\ 2,$$

$$\langle \phi_j, \phi_k^* \rangle = \delta_{j,k}, \qquad j,\ k=1,\ 2.$$

Using ϕ_j, ϕ_k^* $(j, k=1, 2)$ we decompose X, X_β into direct sums

$$X = \text{span}\{\phi_1, \phi_2\} \oplus Z,$$

$$X_\beta = \text{span}\{\phi_1, \phi_2\} \oplus Z_\beta,$$

where

$$Z = \{x \in X:\ \langle x, \phi_j^* \rangle = 0, \qquad j=1,\ 2\},$$

$$Z_\beta = \{x \in X_\beta:\ \langle x, \phi_j^* | X_\beta \rangle = 0, \qquad j=1,\ 2\}.$$

We define projections P and Q from X onto span$\{\phi_1, \phi_2\}$ and Z, respectively, by

$$Px = \langle x, \phi_1^* \rangle \phi_1 + \langle x, \phi_2^* \rangle \phi_2 \quad \text{and} \quad Qx = x - Px.$$

The restriction $P|X_\beta$ (resp. $Q|X_\beta$) of P (resp. Q) to X_β is also the projection from X_β onto span$\{\phi_1, \phi_2\}$ (resp. Z_β). For simplicity we also denote $P|X_\beta$ (resp. $Q|X_\beta$) by P (resp. Q).

For later convenience we replace the norm $\|\ \|_\beta$ on X_β by the equivalent norm:

$$\|x\| = \max\{\sqrt{\langle x, \phi_1^* \rangle^2 + \langle x, \phi_2^* \rangle^2},\ \|Qx\|_\beta\}.$$

Then we have

$$\|x\| = \max\{\|Px\|,\ \|Qx\|\}.$$

Finally we denote by $B_Y(d)$ the open ball having radius d and centered at the origin in a Banach space Y.

1.3. *Existence theorem and some definitions* We begin with the definition of a solution of (E). Let λ be fixed.

DEFINITION 1.1. For fixed $T>0$, a function $u(t)$ is called a solution of (E) on $[0, T)$ if

(i) $u(t)\in C([0, T), X_\beta)$,

(ii) $\dfrac{du(t)}{dt}\in C((0, T), X)$,

(iii) $u(t)$ is contained in the domain of L for $0<t<T$ and in that of $N(\cdot, \lambda)$ for $0\leq t<T$, and

(iv) $u(t)$ satisfies (E) for $0<t<T$ and $u(0)=x_0$.

Then we have the following existence theorem.

THEOREM 1.2. *Under Hypotheses* 1 *and* 2, *for any* T, $0<T<\infty$, *there exists an open neighborhood* $W\times\Lambda$ *of* $(0, 0)$ *in* $X_\beta\times \boldsymbol{R}^n$ *such that for each* $(x_0, \lambda)\in W\times\Lambda$, (E) *has a unique solution* $u(t)$ *on* $[0, T)$.

The proof is done by the standard argument. See, e.g., Henny [13].

We shall introduce some geometrical terminology in the theory of dynamical systems [4, 11, 13].

Let $T>0$ be fixed. Let W and Λ be as in Theorem 1.2 with $\sup_{x\in W,\lambda\in\Lambda}\|N(x, \lambda)\|$ bounded. Let $\lambda\in\Lambda$ be fixed. Then it follows that for each $x_0\in W$ there exists a maximal number $0<\tau(x_0, \lambda)\leq\infty$ such that a solution $u(t)$ of (E) uniquely exists on $[0, \tau(x_0, \lambda))$ and satisfies $u(t)\in W$ for $t\in[0, \tau(x_0, \lambda))$. See [13, Th. 3.3.4].

In the following we fix such W and Λ, and denote a unique solution $u(t)$ of (E) on $[0, \tau(x_0, \lambda))$ by $u(t, x_0, \lambda)$.

By an orbit of the solution $u(t, x_0, \lambda)$, we mean the set $\{u(t, x_0, \lambda): 0\leq t<\tau(x_0, \lambda)\}$ in X_β. An orbit of a stationary solution (resp. a periodic solution) of (E) is called an equilibrium point (resp. a periodic orbit) of (E).

We say that a set $S\subset W$ is invariant if for any $x_0\in S$, $\tau(x_0, \lambda)=\infty$ and $u(t, x_0, \lambda)\in S$ for all $t\geq 0$. Then it is obvious that an equilibrium point and a periodic orbit in W are invariant sets.

We say that an invariant set S is stable if for any neighborhood U of S there exists a neighborhood V of S such that for any $x_0\in V$, $\tau(x_0, \lambda)=\infty$ and $u(t, x_0, \lambda)\in U$ for all $t\geq 0$. We say that an invariant set S is unstable if it is not stable. We say that an invariant set S is asymptotically stable if it is stable and there exists a neighborhood V of S such that for any $x_0\in V$, $\tau(x_0, \lambda)=\infty$ and $u(t, x_0, \lambda)$ converges to S as $t\longrightarrow\infty$.

We say that a stationary solution v (resp. a periodic solution $w(t)$) in W is asymptotically stable, stable, and unstable (resp. orbitally asymptotically stable, orbitally stable, and orbitally unstable), if an equilibrium point $\{v\}$ (resp. a peri-

odic orbit $\{w(t): t \geq 0\}$) is asymptotically stable, stable, and unstable, respectively.
Finally we define a local ω-limit set of $u(t, x_0, \lambda)$.

DEFINITION 1.3. Let U and V be some neighborhoods of 0 in X_β with $U \subset V \subset W$. Let $x_0 \in U$. The local ω-limit set $\Omega_{U,V}(x_0, \lambda)$ of $u(t, x_0, \lambda)$ with respect to $\{U, V\}$ is defined as follows:

$$\Omega_{U,V}(x_0, \lambda) = \begin{cases} \bigcap_{s \geq 0} \overline{\{u(t, x_0, \lambda): t \in [s, \infty)\}} & \text{if } \tau(x_0, \lambda) = \infty,\ u(t, x_0, \lambda) \in V,\ t > 0 \\ \emptyset & \text{otherwise} \end{cases}$$

where the bar denotes the closure in the topology of X_β.

§ 2. Reduction

2.1. *The center manifold theorem* We state the center manifold theorem. It plays an essential role in the study of the existence and stability of periodic orbits of (E). Indeed it has been used by a number of authors, e.g., [2, 3, 4, 12, 13, 26, 29, 17].

THEOREM 2.1. (*Center manifold theorem*) *Under Hypotheses* 1 *and* 2, *there exist* $d_1 > 0$ *with* $B_{X_\beta}(d_1) \subset W$ *and* $B_{R^n}(d_1) \subset \Lambda$, *and a* C^2-*mapping* z_β *of* $\{(a_1, a_2, \lambda): (a_1, a_2) \in B_{R^2}(d_1), \lambda \in B_{R^n}(d_1)\}$ *into* $B_{Z_\beta}(d_1)$ *such that*

$$z_\beta(0, 0, \lambda) = 0, \qquad D_\lambda z_\beta(0, 0, 0) = 0, \qquad D_{a_j} z_\beta(0, 0, 0) = 0, \tag{2.1}$$

$$\|D_{a_j} z_\beta(a_1, a_2, \lambda)\| < 1/4 \text{ for } (a_1, a_2, \lambda) \in B_{R^2}(d_1) \times B_{R^n}(d_1), \quad j = 1, 2$$

and such that for each λ, $|\lambda| < d_1$, *a two-dimensional manifold* C_λ *represented by*

$$C_\lambda \equiv \{a_1\phi_1 + a_2\phi_2 + z_\beta(a_1, a_2, \lambda): (a_1, a_2) \in B_{R^2}(d_1)\} \tag{2.2}$$

has the following properties for each $t > 0$:

(i) *Local invariance*: *If* $x_0 \in C_\lambda$ *and* $u(s, x_0, \lambda) \in B_{X_\beta}(d_1)$ *for all* s, $0 \leq s \leq t$, *then* $u(s, x_0, \lambda) \in C_\lambda$, $0 \leq s \leq t$.

(ii) *Local attractivity*: *If* $u(s, x_0, \lambda) \in B_{X_\beta}(d_1)$ *for all* s, $0 \leq s \leq t$, *then*

$$\begin{aligned} &\|z_\beta(a_1(s), a_2(s), \lambda) - Qu(s, x_0, \lambda)\| \\ &\leq K_1 e^{-\alpha s} \|z_\beta(a_1(0), a_2(0), \lambda) - Qx_0\|, \quad 0 \leq s \leq t, \end{aligned} \tag{2.3}$$

where $a_j(s) = \langle u(s, x_0, \lambda), \phi_j^* \rangle$ $(j = 1, 2)$, α *is as in Hypothesis* 1 (ii), *and* K_1 *is a constant independent of* t, λ *and* x_0.

For the proof of Theorem 2.1, see, e.g., [2], [13, Ths. 6.1.2, 6.1.4, and 6.1.7], [22], and [18].

Using the center manifold theorem, we reduce the existence problem of periodic orbits in a (possibly infinite-dimensional) Banach space to that of fixed points of a one-dimensional map. This reduction is well known (see, e.g., [12, 27, 29]). In what follows we state it following Marsden and McCracken [27, Sec. 3]. First in the rest of this section we reduce it to that of periodic orbits of a two-dimensional space. Then, in Section 2.2, we further reduce it to that of fixed points of a one-dimensional map.

We fix λ, $|\lambda|<d_1$, and simply write $u(t, x_0)$ for $u(t, x_0, \lambda)$. Using the decomposition $X_\beta=\text{span}\{\phi_1, \phi_2\}\oplus Z_\beta$, we decompose $u(t, x_0)$ into the form

$$u(t, x_0, \lambda)=a_1(t)\phi_1+a_2(t)\phi_2+v(t).$$

Then the equation (E) is decomposed into the system of differential equations

$$(2.4)\qquad \begin{cases} da_j(t)/dt=(-1)^{3-j}a_{3-j}(t)+\langle N(a_1(t)\phi_1+a_2(t)\phi_2+v(t), \lambda), \phi_j^*\rangle, & j=1, 2, \\ dv(t)/dt=Lv(t)+QN(a_1(t)\phi_1+a_2(t)\phi_2+v(t), \lambda) \end{cases}$$

with the initial value

$$(2.5)\qquad a_j(0)=\langle x_0, \phi_j^*\rangle, \quad j=1, 2, \quad v(0)=Qx_0.$$

In the case that $x_0\in\mathcal{C}_\lambda$, (E) can be reduced to the two-dimensional ordinary differential equation

$$(2.6)\qquad da_j(t)/dt=(-1)^{3-j}a_{3-j}(t)+f_j(a_1(t), a_2(t), \lambda), \quad j=1, 2,$$

$$(2.7)\qquad a_j(0)=a_{j,0}, \quad j=1, 2,$$

where f_j is defined by

$$(2.8)\qquad f_j(a_1, a_2, \lambda)\equiv\langle N(a_1\phi_1+a_2\phi_2+z_\beta(a_1, a_2, \lambda), \lambda), \phi_j^*\rangle, \quad j=1, 2,$$

for $(a_1, a_2)\in B_{R^2}(d_1)$, $\lambda\in B_{R^n}(d_1)$.

Indeed the following propositions hold.

PROPOSITION 2.2. *Assume that Hypotheses* 1 *and* 2 *are satisfied. Let* $x_0\in\mathcal{C}_\lambda$. *Then if, for some* $s>0$, *a solution* $u(t, x_0)$ *of* (E) *satisfies* $u(t, x_0)\in B_{X_\beta}(d_1)$ *for all* $t, 0\le t\le s$, *then* $u(t, x_0)\in\mathcal{C}_\lambda$, $0\le t\le s$, *and* $a_j(t)=\langle u(t, x_0), \phi_j^*\rangle$, $j=1, 2$, *is a solution of* (2.6)-(2.7) *with the initial value* $a_{j,0}=\langle x_0, \phi_j^*\rangle$, $j=1, 2$. *Conversely if, for some* $s>0$, *a solution* $(a_1(t), a_2(t))$ *of* (2.6)-(2.7) *satisfies* $(a_1(t), a_2(t))\in B_{R^2}(d_1)$ *for all* t, $0\le t\le s$, *then* $u(t)=a_1(t)\phi_1+a_2(t)\phi_2+z_\beta(a_1(t), a_2(t), \lambda)$ *is a solution of* (E) *with the initial value* $x_0=a_{1,0}\phi_1+a_{2,0}\phi_2+z_\beta(a_{1,0}, a_{2,0}, \lambda)$.

PROPOSITION 2.3. *Under Hypotheses* 1 *and* 2, *if* $w(t)$ *is a periodic solution of* (E) *in* $B_{X_\beta}(d_1)$, *then* $a_j(t)=\langle w(t), \phi_j^*\rangle$, $j=1, 2$, *is a periodic solution of* (2.6) *and* $w(t)$ *satisfies*

$$w(t)=a_1(t)\phi_1+a_2(t)\phi_2+z_\beta(a_1(t),\ a_2(t),\ \lambda)\,.$$

Conversely if $(a_1(t),\ a_2(t))$ *is a periodic solution of* (2.6) *in* $B_{R^2}(d_1)$, *then* $w(t)=a_1(t)\phi_1+a_2(t)\phi_2+z_\beta(a_1(t),\ a_2(t),\ \lambda)$ *is a periodic solution of* (E).

Propositions 2.2 and 2.3 can easily be proved by the local invariance and the local attractivity of a center manifold C_λ. See, e.g., [3, 27].

2.2. *Reduction to the one-dimensional problem* In this section we reduce the existence problem of periodic orbits of (2.6) to that of fixed points of a one-dimensional map. Then we give conditions which completely characterize the stability of fixed points of the one-dimensional map.

Following Marsden and McCracken [27, Sect. 3] we consider (2.6) in the polar coordinate:

$$a_1=r\cos\theta\,,\qquad a_2=r\sin\theta\,.$$

Then (2.6) has the form

$$\text{(2.9)}\qquad \begin{cases} dr/dt=R(r,\ \theta,\ \lambda)\\ d\theta/dt=-1+\Theta(r,\ \theta,\ \lambda)\,, \end{cases}$$

where

$$R(r,\ \theta,\ \lambda)=f_1(r\cos\theta,\ r\sin\theta,\ \lambda)\cos\theta+f_2(r\cos\theta,\ r\sin\theta,\ \lambda)\sin\theta\,,$$

$$\Theta(r,\ \theta,\ \lambda)$$

$$=\begin{cases} 1/r\{-f_1(r\cos\theta,\ r\sin\theta,\ \lambda))\sin\theta+f_2(r\cos\theta,\ r\sin\theta,\ \lambda)\cos\theta\}, & \text{if } r\neq 0,\\ -\partial f_1/\partial a_1(0,\ 0,\ \lambda)\cos\theta\sin\theta-\partial f_1/\partial a_2(0,\ 0,\ \lambda)\sin^2\theta & \\ \quad+\partial f_2/\partial a_1(0,\ 0,\ \lambda)\cos^2\theta+\partial f_2/\partial a_2(0,\ 0,\ \lambda)\sin\theta\cos\theta, & \text{if } r=0. \end{cases}$$

We denote the solution $(r(t),\ \theta(t))$ of (2.9) with $r(0)=r_0$ and $\theta(0)=\theta_0$ by $(r(t\,;\ r_0,\ \theta_0,\ \lambda),\ \theta(t\,;\ r_0,\ \theta_0,\ \lambda))$.

Lemma 2.4. There exist d_1' $(0<d_1'<d_1)$ and a C^1-mapping τ of $B_{R^1}(d_1')\times(-2\pi,\ 0]\times B_{R^n}(d_1')$ into $\boldsymbol{R}^1$ such that

$$\theta(\tau(r_0,\ \theta_0,\ \lambda)\,;\ r_0,\ \theta_0,\ \lambda)=-2\pi\,,\qquad \tau(0,\ \theta_0,\ 0)=2\pi+\theta_0\,.$$

Proof. By (2.1) and (2.8) we have $f_j\in C^2$, and

$$f_j(0,\ 0,\ \lambda)=0,\qquad D_\lambda f_j(0,\ 0,\ 0)=0,\qquad D_{a_k}f_j(0,\ 0,\ 0)=0,$$

for $j,\ k=1,\ 2$. Therefore

$$\text{(2.10)}\qquad R(0,\ \theta,\ \lambda)=0,\qquad \partial R/\partial r(0,\ \theta,\ 0)=0,\qquad \Theta(0,\ \theta,\ 0)=0\,.$$

Hence it follows that r, θ are C^1 in t, θ_0, and λ, and that

$$r(t;0,\theta_0,0)=0, \qquad \theta(t;0,\theta_0,0)=-t+\theta_0,$$
$$d\theta/dt(2\pi+\theta_0;0,\theta_0,0)=-1.$$

Thus, applying the implicit function theorem, we get the lemma.

By (2.10) there exists a positive constant K such that for any $(r_0, \theta_0, \lambda)\in B_{R^1}(d_1')\times(-2\pi, 0]\times B_{R^n}(d_1')$

$$K^{-1}|r_0|\leq|r(t;r_0,\theta_0,\lambda)|\leq K|r_0|, \qquad 0\leq t\leq\tau(r_0,\theta_0,\lambda). \tag{2.11}$$

Since we are interested only in periodic orbits, it is sufficient, by Lemma 2.4, to consider periodic solutions of (2.6)-(2.7) with $r_0\geq 0$, $\theta_0=0$. We write simply $r(t;r,\lambda)$, $\theta(t;r,\lambda)$, $\tau(r,\lambda)$ for $r(t;r,0,\lambda)$, $\theta(t;r,0,\lambda)$, $\tau(r,0,\lambda)$, respectively.

Now we define a bifurcation function p by

$$p(r,\lambda)=r(\tau(r,\lambda);r,\lambda) \text{ for } (r,\lambda)\in B_{R^1\times R^n}(d_2), \tag{2.12}$$

where d_2 is a positive number with

$$d_2<K^{-1}d_1' \ (K \text{ a positive number as in (2.11)}). \tag{2.13}$$

Then, by Lemma 2.4, $p\in C^1$ and $p(0,\lambda)=0$.

The relation between periodic orbits of (2.6) and fixed points of $p(\cdot,\lambda)$ is clarified by the following two propositions.

PROPOSITION 2.5. *If* r, $0\leq r<d_2$, *is a fixed point of* $p(\cdot,\lambda)$, *then*

$$\begin{cases} a_1(t)=r(t;r,\lambda)\cos\theta(t;r,\lambda) \\ a_2(t)=r(t;r,\lambda)\sin\theta(t;r,\lambda) \end{cases} \tag{2.14}$$

is a periodic solution of (2.6). *Conversely, if* $(a_1(t), a_2(t))$ *is a periodic solution of* (2.6) *with* $0\leq a_1(0)<d_2$, $a_2(0)=0$, *then* $a_1(0)$ *is a fixed point of* $p(\cdot,\lambda)$.

PROPOSITION 2.6. *The periodic solution of* (2.6) *given by* (2.14) *is orbitally (asymptotically) stable if and only if the fixed point* r, $0\leq r<d_2$, *of* $p(\cdot,\lambda)$ *is (asymptotically) stable.*

Thus the existence problem of periodic orbits of (E) is reduced to that of fixed points of $p(\cdot,\lambda)$. Since we are only concerned with periodic orbits of (E), Proposition 2.3 and Lemma 2.4 show that it is sufficient to consider a periodic solution $u(t,x_0,\lambda)$ of (E) with an initial value of the form $x_0=r\phi_1+z_\beta(r,0,\lambda)$ for our purposes. In what follows, we denote a periodic solution $u(t,x_0,\lambda)$ with $x_0=r\phi_1+z_\beta(r,0,\lambda)$ by $w(t;r,\lambda)$, that is,

$$w(t;r,\lambda)=u(t,x_0,\lambda), \quad x_0=r\phi_1+z_\beta(r,0,\lambda), \quad p(r,\lambda)=r, \ r\geq 0. \tag{2.15}$$

By $w(\cdot\,; r, \lambda)$ we denote the orbit of a periodic solution $w(t; r, \lambda)$. Note that $w(\cdot\,; r, \lambda)$ is a non-equilibrium periodic orbit if and only if $r>0$ is a fixed point, that $w(\cdot\,; 0, \lambda)=\{0\}$, and that $\tau(r, \lambda)$ is the period of a periodic orbit $w(\cdot\,; r, \lambda)$ if $r>0$.

2.3. The aim of this paper is not only to show that there eixsts a one-to-one correspondence between periodic orbits of (E) and fixed points of $p(\cdot, \lambda)$, but also to show that a periodic orbit of (E) is (asymptotically) stable if and only if the corresponding fixed point of $p(\cdot, \lambda)$ is (asymptotically) stable. In the rest of this section, we give the conditions which completely characterize the stability of a fixed point r of $p(\cdot, \lambda)$. Then, in Section 3, we shall show that those conditions also characterize the stability of the corresponding periodic orbit of (E).

Since $(\partial p/\partial r)(0, 0)=1$ (see [27, Lemma 3.7), we can, if necessary, choose $d_2>0$ so small that

$$(2.16) \qquad (\partial p/\partial r)(r, \lambda)>0, \qquad (r, \lambda)\in B_{R^{n+1}}(d_2).$$

Since $p(\cdot, \lambda)$ is a map on a one-dimensional space and since $\partial p/\partial r$ is positive, we easily see that the stability of a fixed point r is determined by the sign of $p(\gamma, \lambda)-\gamma$ near $\gamma=r$. To be more precise, we introduce the following terminology, which completely describes the behavior of $p(\gamma, \lambda)-\gamma$ near $\gamma=r$.

DEFINITION 2.7. (i) A function $q(\gamma, \lambda)$ is said to be outer-(resp. inner-) positive at $(\gamma, \lambda)=(r, \lambda)$ if $q(\gamma, \lambda)>0$ for all $\gamma>r$ (resp. $\gamma<r$) near r.

(ii) A function $q(\gamma, \lambda)$ is said to be outer-(resp. inner-) oscillatory at $(\gamma, \lambda)=(r, \lambda)$ if there exists a sequence $\{r_n\}_{n=1}^{\infty}$ with $r_n\downarrow r$ (resp. $r_n\uparrow r$) and $q(r_n, \lambda)=0$.

(iii) A function $q(\gamma, \lambda)$ is said to be outer-(resp. inner-) negative if $q(\gamma, \lambda)<0$ for all $\gamma>r$(resp. $\gamma<r$) near r.

Then we have

PROPOSITION 2.8. *A fixed point r of $p(\gamma, \lambda)$ is unstable if and only if $p(\gamma, \lambda)-\gamma$ is outer-positive or inner-negative at $(\gamma, \lambda)=(r, \lambda)$. Moreover it is asymptotically stable if and only if $p(\gamma, \lambda)-\gamma$ is outer-negative and inner-positive at $(\gamma, \lambda)=(r, \lambda)$.*

REMARK 2.9. Proposition 2.8 can be summarized by the following table:

inner \ outer	negative	oscillatory	potitive
positive	asymptotically stable	stable	unstable
oscillatory	stable	stable	unstable
negative	unstable	unstable	unstable

DEFINITION 2.10. We say that a periodic solution $w(t; r, \lambda)$ of (E) is outer-(resp. inner-) positive, oscillatory, and negative if $p(\gamma, \lambda)-\gamma$ is outer-(resp. inner-) positive, oscillatory, and negative at $\gamma=r$, respectively.

In Section 3, we shall show that the above table also applies to the orbital stability of $w(\cdot; r, \lambda)$ (Th. 3.1).

The stability of periodic orbits are usually determined by the critical (i.e., near one) eigenvalue of the linearized Poincaré map [7, 10, 13, 15, 27, 28, 35]. Therefore Theorem 3.1 suggests that there exists close relation between the critical eigenvalue and the bifurcation function $p(\cdot, \lambda)$. Before proceeding to Section 3, we investigate it in the next section 2.4.

2.4. *The eigenvalue of the linearized Poincaré map and the bifurcation function* $p(\cdot, \lambda)$ In this section we establish a fundamental relation between the critical eigenvalue of the linearized Poincaré map and the bifurcation function. Let $w(\cdot; r, \lambda)$ $(r>0)$ be fixed. First, we define a Poincaré map of $w(\cdot; r, \lambda)$. To this end, we decompose (E) as follows, using the cylindrical coordinate (r, θ, z):

$$\begin{cases} \dfrac{dr(t)}{dt}=\langle N, \phi_1^*\rangle \cos\theta(t)+\langle N, \phi_2^*\rangle \sin\theta(t), \\ \dfrac{d\theta(t)}{dt}=-1+\dfrac{1}{r(t)}\{-\langle N, \phi_1^*\rangle \sin\theta(t)+\langle N, \phi_2^*\rangle \cos\theta(t)\}, \\ \dfrac{dz(t)}{dt}=Lz(t)+QN, \end{cases}$$

where $\langle u(t), \phi_1^*\rangle = r(t)\cos\theta(t)$, $\langle u(t), \phi_2^*\rangle = r(t)\sin\theta(t)$, $z(t)=Qu(t)$, $N=N(r(t)\cos\theta(t)\varphi_1+r(t)\sin\theta(t)\varphi_2+z(t), \lambda)$. Let $Y_\beta=\text{span}\{\phi_1\}\oplus Z_\beta$. Then, by similar arguments as in the proof of Lemma 2.4, we see that there exists a C^2-mapping $\hat{\tau}$ of a neighborhood $\hat{U}$ of $r\phi_1+z_\beta(r, 0, \lambda)$ in Y_β such that

$$\langle u(\hat{\tau}(y, \lambda), y, \lambda), \phi_2^*\rangle=0, \qquad y\in\hat{U},$$

$$\hat{\tau}(r\phi_1+z_\beta(r, 0, \lambda), \lambda)=\tau(r, \lambda).$$

A Poincaré map of $w(\cdot; r, \lambda)$ is defined by $\hat{P}(y)=u(\hat{\tau}(y, \lambda), r, \lambda)$, $y\in\hat{U}$. Then the following theorem holds.

THEOREM 2.11. *Let $w(\cdot; r, \lambda)$ be a non-equilibrium periodic orbit of* (E). *Then*

$$D_y\hat{P}(r\phi_1+z_\beta(r, 0, \lambda))[\phi_1+D_{a_1}z_\beta(r, 0, \lambda)]=(\partial p(r, \lambda)/\partial r)[\phi_1+D_{a_1}z_\beta(r, 0, \lambda)]. \tag{2.17}$$

Proof. By definitions of p and $\hat{P}$, we have for r' near r

$$p(r', \lambda)=\langle \hat{P}(r'\phi_1+z_\beta(r', 0, \lambda)), \phi_1^*\rangle, \tag{2.18}$$

$$\hat{P}(r'\phi_1+z_\beta(r', 0, \lambda))=\langle \hat{P}, \phi_1^*\rangle\phi_1+z_\beta(\langle \hat{P}, \phi_1^*\rangle, 0, \lambda). \tag{2.19}$$

Differentiating both sides of (2.19) with respect to r' at $r'=r$, and noticing (2.18) and

$$\hat{P}(r\phi_1+z_\beta(r,\ 0,\ \lambda))=r\phi_1+z_\beta(r,\ 0,\ \lambda),$$

we obtain (2.17). Q.E.D.

REMARK 2.12. On the stability of a stationary solution $(0, \lambda)$, Marsden and McCracken [27, Lemma 3.7] gives the following relation

$$\frac{\partial p(0,\ \lambda)}{\partial r}=e^{2\pi Re\,\kappa(\lambda)/Im\,\kappa(\lambda)},$$

where $\kappa(\lambda)$ is an eigenvalue near i of the linearized operator $L+D_xN(0, \lambda)$.

REMARK 2.13. Since $\kappa(0)=i$, the critical eigenvalue of the linearized Poincaré map $D_y\hat{P}(w(0; r, \lambda))$ is $\partial p/\partial r(r, \lambda)$. It is well know that if the critical eigenvalue $\partial p/\partial r(r, \lambda)$ is less (resp. larger) than 1, then the periodic solution $w(t; r, \lambda)$ is orbitally asymptotically stable (resp. orbitally unstable). By Definitions 2.7 and 2.10, if $\partial p/\partial r(r, \lambda)<1$ (resp. $(\partial p/\partial r)(r, \lambda)>1$) then $w(t; r, \lambda)$ is outer-negative and inner-positive (resp. outer-positive and inner-negative). Theorem 3.1 says that the stability is also determined even if $(\partial p/\partial r)(r, \lambda)=1$.

§ 3. Stability and asymptotic behavior

3.1. *Stability and bifurcation* In the following theorems, by the term "stable", "unstable", or "asymptotically stable" we mean "stable, unstable, or asymptotically stable in the topology of X_β". In this section, we state our main results on stability and bifurcation.

THEOREM 3.1. (*Stability theorem*) *Under Hypotheses* 1 *and* 2, *there exists a positive number d for which the following statement hold*:

(i) *A stationary solution* $(0, \lambda)$ *of* (E) *in* $B_{X_\beta}(d)$ *is unstable if and only if it is outer-positive* (*see Def.* 2.10). *Moreover it is asymptotically stable if and only if it is outer-negative.*

(ii) *A non-stationary periodic solution* $w(t; r, \lambda)$ (*given by* (2.15)) *of* (E) *in* $B_{X_\beta}(d)$ *is orbitally unstable if and only if it is outer-positive or inner-negative. Moreover it is orbitally asymptotically stable if and only if it is outer-negative and inner-positive.*

THEOREM 3.2 (*Bifurcation theorem*) *Assume that Hypotheses* 1 *and* 2 *be satisfied. Let* λ *be a real number. Then if a stationary solution* $(0, \lambda)$ *of* (E) *changes its stability at* $\lambda=0$, *then non-equilibrium periodic orbits of* (E) *bifurcate from* $(x, \lambda)=(0, 0)$.

REMARK 3.3. Theorem 3.1 can be summarized by the table in Remark 2.9.

REMARK 3.4. If the condition (i) $\partial \mathrm{Re}\,\kappa(0)/\partial\lambda \neq 0$, is satisfied, then non-equilibrium periodic orbits of (E) bifurcate from $(x, \lambda)=(0, 0)$ (see, e.g., [7], [27]). The condition (i) can be replaced by one of the following conditions (ii)-(iv):

(ii) $\mathrm{Re}\,\kappa(\lambda)>0$ for $\lambda>0$ and a stationary solution $(0, 0)$ is asymptotically stable [3, 4].

(iii) $\mathrm{Re}\,\kappa(\lambda)$ changes its sign at $\lambda=0$ [35].

(iv) A stationary solution $(0, \lambda)$ is stable for $\lambda<0$ and unstable for $\lambda>0$ [6, Chap. 9].

All the above conditions (i)-(iv) are sufficient for a stationary solution $(0, \lambda)$ to change its stability at $\lambda=0$. (Note Theorem 3.1 and Remark 2.13).

3.2. *Reduction of Theorem* 3.2 *to Theorem* 3.1 We claim that there exists a sequence $\{(r_n, \lambda_n)\}_{n=1}^{\infty}$ such that $r_n>0$, $p(r_n, \lambda_n)=r_n$, and $(r_n, \lambda_n) \longrightarrow (0, 0)$ as $n \longrightarrow \infty$. If this is proved, then, by Propositions 2.3 and 2.5, and by recalling that $w(\cdot\,; r, \lambda)$ is non-equilibrium if $r>0$, we have non-equilibrium periodic orbits $w(\cdot\,; r_n, \lambda_n)$. Hence $(x, \lambda)=(0, 0)$ is a bifurcation point.

We now prove the above claim. Suppose that there does not exist such a sequence. Then, by Theorem 3.1 and our assumption on a stationary soltion $(0, \lambda)$, there exists $\varepsilon>0$ such that for each λ, $|\lambda|<\varepsilon$, $p(\gamma, \lambda)-\gamma$ is right-positive or right-negative at $\gamma=0$. Again by Theorem 3.1 and our assumption on $(0, \lambda)$, there exist two sequences $\{\lambda_n^+\}_{n=1}^{\infty}$, $\{\lambda_n^-\}_{n=1}^{\infty}$ such that $\lambda_n^+ \longrightarrow 0$, $\lambda_n^- \longrightarrow 0$, $p(\gamma, \lambda_n^+)-\gamma$ is right-positive at $\gamma=0$, and $p(\gamma, \lambda_n^-)-\gamma$ is right-negative at $\gamma=0$. Hence there exists $r_n>0$ such that

$$p(r_n, \lambda_n^+)-r_n>0, \qquad p(r_n, \lambda_n^-)-r_n<0,$$

and $r_n\downarrow 0$ as $n \longrightarrow 0$. Hence by the mean value theorem, there exists a number λ_n between λ_n^+ and λ_n^- such that $p(r_n, \lambda_n)-r_n=0$. Thus we have a contradiction. This completes the proof of Theorem 3.2.

3.3. *The stable manifold theorem* Let $w(t\,; r, \lambda)$ be a periodic solution of (E). We characterize initial values x_0 such that the solution $u(t, x_0, \lambda)$ of (E) exists on $[0, \infty)$ and converges to the periodic solution $w(t\,; r, \lambda)$ of (E) with the exponential rate α(α is a constant as in Hypothesis 1):

$$\|u(t, x_0, \lambda)-w(t\,; r, \lambda)\| \leq K_2 e^{-\alpha t}, \quad t\geq 0, \tag{3.1}$$

where K_2 is some positive constant. This characterization is known as the stable manifold theorem. In this paper, we use it in the following form.

THEOREM 3.5. (*Stable manifold theorem*) *Under Hypotheses* 1 *and* 2, *there exists a positive constant d_3 for which the following statement holds: For each $(x, \lambda)\in B_{X_\beta}(d_3)\times B_{R^n}(d_3)$ there exists a C^1-mapping $a_j(z\,; x, \lambda)$, $j=1, 2$, of $B_{Z_\beta}(d_3)$*

into $\boldsymbol{R}^1$ *with*

$$a_j(0\,;\,x,\,\lambda)=0, \quad \|D_z a_j(z\,;\,x,\,\lambda)\|<1/2 \quad \textit{for } z\in B_{Z_\beta}(d_3) \tag{3.2}$$

such that for each λ, $|\lambda|<d_3$, *a manifold* $\mathcal{M}(x,\lambda)$ *of codimension two represented by*

$$\mathcal{M}(x,\,\lambda)\equiv\{x+\sum_{j=1,2}a_j(z\,;\,x,\,\lambda)\phi_j+z : z\in B_{Z_\beta}(d_3)\} \tag{3.3}$$

has the following properties:

(i) *If* $u(t,x,\lambda)\in B_{X_\beta}(d_3)$, $t\geq 0$, *and if*

$$\|u(t,\,x_0,\,\lambda)-u(t,\,x,\,\lambda)\|\leq K_2e^{-\alpha t}, \quad t\geq 0, \tag{3.4}$$

then $x_0\in\mathcal{M}(x,\lambda)$, *where* K_2 *is a positive constant independent of* t, x, λ.

(ii) *Suppose that for some* $s>0$, $u(t,x,\lambda)\in B_{X_\beta}(d_3)$, $0\leq t\leq s$. *Then if* $x_0\in\mathcal{M}(x,\lambda)$, *then*

$$\|u(t,\,x_0,\,\lambda)-u(t,\,x,\,\lambda)\|\leq K_3e^{-\alpha t}\|x_0-x\|, \quad 0\leq t\leq s, \tag{3.5}$$

where K_3 *is a positive constant independent of* t, x, λ.

Furthermore if $w(t)$ *is a periodic solution of* (E) *in* $B_{X_\beta}(d_3)$, *then* $a_j(z\,;\,w(t),\lambda)$, $j=1, 2$, *is continuous in* (z,t).

The proof of Theorem 3.5 is standard. See, e.g. [3, 4, 11, 13, 14, and 23].

Let $w(\cdot\,;\,r,\lambda)$, $|\lambda|<d_3$, be a periodic orbit in $B_{X_\beta}(d_3)$. Then for each $x=w(s\,;\,r,\lambda)$, $s\geq 0$, $u(t,x,\lambda)\in B_{X_\beta}(d_3)$ holds for $t\geq 0$, since $u(t,x,\lambda)=u(t,w(s\,;\,r,\lambda),\lambda)=w(t+s\,;\,r,\lambda)$ holds for $t\geq 0$. (See (2.15)). Hence, with x replaced by $w(s\,;\,r,\lambda)$, (3.5) holds for $t\geq 0$. We call $\mathcal{M}(0,\lambda)$ the stable manifold of a stationary solution $(0,\lambda)$.

In what follows, we shall define a stable manifold of a non-equilibrium periodic orbit $w(\cdot\,;\,r,\lambda)$ and give its properties. To this end, we consider the set $\bigcup_{0\leq s\leq\tau(r,\lambda)}\mathcal{M}(w(s\,;\,r,\lambda),\lambda)$. In the following, for simplicity, we write $z_\beta(b_1,b_2)$ and $a_j(z\,;\,x)$ for $z_\beta(b_1,b_2,\lambda)$ and $a_j(z\,;\,x,\lambda)$, respectively. Also, we simply write $w(s)$ and $b_j(s)$ for $w(s\,;\,r,\lambda)$ and $\langle w(s\,;\,r,\lambda),\phi_j^*\rangle$, respectively. We note that $w(s)=\sum_{j=1,2}b_j(s)\phi_j+z_\beta(b_1(s),b_2(s))$. By the definition of $\mathcal{M}(x,\lambda)$, we have

$$\begin{aligned}
&\bigcup_{0\leq s\leq\tau(r,\lambda)}\mathcal{M}(w(s),\,\lambda)\\
&\quad=\bigcup_{0\leq s\leq\tau(r,\lambda)}\{w(s)+\sum_{j=1,2}a_j(z\,;\,w(s))\phi_j+z : z\in B_{X_\beta}(d_3)\}\\
&\quad=\bigcup_{0\leq s\leq\tau(r,\lambda)}\{\sum_{j=1,2}(b_j(s)+a_j(z\,;\,w(s)))\phi_j+z+Qw(s) : z\in B_{X_\beta}(d_3)\}.
\end{aligned}$$

We shall investigate its local structure. Let

$$d_4\equiv\min\{d_3/3,\ K_2K_3^{-1}/3\}. \tag{3.6}$$

Let us consider the set

$$(3.7)\qquad \mathcal{M}(r,\ \lambda)\equiv\Big(\bigcup_{0\le s\le\tau(r,\lambda)}\tilde{\mathcal{M}}(w(s),\ \lambda)\Big)\cap\{x:\ \|Qx\|<d_4\}.$$

Then, by (2.1) and (3.2), we get

$$(3.8)\qquad \mathcal{M}(r,\ \lambda)$$

$$=\bigcup_{0\le s\le\tau(r,\lambda)}\Big\{\sum_{j=1,2}(b_j(s)+a_j(z-z_\beta(b_1(s),\ b_2(s));\ w(s)))\phi_j+z: z\in B_{X_\beta}(d_4)\Big\}.$$

If we set

$$(3.9)\qquad g_j(b_1,\ b_2;\ z)\equiv b_j+a_j(z-z_\beta(b_1,\ b_2);\ \sum_{k=1,2}b_k\phi_k+z_\beta(b_1,\ b_2)),\ j=1,\ 2,$$

for $(b_1,\ b_2)\in B_{\boldsymbol{R}^2}(d_4)$, then

$$(3.10)\qquad \mathcal{M}(r,\ \lambda)$$

$$=\bigcup_{0\le s\le\tau(r,\lambda)}\Big\{\sum_{j=1,2}g_j(b_1(s),\ b_2(s);\ z)\phi_j+z: z\in B_{Z_\beta}(d_4)\Big\}.$$

We consider the intersection of $\mathcal{M}(r,\ \lambda)$ and a plane $\{x:\ Qx=z\}$. For each $z\in B_{Z_\beta}(d_4)$, we set

$$(3.11)\qquad C(r,\ \lambda;\ z)$$

$$\equiv\{(g_1(b_1(s),\ b_2(s);\ z),\ g_2(b_1(s),\ b_2(s);\ z)): 0\le s\le\tau(r,\ \lambda)\}.$$

We note that if we identify the plane $\{x:\ Qx=z\}$ with $\boldsymbol{R}^2$, then $C(r,\ \lambda;\ z)$ is the intersection of $\mathcal{M}(r,\ \lambda)$ with the plane $\{x:\ Qx=z\}$. The structures of $\mathcal{M}(r,\ \lambda)$, $C(r,\ \lambda;\ z)$ are given by

PROPOSITION 3.6. *Let $w(\cdot\,;\ r,\ \lambda)$ be a non-equilibrium periodic orbit in $B_{X_\beta}(d_4)$. Then the following hold:*

(i) *The set $\mathcal{M}(r,\ \lambda)$ defined by* (3.7) *is a C^0-manifold of codimension one.*
(ii) *For each $z\in B_{Z_\beta}(d_4)$, $C(r,\ \lambda;\ z)$ is a Jordan curve in $B_{\boldsymbol{R}^2}(d_3)$.*

To prove this we require the following lemma, which will repeatedly be used later on.

LEMMA 3.7. *For each $z\in B_{Z_\beta}(d_4)$, a mapping $g_z=(g_1(\cdot,\cdot\,;\ z))$ of $\{(b_1,\ b_2)\in B_{\boldsymbol{R}^2}(d_4): for\ x\equiv b_1\phi_1+b_2\phi_2+z_\beta(b_1,\ b_2,\ \lambda),\ u(t,\ x,\ \lambda)\in B_{X_\beta}(d_3),\ t\ge 0\}$ into $\boldsymbol{R}^2$ is one-to-one, where g_j is defined by* (3.9).

Proof. Suppose that $g_j(b_1,\ b_2;\ z)=g_j(b_1',\ b_2';\ z)\ (=c_j)$, $j=1,\ 2$. Then if we set

$$x_1=\sum_{j=1,2} b_j\phi_j+z_\beta(b_1,\ b_2,\ \lambda), \qquad x_2=\sum_{j=1,2} b'_j\phi_j+z_\beta(b'_j b'_2,\ \lambda),$$

then we have by (3.3) and (3.9),

$$x\equiv\sum_{j=1,2} c_j\phi_j+z\in\hat{\mathcal{M}}(x_1,\ \lambda)\cap\hat{\mathcal{M}}(x_2,\ \lambda),$$

and by (2.1) and (3.2), we get

$$\|x-x_j\|<3d_4/2 \qquad (j=1,\ 2). \tag{3.12}$$

Hence, applying Theorem 3.5 (ii), we have the inequalities

$$\|u(t,\ x,\ \lambda)-u(t,\ x_j,\ \lambda)\|\leq K_3e^{-\alpha t}\|x-x_j\|, \quad t\geq 0 \qquad (j=1,\ 2).$$

Adding both sides of the inequalities, we get by (3.6) and (3.12),

$$\begin{aligned}\|u(t,\ x_1,\ \lambda)-u(t,\ x_2,\ \lambda)\| &\leq K_3e^{-\alpha t}\|x_1-x_2\| \\ &\leq 3K_3d_4e^{-\alpha t}\leq K_2e^{-\alpha t}, \qquad t\geq 0.\end{aligned}$$

Hence, by Theorem 3.5 (i), we obtain $x_1\in\hat{\mathcal{M}}(x_2,\ \lambda)$. On the other hand, by (2.1) and (3.2), we get $C_\lambda\cap\hat{\mathcal{M}}(x_2,\ \lambda)=\{x_2\}$. Therefore, since $x_1\in C_\lambda$, it follows that $x_1=x_2$, and so $b_j=b'_j$ $(j=1,\ 2)$. Q.E.D.

Proof of Proposition 3.6. *Proof of* (i): Let $(\hat{c}_1,\ \hat{c}_2,\ \hat{z})$ be a point in $\mathcal{M}(r,\ \lambda)$. Then by (3.10) there exists a $t>0$ such that $\hat{c}_j=g_j(b_1(t),\ b_2(t)\,;\ \hat{z})$, $j=1,\ 2$. Here we can assume that the domain of definition of $b_j(t)=\langle w(t),\ \phi_j^*\rangle$ is extended on the whole of $\boldsymbol{R}^1$. Let $\mathcal{I}$ be an open interval containing t with length smaller than $\tau(r,\ \lambda)$. If we construct a homeomorphism $\mathfrak{g}$ of $\mathcal{I}\times B_{Z_\beta}(d_4)$ onto a neighborhood of $(\hat{c}_1,\ \hat{c}_2,\ \hat{z})$ in $\mathcal{M}(r,\ \lambda)$, then we can conclude that $\mathcal{M}(r,\ \lambda)$ is a C^0-manifold of codimension one. We define a mapping $\mathfrak{g}=(\mathfrak{g}_1,\ \mathfrak{g}_2,\ \mathfrak{g}_3)$ by setting

$$\mathfrak{g}_j(s,\ z)=g_j(b_1(s),\ b_2(s)\,;\ z) \quad (j=1,\ 2), \quad \mathfrak{g}_3(s,\ z)=z \tag{3.13}$$

for $(s,\ z)\in\mathcal{I}\times B_{Z_\beta}(d_4)$. We shall show that $\mathfrak{g}$ is a homeomorphism. First, by Theorem 3.5, we see that $\mathfrak{g}$ is continuous, and by (3.10), the image of $\mathfrak{g}$ is a neighborhood of $(\hat{c}_1,\ \hat{c}_2,\ \hat{z})$. We next show that $\mathfrak{g}$ is one-to-one. Suppose that $\mathfrak{g}(s,\ z)=\mathfrak{g}(s',\ z')$, i.e., $\mathfrak{g}_j(s,\ z)=\mathfrak{g}_j(s',\ z')$, $j=1,\ 2$, and $z=z'$. Then, by the definition of $\mathfrak{g}_j$, we have $g_j(b_1(s),\ b_2(s)\,;\ z)=g_j(b_1(s'),\ b_2(s')\,;\ z)$. Hence, by Lemma 3.7, we get $b_j(s)=b_j(s')$. Since $b_j(s)$, $j=1,\ 2$, is periodic with period $\tau(r,\ \lambda)$ and since the length of $\mathcal{I}$ is smaller than $\tau(r,\ \lambda)$, we obtain that $s=s'$. Lastly, we show that $\mathfrak{g}^{-1}$ is continuous. Let $(c_1,\ c_2,\ z)$, $(c'_1,\ c'_2,\ z')$ be points in the image of $\mathfrak{g}$. Set

$$\mathfrak{g}^{-1}(c_1,\ c_2,\ z)=(b_1(s),\ b_2(s),\ z), \qquad \mathfrak{g}^{-1}(c'_2,\ c'_2,\ z')=(b_1(s'),\ b_2(s'),\ z').$$

Then, by (3.2), (3.9), and (3.13), the inequalities

$$|c_j - c'_j| \geq |\hat{g}_j(s, z) - \hat{g}_j(s', z)| - |\hat{g}_j(s', z) - \hat{g}_j(s', z')|$$
$$\geq |\hat{g}_j(s, z) - \hat{g}_j(s', z)| - 1/2\|z - z'\|$$

hold. Hence we get

$$|\hat{g}_j(s, z) - \hat{g}_j(s', z)| \leq |c_j - c'_j| + 1/2\|z - z'\|. \tag{3.14}$$

Let $z \in B_{Z_\beta}(d_4)$ be fixed. We consider a mapping $\hat{g}_z = (\hat{g}_1(\cdot, z), \hat{g}_2(\cdot, z)) : \mathcal{J} \longrightarrow \boldsymbol{R}^2$. Since $\hat{g}$ is continuous and one-to-one, so is $\hat{g}_z$. Therefore, by the domain invariance theorem [34], we have that $\hat{g}_z^{-1}$ is continuous. Hence, together with the inequality (3.14), we can conclude that $\hat{g}^{-1}$ is continuous. Thus the proof of (i) is complete.

Proof of (ii): By (2.1), (3.2), and (3.11), we have $\mathcal{C}(r, \lambda; z) \subset B_{R^2}(d_3)$. Since $g_j(b_1(s), b_2(s); z)$ is continuous in s and since $\{(b_1(s), b_2(s)) : 0 \leq s \leq \tau(r, \lambda)\}$ is a Jordan curve, we see by Lemma 3.7 that $\mathcal{C}(r, \lambda; z)$ is a Jordan curve. Q.E.D.

In what follows we exclusively consider a periodic orbit (w, λ) in $B_{X_\beta}(d_4) \times B_{R^n}(d_3)$.

DEFINITION 3.8. We call the manifold $\mathcal{M}(r, \lambda)$ defined by (3.7) the stable manifold of the non-equilibrium periodic orbit $w(\cdot\,; r, \lambda)$ of (E). We also call $\mathcal{M}(0, \lambda) \equiv \hat{\mathcal{M}}(0, \lambda) \cap \{x : \|Qx\| < d_4\}$ the stable manifold of the stationary solution $(0, \lambda)$ of (E).

In virtue of Proposition 3.6 (ii), we can define the inside (outside) of the stable manifold $\mathcal{M}(r, \lambda)$ of codimension one as follows.

DEFINITION 3.9. A point x with $\|Px\| < d_3$ and $\|Qx\| < d_4$ is said to be contained inside (resp. outside) $\mathcal{M}(r, \lambda)$ if $(\langle x, \phi_1^*\rangle, \langle x, \phi_2^*\rangle)$ is contained inside (resp. outside) the Jordan curve $\mathcal{C}(r, \lambda; Qx)$ defined by (3.11).

The set of all points inside (resp. outside) $\mathcal{M}(r, \lambda)$ is denoted by $\mathcal{M}_{in}(r, \lambda)$ (resp. $\mathcal{M}_{out}(r, \lambda)$). The set $\{x : \|Px\| < d_3, \|Qx\| < d_4\} \setminus \mathcal{M}(0, \lambda)$ is denoted by $\mathcal{M}_{out}(0, \lambda)$. For later convenience, we understand that $\mathcal{M}_{in}(0, \lambda)$ means $\emptyset$.

In the rest of this section, we prove two propositions on the properties of $\mathcal{M}(r, \lambda)$. The former gives the relation between the positions of two stable manifolds of periodic orbits of (E). The latter concerns that between a solution of (E) and the stable manifold of a periodic orbit.

PROPOSITION 3.10. *Let* λ, $|\lambda| < d_3$, *be fixed. Let* $w(\cdot, r_j, \lambda)$, $j = 1, 2$, *be periodic orbits in* $B_{X_\beta}(d_4)$ *with* $0 \leq r_1 < r_2$. *Then*

$$\mathcal{M}(r_1, \lambda) \subset \mathcal{M}_{in}(r_2, \lambda). \tag{3.15}$$

PROPOSITION 3.11. *Let* λ, $|\lambda| < d_3$, *be fixed. Let* $w(\cdot\,; r, \lambda)$ *be a non-equi-*

librium periodic orbit in $B_{X_\beta}(d_4)$. *Then if* $x_0 \in \mathcal{M}_{in}(r, \lambda)$ (*resp.* $\mathcal{M}(r, \lambda)$, $\mathcal{M}_{out}(r, \lambda)$) *and if, for some* $s>0$, $\|Pu(t, x_0, \lambda)\|<d_3$, $\|Qu(t, x_0, \lambda)\|<d_4$, $0\le t\le s$, *then*

$$(3.16) \qquad u(t, x_0, \lambda)\in \mathcal{M}_{in}(r, \lambda) \quad (\textit{resp. } \mathcal{M}(r, \lambda),\ \mathcal{M}_{out}(r, \lambda)), \quad 0\le t\le s\,.$$

For the proofs of Propositions 3.10 and 3.11, we need the following lemma.

LEMMA 3.12. (i) (c_1, c_2) *is contained inside* (*resp. outside*) *the Jordan curve* $\{(b_1(s), b_2(s)) : 0\le s\le \tau(r, \lambda)\}$ *if and only if* (c_1, c_2) *is contained inside* (*resp. outside*) *the Jordan curve* $C(r, \lambda ; z_\beta(c_1, c_2))$.

(ii) *Let* $z\in B_{Z_\beta}(d_4)$. *If* (c_1, c_2) *is contained inside* $C(r, \lambda ; z)$ *then there exists* $d'>0$ *such that for each* $z'\in Z_\beta$ *with* $\|z-z'\|<d'$, (c_1, c_2) *is contained inside* $C(r, \lambda ; z')$.

The proof of Lemma 3.12 will be given after the proofs of Propositions 3.10 and 3.11.

Proof of Proposition 3.10. By Lemma 3.12 (i) and by the hypothesis that $0\le r_1<r_2$, we have

$$(3.17) \qquad w(\cdot\, ; r_1, \lambda)\subset \mathcal{M}_{in}(r_2, \lambda)\,.$$

Suppose that (3.15) does not hold. Then, by the definition of $\mathcal{M}(r_1, \lambda)$, there exist $x_1\in w(\cdot\, ; r_1, \lambda)$ and x, $\|Qx\|<d_4$, such that $x\in \hat{\mathcal{M}}(x_1, \lambda)$ and $x\notin \mathcal{M}_{in}(r_2, \lambda)$. Hence, by Lemma 3.12 (ii) and (3.17), there exists $\hat{x}$ such that $\hat{x}\in \hat{\mathcal{M}}(x_1, \lambda)\cap \mathcal{M}(r_2, \lambda)$. This implies that $Q\hat{x}\in Z_\beta(d_4)$ and that there exists $x_2\in w(\cdot\, ; r_2, \lambda)$ such that $\hat{x}\in \hat{\mathcal{M}}(x_1, \lambda)\cap \hat{\mathcal{M}}(x_2, \lambda)$. Since $x_j\in \mathcal{C}_\lambda$, we get by (3.3), (3.10),

$$\langle \hat{x}, \phi_j^*\rangle = g_j(\langle x_1, \phi_1^*\rangle, \langle x_1, \phi_2^*\rangle ; Q\hat{x}) = g_j(\langle x_2, \phi_1^*\rangle, \langle x_2, \phi_2^*\rangle ; Q\hat{x}),\ j=1,\ 2.$$

Hence, by Lemma 3.7, we have $\langle x_1, \phi_k^*\rangle=\langle x_2, \phi_k^*\rangle$, $k=1, 2$. Since $x_j\in \mathcal{C}_\lambda$, we get $x_1=x_2$. This, however, contradicts the hypothesis that $0\le r_1<r_2$, since $x_1\in w(\cdot\, ; r_j, \lambda)$, $j=1, 2$. Q.E.D.

Proof of Proposition 3.11. We shall consider only the case that $x_0\in \mathcal{M}_{in}(r, \lambda)$. In the case that $x_0\in \mathcal{M}(r, \lambda)\cup \mathcal{M}_{out}(r, \lambda)$, the proof is similar. Suppose that (3.16) does not hold. We set

$$(3.18) \qquad T=\sup\{\tau : \ u(t, x_0, \lambda)\in \mathcal{M}_{in}(r, \lambda), \quad 0\le t\le \tau\}\,.$$

Then, by Lemma 3.12 (ii), we have $0<T\le s$ and $u(T, x_0, \lambda)\in \mathcal{M}(r, \lambda)$. By the definition of $\mathcal{M}(r, \lambda)$, this means that $u(T, x_0, \lambda)\in \hat{\mathcal{M}}(w(\hat{s}\, ; r, \lambda), \lambda)$ for some $\hat{s}\ge 0$, and so by (3.2), (3.3), and (3.6), we get

$$\|u(T, x_0, \lambda)-w(\hat{s}\, ; r, \lambda)\| = \|Qu(T, x_0, \lambda)-Qw(\hat{s}\, ; r, \lambda)\|\le 2d_4<d_3\,.$$

Therefore we can apply Theorem 3.5 (ii). We recall here that $w(t\, ; r, \lambda)=u(t, x, \lambda)$ and $x=r\phi_1+z_\beta(r, 0)$, then using the semigroup property of a solution $u(t+s, x, \lambda)$

$=u(t, u(s, x, \lambda), \lambda)$, we get by (3.6),

$$\|u(t, x_0, \lambda)-w(t-T+\hat{s}; r, \lambda)\|$$
$$=\|u(t-T, u(T, x_0, \lambda), \lambda)-u(t-T, w(\hat{s}; r, \lambda), \lambda)\|$$
$$\leq K_3 e^{-\alpha(t-T)}\|u(T, x_0, \lambda)-w(\hat{s}; r, \lambda)\|$$
$$\leq 2d_4 K_3 e^{-\alpha(t-T)}<K_2 e^{-\alpha(t-T)}, \quad t\geq T.$$

Hence, by the continuity of $u(\cdot, x_0, \lambda)$ there exists $\hat{T}$, $0<\hat{T}<T$, such that $\|u(t, x_0, \lambda)-w(t-T+\hat{s}; r, \lambda)\|<K_2 e^{-\alpha(t-\hat{T})}$, $t\geq\hat{T}$. Thus, using the semigroup property, we obtain by Theorem 3.5 (i)

$$u(\hat{T}, x_0, \lambda)\in\hat{\mathcal{M}}(w(\hat{T}-T+\hat{s}; r, \lambda), \lambda). \tag{3.18}$$

(Here we assume that the domain of $w(t; r, \lambda)$ is extended on the whole of $\boldsymbol{R}^1$). Since $\|Qu(T, x_0, \lambda)\|<d_4$, we get $u(\hat{T}, x_0, \lambda)\in\mathcal{M}(r, \lambda)$. This contradicts (3.18) and $\hat{T}<T$. Q.E.D.

Proof of Lemma 3.12. We first recall the following fact from the degree theory [34]:

Let C be a Jordan curve and let C_{in} (resp. C_{out}) denote the inside (resp. outside) of C. Suppose that $c\equiv(c_1, c_2)\notin C$. Then if $c\in C_{in}$ (resp. C_{out}), then $\deg(c, 1, C_{in})=1$ (resp. $\deg(c, 1, C_{in})=0$).

By (2.1) and (3.2), we get the following inequalities.

$$(3/4)\sum_{j=1,2}|c_j-b_j(s)|\leq\sum_{j=1,2}|c_j-g_j(b_1(s), b_2(s); z_\beta(c_1, c_2))|$$
$$\leq(4/5)\sum_{j=1,2}|c_j-b_j(s)|.$$

Using these inequalities, we can see that $c\equiv(c_1, c_2)\in C(r, \lambda; z_\beta(c_1, c_2))$ if and only if $c\in C\equiv\{(b_1(s), b_2(s)): 0\leq s\leq\tau(r, \lambda)\}$. For the proof of (i), it therefore suffices to show that

$$\deg(c, 1, C_{in})=\deg(c, 1, C_{in}(r, \lambda; z_\beta(c_1, c_2))).$$

This can be proved by applying the standard arguments as in the proof of the homotopy invariance property of degree to the above inequalities [34, Chap. III]. For the proof of (ii), we only need to note that $C(r, \lambda; z)$ is compact. Then we can prove (ii) by the same way as in the proof of (i). Q.E.D.

3.4. *Asymtotic behavior* We now state our results on the asymptotic behavior of solutions of (E).

THEOREM 3.13. *Under Hypotheses* 1 *and* 2, *there exist a positive number d* *and neighborhoods U, $U(\lambda)$, $|\lambda|<d$, of 0 in X_β with U contained in $U(\lambda)$ and independent of λ such that the local ω-limit set $\Omega_{U,U(\lambda)}(x_0, \lambda)$ of a solution $u(t, x_0, \lambda)$ of* (E) *(with respect to $\{U, U(\lambda)\}$) consists of a single periodic orbit $\gamma(x_0, \lambda)$ (which may be an equilibrium $\{0\}$) in $U(\lambda)$ if $\Omega_{U,U(\lambda)}(x_0, \lambda)$ is not empty. Furthermore, $u(t, x_0, \lambda)$ converges to the periodic orbit $\lambda(x_0, \lambda)$ as $t \longrightarrow \infty$.*

The next theorem shows that the mapping $\Omega_{U,U(\lambda)}(\cdot, \lambda)$: $U \longrightarrow 2^{X_\beta}$ can be determined by the behavior of the bifurcation function $p(\cdot, \lambda)$ and the position of an initial value x_0 situated around the stable manifolds of periodic orbits of (E).

To describe these circumstances, we need to consider the following sets:

$$R_1(x_0, \lambda) \equiv \{r : x_0 \in \mathcal{M}_{out}(r, \lambda) \text{ or } x_0 \in \mathcal{M}(r, \lambda),\ 0 \leq r \leq d\}, \tag{3.20}$$

$$R_2(x_0, \lambda) \equiv \{r : x_0 \in \mathcal{M}_{in}(r, \lambda) \text{ or } x_0 \in \mathcal{M}(r, \lambda),\ 0 \leq r \leq d\}, \tag{3.21}$$

where $d>0$ is as in Theorem 3.13. We set

$$r_1 = r_1(x_0, \lambda) \equiv \sup_{r \in R_1(x_0, \lambda)} r, \qquad r_2 = r_2(x_0, \lambda) \equiv \inf_{r \in R_2(x_0, \lambda)} r$$

if $R_2(x_0, \lambda) \neq \emptyset$. Then, by the continuity of $p(\cdot, \lambda)$, it follows that $0 \leq r_1 \leq r_2$ and $p(r_j, \lambda) = r_j$, $j=1, 2$. Note that $R_1(x_0, \lambda) \ni 0$.

Using the above $R_j(x_0, \lambda)$, $r_j(x_0, \lambda)$, the mapping $\Omega_{U,U(\lambda)}(\cdot, \lambda)$ can be determined as follows.

Theorem 3.14. *Let d, U, and $U(\lambda)$ be as in Theorem 3.13. Then the following statements hold: Let $x_0 \in U$. If $R_2(x_2, \lambda)$ is not empty, then*

(i) $r_1(x_0, \lambda) = r_2(x_0, \lambda)$ $(=\hat{r})$, *or*

(ii) $p(\gamma, \lambda) < \gamma$ *for* $r_1(x_0, \lambda) < \gamma < r_2(x_0, \lambda)$, *or*

(iii) $p(\gamma, \lambda) > \gamma$ *for* $r_1(x_0, \lambda) < \gamma < r_2(x_0, \lambda)$.

If $R_2(x_0, \lambda)$ is empty, then either

(iv) $p(\gamma, \lambda) < \gamma$ *for* $r_1(x_0, \lambda) < \gamma \leq d$, *or*

(v) $p(\gamma, \lambda) > \gamma$ *for* $r_1(x_0, \lambda) < \gamma \leq d$.

We have $\Omega_{U,U(\lambda)}(x_0, \lambda) \neq \emptyset$, in Cases (i)-(iv) *and $\Omega_{U,U(\lambda)}(x_0, \lambda) = \emptyset$ in Case* (v). *Furthermore, $\gamma(x_0, \lambda) = w(\cdot\,; r_1(x_0, \lambda), \lambda)$ in Cases* (i), (ii), *and* (iv), *and $\gamma(x_0, \lambda) = w(\cdot\,; r_2(x_0, \lambda), \lambda)$ in Case* (iii).

Remark 3.15. Chafee [3, 4] showed, under the additional condition (ii) in Remark 3.4, that $u(t, x_0, \lambda)$ converges either to $\{0\}$ or to an invariant set on C_λ, which are bounded by two periodic orbits of (E), as $t \longrightarrow \infty$.

3.5. *Proofs of Theorems* 3.1, 3.2, and 3.13 Theorem 3.1 is an immediate consequence of Theorems 3.13 and 3.14. In the following, we shall give the proofs of Theorems 3.13 and 3.14.

Proofs of Theorems 3.13 *and* 3.14. Let K_1, K_2, $K\geq 1$ be constants as in Theorems 2.1, 3.5, and (2.11), respectively. We set $\hat{K}=\max\{K_1, K_2, K\}$. Let h be a positive number and set

$$\ell_1=h/(4\hat{K}), \qquad \ell_2=2h\hat{K}, \qquad \ell_3=4h\hat{K}^2, \qquad d=6h\hat{K}. \tag{3.22}$$

We choose h so small that

$$Kd<d_4, \qquad \|D_{b_j}z_\beta(b_1, b_2, \lambda)\|<1/(24\hat{K}), \qquad j=1, 2, \tag{3.23}$$

for $(b_1, b_2, \lambda)\in B_{R^2}(\hat{K}d)\times B_{R^n}(d)$.

In what follows, we fix such a number h, and we shall define two sets U, $U(\lambda)$ as in the statement of Theorem 3.13. Then we shall show that the assertions of Theorems 3.13 and 3.14 hold with such d, U, and $U(\lambda)$.

Before proceeding to define U, $U(\lambda)$, we state the following lemmas.

LEMMA 3.16. *Let r be a fixed point of $p(\cdot, \lambda)$. Then the following statements hold:*

(i) *If $0\leq r\leq \ell_2$, then*

$$\mathcal{M}(r, \lambda)\cap\{x: \|Qx\|<h\}\subset\{x: \|Px\|<\ell_3\}.$$

(ii) *If $d\leq r<K^{-1}d_4$, then*

$$B(\ell_3, h)\equiv\{x: \|Px\|<\ell_3, \|Qx\|<h\}\subset\mathcal{M}_{in}(r, \lambda).$$

(iii) *If $\ell_2\leq r<K^{-1}d_4$, then $B_{X_\beta}(\ell_1)\subset\mathcal{M}_{in}(r, \lambda)$.*

LEMMA 3.17. *Let r be a fixed point of $p(\cdot, \lambda)$ with $0\leq r\leq d$. then if $x_0\in\mathcal{M}_{in}(r, \lambda)\cap B_{X_\beta}(\ell_1)$ (resp. $\mathcal{M}(r, \lambda)\cap B_{X_\beta}(\ell_1)$), then for $t\geq 0$*

$$u(t, x_0, \lambda)\in\mathcal{M}_{in}(r, \lambda)\cap\{x: \|Qx\|<h\}$$

$$(\text{resp. } u(t, x_0, \lambda)\in\mathcal{M}(r, \lambda)\cap\{x: \|Qx\|<h\}).$$

LEMMA 3.18. *For each $x\in B_{X_\beta}(\ell_1)$, $\tilde{\mathcal{M}}(x, \lambda)\cap C_\lambda$ consists of a single point $\hat{x}$ and is included in $B_{X_\beta}(\ell_2)$. Furthermore if $x\in\mathcal{M}(r, \lambda)$ (resp. $\mathcal{M}_{in}(r, \lambda)$, $\mathcal{M}_{out}(r, \lambda)$), then $\hat{x}\in\mathcal{M}(r, \lambda)$ (resp. $\mathcal{M}_{in}(r, \lambda)$, $\mathcal{M}_{out}(r, \lambda)$).*

The proofs of Lemmas 3.16, 3.17, and 3.18 will be given after the proofs of Theorems 3.13 and 3.14.

On the basis of Lemma 3.16, we define U, $U(\lambda)$ as follows. We set $U\equiv B_{X_\beta}(\ell_1)$. For the difinition of $U(\lambda)$, we consider the set

(3.24) $$R(\lambda)=\{r: p(r, \lambda)=r, \ \ell_2 \leq r \leq d\}.$$

In the case that $R(\lambda)=\emptyset$, we define

$$U(\lambda) \equiv B(\ell_3, h)=\{x: \|Px\|<\ell_3, \ \|Qx\|<h\},$$

and in the case that $R(\lambda) \neq \emptyset$, we define

$$U(\lambda) \equiv (\mathcal{M}_{in}(r(\lambda), \lambda) \cup \mathcal{M}(r(\lambda), \lambda)) \cap \{x: \|Qx\|<h\},$$

where $r(\lambda)=\sup_{r \in R(\lambda)} r$. Then, from our definitions of U, $U(\lambda)$, it follows by Lemma 3.16 that $U \subset U(\lambda) \subset B_{X_\beta}(d_4)$ and that

(3.25) $$(\mathcal{M}(r, \lambda) \cup \mathcal{M}_{in}(r, \lambda)) \cap \{x: \|Qx\|<h\} \subset U(\lambda)$$

for any $0 \leq r \leq d$ with $p(r, \lambda)=r$.

Now we shall show that the U, $U(\lambda)$ defined above satisfy the assertions of Theorems 3.13 and 3.14. Let $x_0 \in U$ be given. Then, from the definition of $R_j(x_0, \lambda)$ $(j=1, 2)$ (see (3.20), (3.31)) and from the continuity of $p(\cdot, \lambda)$, it follows that in the case that $R_2(x_0, \lambda) \neq \emptyset$, (i), (ii), or (iii) of Theorem 3.14 holds and in the case that $R_2(x_0, \lambda)=\emptyset$, (iv) or (v) of Theorem 3.14 holds. We shall prove Theorems 3.13 and 3.14 considering cases (i)-(v) separately.

Case (i). In this case we have by (3.20), (3.21),

(3.26) $$R_1(x_0, \lambda)=R_2(x_0, \lambda)=\{\hat{r}\}, \qquad x_0 \in \mathcal{M}(\hat{r}, \lambda).$$

We first show that $\Omega_{U, U(\lambda)}(x_0, \lambda) \neq \emptyset$. By (3.26) and by Lemma 3.16 (iii), we get $0 \leq \hat{r} \leq \ell_2$. Hence, by Lemma 3.17, we obtain

$$u(t, x_0, \lambda) \in M(\hat{r}, \lambda) \cap \{x: \|Qx\|<h\}, \qquad t \geq 0.$$

Therefore, by (3.25), it follows that $u(t, x_0, \lambda) \in U(\lambda)$, $t \geq 0$, i.e., $\Omega_{U, U(\lambda)}(x_0, \lambda) \neq \emptyset$.

We next show that $u(t, x_0, \lambda)$ converges to the periodic orbit $w(\cdot\,; \hat{r}, \lambda)$ as $t \longrightarrow \infty$. By (3.26), there exists s such that $x_0 \in \tilde{\mathcal{M}}(w(s; \hat{r}, \lambda), \lambda)$. Since $w(\cdot\,; \hat{r}, \lambda) \subset B_{X_\beta}(d_3)$, (3.5) holds for $t \geq 0$ with x replaced by $w(s; \hat{r}, \lambda)$. Therefore $u(t, x_0, \lambda)$ converges to the periodic orbit $w(\cdot : \hat{r}, \lambda)$ as $t \longrightarrow \infty$.

Cases (ii), (iii). In these cases, we have

(3.27) $$x_0 \in \mathcal{M}_{out}(r_1, \lambda) \cap \mathcal{M}_{in}(r, \lambda).$$

We first show that $\Omega_{U, U(\lambda)}(x_0, \lambda) \neq \emptyset$. By Lemma 3.17, we obtain

$$u(t, x_0, \lambda) \in \mathcal{M}_{in}(r_2, \lambda) \cap \{x: \|Qx\|<h\}, \qquad t \geq 0.$$

Hence, by (3.25), it follows that $u(t, x_0, \lambda) \in U(\lambda)$, $t \geq 0$, i.e., $\Omega_{U, U(\lambda)}(x_0, \lambda) \neq \emptyset$.

We next show that $u(t, x_0, \lambda)$ converges to the periodic orbit $w(\cdot\,; r_1, \lambda)$ (resp. $w(\cdot\,; r_2, \lambda)$) as $t \longrightarrow \infty$ in the case (ii) (resp. (iii)). By Lemma 3.18 and

(3.28), there exists $\hat{x}$ such that

$$\hat{x}\in\hat{\mathcal{M}}(x_0,\ \lambda)\cap C_\lambda,\qquad \hat{x}\in\mathcal{M}_{out}(r_1,\ \lambda)\cap\mathcal{M}_{in}(r_2,\ \lambda).$$

Therefore, by Propositions 2.2, 2.5, 2.6, and 2.8, it follows that $u(t,\ \hat{x},\ \lambda)$ converges to the periodic orbit $w(\cdot\ ;\ r_1,\ \lambda)$ (resp. $w(\cdot\ ;\ r_2,\ \lambda)$) in the case (ii) (resp. case (iii)). Since $\hat{x}\in\hat{\mathcal{M}}(x_0,\ \lambda)$ and since $u(t,\ x_0,\ \lambda)\in B_{X_\beta}(d_3)$, $t\geq 0$, we conclude, by Theorem 3.5 (ii), that $u(t,\ x_0,\ \lambda)$ converges to the same periodic orbit as $u(t,\hat{x},\lambda)$ does.

Case (iv). In this case, we have

$$x_0\in\mathcal{M}_{out}(r_1,\ \lambda). \tag{3.28}$$

Since $R_2(x_0,\ \lambda)=\emptyset$, we get by Lemma 3.16 (iii) and (3.24), $R(\lambda)=\emptyset$. Hence, by definition of $U(\lambda)$, we have $U(\lambda)=B(\ell_3,\ h)$.

We first show that $\Omega_{U,U(\lambda)}(x_0,\ \lambda)\neq\emptyset$, i.e., $u(t,\ x_0,\ \lambda)\in B(\ell_3,\ h)$, $t\geq 0$. Since $x_0\in U(=B_{X_\beta}(\ell_1))$, there exists $s>0$ such that

$$\|u(t,\ x_0,\ \lambda)\|<\ell_3,\qquad 0\leq t\leq s. \tag{3.29}$$

We claim that (3.29) holds for all $t\geq 0$. Assume the contrary. Then there exists $T>0$ such that

$$\|u(t,\ x_0,\ \lambda)\|<\ell_3,\qquad 0\leq t\leq T;\qquad \|u(T,\ x_0,\ \lambda)\|=\ell_3. \tag{3.30}$$

On the other hand, by Lemma 3.18, there exists $\hat{x}$ such that $\hat{x}\in C_\lambda\cap\hat{\mathcal{M}}(x_0,\ \lambda)\cap B_{X_\beta}(\ell_2)$. Hence, applying Theorem 3.5 (ii), we get

$$\|u(t,\ \hat{x},\ \lambda)-u(t,\ x_0,\ \lambda)\|\leq K_3 e^{-\alpha t}\|\hat{x}-x_0\|,\qquad 0\leq t<T. \tag{3.31}$$

We estimate the right-hand side of (3.31). Since $\hat{x}\in\hat{\mathcal{M}}(x_0,\ \lambda)$, we have by definition of $\hat{\mathcal{M}}(x_0,\ \lambda)$,

$$\hat{x}-x_0=\sum_{j=1,2}a_j(Q\hat{x}-Qx_0\ ;\ x_0)\phi_j+Q(x-x_0).$$

Since $\hat{x}\in C_\lambda$, we get by (3.23),

$$\|Q\hat{x}\|\leq\ell_2/(12\,\hat{K}^4).$$

Hence, by (3.2), we have

$$\|\hat{x}-x_0\|=\|Q\hat{x}-Qx_0\|<\ell_1+\ell_2/(12\,\hat{K}^4).$$

Therefore we obtain by (3.22), (3.23), and (3.31),

$$\|u(t,\ \hat{x},\ \lambda)-u(t,\ x_0,\ \lambda)\|\leq K_3(\ell_1+\ell_2/(12\,\hat{K}^4))<\ell_2<\ell_3,\quad 0\leq t\leq T. \tag{3.32}$$

Since $\hat{x}\in C_\lambda\cap B_{X_\beta}(\ell_2)$ and since the case (iv) holds, we get by Proposition 2.2 and (2.11), (3.33) $$u(t, \hat{x}, \lambda) \in B_{X_\beta}(K\ell_2) \cap C_\lambda, \qquad t \geq 0.$$

Hence, by (3.32), we get for $0 \leq t \leq T$,

(3.34) $$\begin{aligned} \|u(t, x_0, \lambda)\| &\leq \|u(t, \hat{x}, \lambda) - u(t, x_0, \lambda)\| + \|u(t, \hat{x}, \lambda)\| \\ &\leq K_3(\ell_1 + \ell_2/(12\hat{K}^4)) + K\ell_2 < \ell_3. \end{aligned}$$

This, however, contradicts (3.30). Thus we can conclude that $\|u(t, x_0, \lambda)\| < \ell_3$, $t \geq 0$. Using this instead of (3.30), the similar argument allows us to conclude that (3.31), (3.32), and (3.34) hold for all $t \geq 0$. Therefore it will follow that $\Omega_{U,U(\lambda)}(x_0, \lambda) \neq \emptyset$ once we show that $\|Pu(t, x_0, \lambda)\| < h$, $t \geq 0$, since $U(\lambda) = B(\ell_3, h)$. This inequality is proved as follows. Since (3.32) holds for $t \geq 0$, we have

$$\|Qu(t, \hat{x}, \lambda) - Qu(t, x_0, \lambda)\| \leq K_3(\ell_1 + \ell_2/(12\hat{K}^4)), \qquad t \geq 0.$$

On the other hand, since $u(t, \hat{x}, \lambda) \in B_{X_\beta}(K\ell_2) \cap C_\lambda$, we get $\|Qu(t, \hat{x}, \lambda)\| \leq \ell_2/(12\hat{K}^3)$, $t \geq 0$. Therefore we have

$$\begin{aligned} \|Qu(t, x_0, \lambda)\| &\leq \|Qu(t, x_0, \lambda) - Qu(t, \hat{x}, \lambda)\| + \|Qu(t, \hat{x}, \lambda)\| \\ &\leq K_3(\ell_1 + \ell_2/(12\hat{K})) + \ell_2/(12\hat{K}^3) \\ &\leq h, \quad t \geq 0. \end{aligned}$$

Thus we obtain $\Omega_{U,U(\lambda)}(x_0, \lambda) \neq \emptyset$.

We next show that $u(t, x_0, \lambda)$ converges to the periodic orbit $w(\cdot\,; r_1, \lambda)$ as $t \longrightarrow \infty$. Since $\hat{x} \in C_\lambda \cap \mathcal{M}_{out}(r_1, \lambda) \cap B_{X_\beta}(\ell_2)$ and we are treating the case (iv), it follows, by Propositions 2.2, 2.5, 2.6, and 2.8, that $u(t, \hat{x}, \lambda)$ converges to $w(\cdot\,; r_1, \lambda)$ as $t \longrightarrow \infty$. Hence $u(t, x_0, \lambda)$ converges to $w(\cdot\,; r_1, \lambda)$, since (3.31) holds for all $t \geq 0$.

Case (v). As in the case (iv), it follows that

$$x_0 \in \mathcal{M}_{out}(r_1, \lambda), \qquad U(\lambda) = B(\ell_3, h),$$

and there exists $\hat{x}$ such that $\hat{x} \in C_\lambda \cap \hat{\mathcal{M}}(x_0, \lambda) \cap B_{X_\beta}(\ell_2)$, $\hat{x} \in \mathcal{M}_{out}(r_1, \lambda)$ and

(3.35) $$\|x_0 - \hat{x}\| \leq \ell_1 + \ell_2/(12\hat{K}^4).$$

We show that $\Omega_{U,U(\lambda)}(x_0, \lambda) = \emptyset$, i.e., for some $s > 0$, $u(s, x_0, \lambda) \notin B(\ell_3, h)$. Assume the contrary. Then we have

(3.36) $$u(t, x_0, \lambda) \in B(\ell_3, h), \qquad t > 0.$$

Hence, applying Theorem 3.5 (ii), we get

$$\|u(t, \hat{x}, \lambda) - u(t, x_0, \lambda)\| \leq K_3 e^{-\alpha t}\|\hat{x} - x_0\|, \qquad t \geq 0.$$

By (3.22), (3.25), and (3.36), we get

$$(3.37) \qquad \|u(t, \hat{x}, \lambda)\| \leq \|u(t, \hat{x}, \lambda) - u(t, x_0, \lambda)\| + \|u(t, x_0, \lambda)\|$$
$$\leq K_3(\ell_1 + \ell_3/(12\hat{K}^4)) + \ell_3 < d.$$

On the other hand, since $\hat{x} \in \mathcal{C}_\lambda \cap \mathcal{M}_{out}(r_1, \lambda)$ and since the case (v) holds, it follows, by Propositions 2.2, 2.5, 2.6, and 2.8, that there exists $s>0$ such that

$$\langle u(t, \hat{x}, \lambda_1^*), \varphi_1^* \rangle > d.$$

This, however, contradicts (3.37). Thus we have $\Omega_{U, U(\lambda)}(x_0, \lambda) = \emptyset$. Q.E.D.

Thus the proofs of Theorems 3.13 and 3.14 will be complete once we prove Lemmas 3.16, 3.17, and 3.18. In the rest of this section we prove them.

Proof of Lemma 3.16. By (3.10) and (3.11), for the proofs of (i), (ii), and (iii) it suffices to show that for each $z \in B_{Z_\beta}(h)$, the following statements hold, respectively: $\mathcal{C}(r, \lambda; z) \subset B_{R^2}(\ell_3)$, $0 \leq r \leq \ell_2$; $B_{R^2}(\ell_3)$ is included inside $\mathcal{C}(r, \lambda; z)$, $d \leq r < K^{-1}d_4$; and $B_{R^2}(\ell_1)$ is included inside $\mathcal{C}(r, \lambda; z)$, $\ell_2 \leq r < K^{-1}d_4$. These follow immediately from (3.2), (3.22), and (3.23). Q.E.D.

Proof of Lemma 3.17. We prove the lemma in the case that $x_0 \in \mathcal{M}_{in}(r, \lambda) \cap B_{X_\beta}(\ell_1)$, since the proof is simpler than in the case that $x_0 \in \mathcal{M}(r, \lambda) \cap B_{X_\beta}(\ell_1)$. In that case there exists $s>0$ such that

$$(3.38) \qquad u(t, x_0, \lambda) \in \mathcal{M}_{in}(r, \lambda) \cap \{x : \|Qx\| < h\}, \qquad 0 \leq t \leq s.$$

We show that (3.38) holds for all $t \geq 0$. Assume the contrary. Then there exists $T>0$ such that (3.38) holds for $0 \leq t \leq T$, and $u(T, x_0, \lambda) \in \mathcal{M}(r, \lambda)$ or $\|Qu(T, x_0, \lambda)\| = h$. Since, by Proposition 3.6 (ii), (3.22), (3.23), and (3.38),

$$(3.39) \qquad u(t, x_0, \lambda) \in B(d_3, d_4), \qquad 0 \leq t \leq T,$$

we have by Proposition 3.11, $u(t, x_0, \lambda) \in \mathcal{M}_{in}(r, \lambda)$ for $0 \leq t \leq T$. Therefore we obtain $\|Qu(T, x_0, \lambda)\| = h$. In the following, we show that this is impossible. By Theorem 2.1 and (3.39), we have for $0 \leq t \leq T$,

$$(3.40) \qquad \|Qu(t, x_0, \lambda) - z_\beta(a_1(t), a_2(t))\| \leq K_1 e^{-\alpha t} \|Qx_0 - z_\beta(a_1(0), a_2(0))\|,$$

where $a_j(t) = \langle u(t, x_0, \lambda), \phi_j^* \rangle$ $(j=1, 2)$. On the other hand, since $u(t, x_0, \lambda) \in \mathcal{M}_{in}(r, \lambda)$ for $0 \leq t \leq T$, we get by (3.2), (3.8), and (3.23),

$$(3.41) \qquad \|Pu(t, x_0, \lambda)\|$$
$$\leq \sup_{0 \leq s \leq \tau(r, \lambda)} \|Pw(s; r, \lambda)\| + \sup_{0 \leq s \leq \tau(r, \lambda)} \Big\| \sum_{j=1,2} a_j(Qu - Qw(s; r, \lambda); w(s; r, \lambda)) \phi_j \Big\|$$
$$\leq Kr + (\|Qu(t, x_0, \lambda)\| + Kr/(12\hat{K}^4))$$

$$\leq 13\,Kd/12+h\,, \qquad 0\leq t\leq T\,.$$

Hence, by (2.22), (3.23), and (3.40) we have for $0\leq t\leq T$

$$\begin{aligned}&||Qu(t,\ x_0,\ \lambda)||\\ &\leq K_1||Qx_0-z_\beta(a_1(0),\ a_2(0))||+||z_\beta(a_1(t),\ a_2(t))||\\ &\leq K_1(||Qx_0||+||Px_0||/(12\,\hat{K}^4))+||Pu(t,\ x_0,\ \lambda)||/(12\,\hat{K}^4)\\ &\leq 13\,K_1\ell_1/12+(13\,Kd/12+h)/(12\,\hat{K}^4)\\ &<h\end{aligned}$$

Q.E.D.

Proof of Lemma 3.18. Suppose that $\hat{x}\in\mathcal{C}_\lambda\cap\hat{\mathcal{M}}(x,\ \lambda)$. We first show that $\hat{x}\in B_{X_\beta}(\ell_2)$. Since $\hat{x}\in\mathcal{C}_\lambda$, we have

$$\hat{x}=\sum_{j=1,2}c_j\phi_j+z_\beta(c_1,\ c_2)\,; \qquad c_j=\langle\hat{x},\ \phi_j^*\rangle\,, \qquad j=1,\ 2\,. \tag{3.42}$$

On the other hand, since $\hat{x}\in\hat{\mathcal{M}}(x,\ \lambda)$, we have for some $\hat{z}\in B_{Z_\beta}(d_3)$

$$\hat{x}=x+\sum_{j=1,2}a_j(\hat{z}\,;\ x)\phi_j+\hat{z}\,. \tag{3.43}$$

By (3.42) and (3.43), we get

$$c_j=\langle x,\ \phi_j^*\rangle+a_j(z_\beta(c_1,\ c_2)-Qx\,;\ x,\ \lambda)\,, \qquad j=1,\ 2\,. \tag{3.44}$$

Using the equality $||P\hat{x}||=\sqrt{c_1^2+c_2^2}$, we have by (2.1), (3.2),

$$||P\hat{x}||\leq||Px||+||z_\beta(c_1,\ c_2)||+||Qx||<2||x||+||P\hat{x}||/2\,.$$

Hence we get by (3.22),

$$||P\hat{x}||\leq 4||x||\leq 4\ell_1<\ell_2\,.$$

Since $||Q\hat{x}||\leq||P\hat{x}||/2$ by (2.1) and (3.42), we obtain $\hat{x}\in B_{X_\beta}(\ell_2)$. We next show that $\mathcal{C}_\lambda\cap\hat{\mathcal{M}}(x,\ \lambda)$ consists of a single point. Since (3.44) holds for $\hat{x}\in\mathcal{C}_\lambda\cap\hat{\mathcal{M}}(x,\ \lambda)$, we consider a mapping $\tilde{g}=(\tilde{g}_1,\ \tilde{g}_2)$ defined by

$$\tilde{g}_j(c_1,\ c_2)=\langle x,\ \phi_j^*\rangle+\sum_{j=1,2}a_j(z_\beta(c_1,\ c_2)-Qx\,;\ x)\,, \qquad j=1,\ 2$$

for $(c_1,\ c_2)\in B_{R^2}(\ell_2)$. Then, by (2.1), (3.2), and (3.22), it follows that $\tilde{g}$ maps $B_{R^2}(\ell_2)$ into itself and is a contraction on $B_{R^2}(\ell_2)$. Hence there exists a unique fixed point $\hat{x}$ of $\tilde{g}$, which implies that $\{\hat{x}\}=\mathcal{C}_\lambda\cap\mathcal{M}(x,\ \lambda)\cap B_{X_\beta}(\ell_2)$. Thus, together with the result obtained in the preceding paragraph, we get $\{\hat{x}\}=\mathcal{C}_\lambda\cap\hat{\mathcal{M}}(x,\ \lambda)$.

Lastly we show that $\hat{x}\in\mathcal{M}_{out}(r,\ \lambda)$ (resp. $\mathcal{M}_{in}(r,\ \lambda)$, $\mathcal{M}(r,\ \lambda)$) if $x\in\mathcal{M}_{out}(r,\ \lambda)$ (resp. $\mathcal{M}_{in}(r,\ \lambda)$, $\mathcal{M}(r,\ \lambda)$). We consider only the case $x\in\mathcal{M}_{out}(r,\ \lambda)$. In other cases, the proofs are similar. Suppose that $\hat{x}\notin\mathcal{M}_{out}(r,\ \lambda)$. Then either $\hat{x}\in\mathcal{M}_{in}(r,\lambda)$ or $\hat{x}\in\mathcal{M}(r,\ \lambda)$. In either case, since $\hat{x}\in\mathcal{C}_\lambda$, we have by Proposition 2.2

and (2.11),

(3.45) $$u(t,\ \hat{x},\ \lambda)\in B_{X_\beta}(Kd),\qquad t>0.$$

On the other hand, since $x\in B_{X_\beta}(\ell_1)$, there exists $s>0$ such that

(3.46) $$u(t,\ x,\ \lambda)\in B_{X_\beta}(d_3),\qquad 0\leq t\leq s.$$

Since $\hat{x}\in\mathcal{M}(x,\ \lambda)$, Theorem 3.5 (ii) yields

(3.47) $$\|u(t,\ \hat{x},\ \lambda)-u(t,\ x,\ \lambda)\|\leq K_3e^{-\alpha t}\|\hat{x}-x\|,\qquad 0\leq t\leq s.$$

We show that (3.46) and (3.47) hold for all $t\geq 0$. Suppose that (3.46) does not hold. Then there exists $T>0$ such that $u(t,\ x,\ \lambda)\in B_{X_\beta}(d_3)$, $0\leq t\leq T$;

(3.48) $$\|u(T,\ x,\ \lambda)\|=d_3.$$

Since (3.47) holds for $0\leq t<T$, we get by (3.22), (3.23), and (3.45),

$$\begin{aligned}\|u(t,\ x,\ \lambda)\| &\leq \|u(t,\ \hat{x},\ \lambda)\|+K_3(\|\hat{x}\|+\|x\|)\\ &\leq Kd+(\ell_2+\ell_1)<Kd+Kd<d_3\end{aligned}$$

for $0\leq t\leq T$. This contradicts (3.48). Thus (3.46) holds for all $t\geq 0$, and so (3.47) holds for all $t\geq 0$. Hence by (3.22), (3.23), and (3.6), we get for $t\geq 0$

$$\begin{aligned}\|u(t,\ \hat{x},\ \lambda)-u(t,\ x,\ \lambda)\| &\leq K_3e^{-\alpha t}\|\hat{x}-x\|\\ &\leq K_3(\ell_2+\ell_1)e^{-\alpha t}<K_2e^{-\alpha t}.\end{aligned}$$

Therefore, applying Theorem 3.5 (i), we obtain that $x\in\hat{\mathcal{M}}(\hat{x},\ \lambda)$. Hence, by the same arguments as in the proof of Proposition 3.11, we can conclude that $x\in\mathcal{M}_{in}(r,\ \lambda)$ (resp. $\mathcal{M}(r,\ \lambda)$) if $\hat{x}\in\mathcal{M}_{in}(r,\ \lambda)$ (resp. $\mathcal{M}(r,\ \lambda)$), which contradicts the hypothesis that $x\in\mathcal{M}_{out}(r,\ \lambda)$. Q.E.D.

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