On Pairs of Groups Having a Common 2-Subgroup of Odd Indices

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(Received February 5, 1985)

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In a previous paper [1], we considered pairs of groups G, H satisfying the following conditions.

(a) G and H have a common 2-subgroup S.
(b) Both $|G:S|$ and $|H:S|$ are odd primes.
(c) No nonidentity subgroup of S is normal both in G and in H.
(d) $C_G(O_2(G))\leq O_2(G)$ and $C_H(O_2(H))\leq O_2(H)$.

A closer look at the method of [1] has shown recently that the same method can handle a more general situation. Specifically, we can replace the condition (b) above by the condition

(b′) $|G:S|$ and $|H:S|$ are powers of odd primes q and r, respectively, and Sylow q-subgroups of G and Sylow r-subgroups of H are cyclic and nontrivial,

and carry out the same analysis to reach the same conclusions. Thus, we can obtain the following theorems. (The reader is referred to Section 1 of [1] for the definitions not given below.)

MAIN THEOREM. *Let G, H be a pair of groups satisfying the conditions* (a), (b′), (c), *and* (d). *Let* $S^*=(S\cap O^2(G))(S\cap O^2(H))$, $G^*=S^*O^2(G)$, *and* $H^*=S^*O^2(H)$. *Then the pair* (G^*, H^*) *or* (H^*, G^*) *is of* $GL_3(2)$*-type or* $G_2(2)'$*-type or* M_{12}*-type or* ${}^2F_4(2)'$*-type.*

THEOREM A. *Let G, H be a pair of groups satisfying the conditions* (a), (b′), (c), *and* (d), *and assume that* $\Omega_1(Z(S)) \leq Z(H)$. *Then the pair* (G^*, H^*) *defined in the Main Theorem is of* $G_2(2)'$*-type or* M_{12}*-type or* ${}^2F_4(2)'$*-type.*

THEOREM B. *Let* G, H *be a pair of groups satisfying the conditions* (a), (b′), (c), *and* (d), *and assume that* $Z(G) \not\geq \Omega_1(Z(S)) \not\leq Z(H)$. *Then* $G/O_2(G) \cong H/O_2(H) \cong \boldsymbol{D}_6$ *and* $O_2(G) \cong O_2(H) \cong \boldsymbol{E}_4$ *or* $\boldsymbol{E}_8$.

We can obtain further generalizations of the above theorems in certain cases when the condition (d) is not satisfied. In order to describe them, we require the following definitions. Let X be a group and Y a subgroup. We say that X is *Y-irreducible* (or Y is nearly maximal in X) if Y is contained in a unique maximal subgroup of X. If a finite group X is S-irreducible for some Sylow p-subgroup S, we say that X is *p-irreducible.* The generalizations mentioned above are obtained through the following propositions.

1.1. Let G be a finite group, p a prime, and S a Sylow p-subgroup of G. Assume that G is S-irreducible and $G \neq SO_{p'}(G)$. Then for every subgroup X of S, $XO_{p'}(G)$ is normal in G if and only if X is normal in G.

Proof. Suppose $XO_{p'}(G) \lhd G$. Then $G = N_G(X)SO_{p'}(G)$ by a Frattini argument, and so $G = N_G(X)$ by the S-irreducibility.

1.2. Let G, H be a pair of 2-irreducible groups satisfying the conditions (a), (c), and

(b$^\infty$) both $|G:S|$ and $|H:S|$ are odd, $G \neq SO(G)$, and $H \neq SO(H)$.

Let $g: G \to G/O(G)$ and $h: H \to H/O(H)$ be the natural homomorphisms. Then there exist groups $\bar{G}$ and $\bar{H}$ satisfying the following conditions.

(1) The pair $(\bar{G}, \bar{H})$ satisfies (a), (b$^\infty$), and (c) with respect to a common 2-subgroup $\bar{S}$.

(2) There exist isomorphisms $i: G^g \to \bar{G}$, $j: H^h \to \bar{H}$, and $s: S \to \bar{S}$ such that $x^{gi} = x^{hj} = x^s$ for all $x \in S$.

Proof. As the restrictions $g|S$ and $h|S$ are one to one, we can define the amalgamated product F of G^g and H^h with respect to $g|S$ and $h|S$. Thus, F is a group equipped with monomorphisms $i: G^g \to F$ and $j: H^h \to F$ such that $(g|S)i = (h|S)j$. Let $s = (g|S)i = (h|S)j$. We show that the pair (G^{gi}, H^{hj}) satisfies (a), (b$^\infty$), and (c) with respect to the common 2-subgroup S^s. As the pair (G, H) satisfies (a) and (b$^\infty$), so does the pair (G^{gi}, H^{hj}). Suppose some subgroup Y of S^s is normal both in G^{gi} and in H^{hj}. Let $X = s^{-1}(Y)$. Then $X^g \lhd G^g$ and $X^h \lhd H^h$, and so X is normal both in G and in H by 1.1. Therefore, $X=1$ and (G^{gi}, H^{hj}) satisfies (c).

Now, we can prove a corollary to Theorem B. (The interested reader may observe that we can derive analogous corollaries from the Main Theorem and Theorem A.)

COROLLARY B. *Let G, H be a pair of 2-irreducible groups satisfying* (a), (c), *and*

(b″) $|G:SO(G)|$ *and* $|H:SO(H)|$ *are powers of odd primes q and r, respectively, and Sylow q-subgroups of $G/O(G)$ and Sylow r-subgroups of $H/O(H)$ are cyclic and nontrivial.*

Assume further that $[G, \Omega_1(Z(S))]\nleq O(G)$ *and* $[H, \Omega_1(Z(S))]\nleq O(H)$. *Then*

$$G/O(G)\times O_2(G)\cong H/O(H)\times O_2(H)\cong \boldsymbol{D}_6 \text{ and } O_2(G)\cong O_2(H)\cong \boldsymbol{E}_4 \text{ or } \boldsymbol{E}_8 .$$

Proof. First of all, $O_{2',2}(G)=O(G)\times O_2(G)$ and $O_{2',2}(H)=O(H)\times O_2(H)$ by 1.1. Also, 1.2 shows that there exists a pair of groups $\bar{G}$, $\bar{H}$ satisfying (a), (b′), (c), and (d) with respect to a common 2-subgroup $\bar{S}$ with $\bar{G}\cong G/O(G)$, $\bar{H}\cong H/O(H)$, and $Z(\bar{G})\ngeq \Omega_1(Z(\bar{S}))\nleq Z(\bar{H})$ (here, we must use Burnside's $p^a q^b$-theorem to verify (d)). Therefore, our assertion follows from Theorem B.

The following remark is useful when we apply Corollary B.

1.3. Let G, H be a pair of 2-irreducible groups satisfying (a), (b^∞), and (c), and assume $G\cong H$. Then $[G, \Omega_1(Z(S))]\nleq O(G)$ and $[H, \Omega_1(Z(S))]\nleq O(H)$.

Proof. Suppose, say, $[G, \Omega_1(Z(S))]\leq O(G)$. Then $\Omega_1(Z(T))\leq Z(G)$ for all $T\in \mathrm{Syl}_2(G)$ by 1.1 and Sylow's theorem and so, as $G\cong H$, we have $\Omega_1(Z(S))\leq Z(H)$. As this violates (c), our assertion holds.

In a previous paper [**2**], the first author described a new approach to the thin finite simple groups with many solvable 2-local subgroups, and the main result of [**1**] played an important role in it. Therefore, the improvements on [**1**] have an effect on [**2**]. First, we have the following corollary to the Main Theorem and Theorem A.

COROLLARY C. *Let G, H be a pair of groups satisfying the conditions* (a), (b′), (c), *and* (d). *Then the following holds.*

(1) $3\in\{q, r\}\leq\{3, 5\}$.
(2) *If* $r=5$, *then* $\Omega_1(Z(S))\leq Z(H)$.

This result supersedes Lemma 5.2 of [**2**] which is derived from Glauberman's "triple factorization theorem". The interested reader may observe that we can use Corollary C to condense the proof of Lemma 7.3 of [**2**] to about a third of its length. Secondly, the following corollary to Theorem A can replace Lemmas 7.5, 7.6, 7.7, and 7.8 of [**2**].

COROLLARY D. *Let G be a finite group,* $S\in\mathrm{Syl}_2(G)$, $C=C_G(\Omega_1(Z(S)))$, *and* $\mathcal{M}(S)$ *the set of all maximal 2-local subgroups of G containing S. Assume that* $\mathcal{M}(S)=\{M, C\}$ *with* $M\neq C$ *and that both M and C are solvable groups with* $O(M)$

$=O(C)=1$ *and with cyclic Sylow subgroups for all odd primes. Then we have* $|M:S|=3$, $|C:S|=3$ *or* 5, *and the structure of* M *and* C *is described by Theorem* A.

Proof. Let q and r be prime divisors of $|M:M\cap C|$ and $|C:M\cap C|$, respectively, and pick a Hall $\{2, q\}$-subgroup X of M and a Hall $\{2, r\}$-subgroup Y of C so that $S\leq X\cap Y$. Then our assumptions show that X and Y satisfy (a), (b'), (c), and (d) with respect to S. Therefore, the pair (X, Y) is described by Theorem A. In particular, we have $|X:S|=q=3$ and $|Y:S|=r=3$ or 5. Now let $S^*=(S\cap O^2(X))(S\cap O^2(Y))$, $X^*=S^*O^2(X)$, and $Y^*=S^*O^2(Y)$. If $r=3$, then (X^*, Y^*) is of $G_2(2)'$-type or M_{12}-type and, consequently, X has precisely two nontrivial chief factors within $O_2(X)$, while if $r=5$ then (X^*, Y^*) is of ${}^2F_4(2)'$-type and X has four nontrivial chief factors within $O_2(X)$. This shows that $|C:M\cap C|$ is a power of r. Therefore, we conclude that $|M:M\cap C|=3$, $|C:M\cap C|=r$, and $(3r, |M\cap C|)=1$. Now assume $M\cap C\neq S$ and pick a prime divisor p of $|M\cap C:S|$. By theorems of P. Hall on solvable groups, we can pick an S_3-subgroup Q of M, an S_r-subgroup R of Y, and an S_p-subgroup P of $M\cap C$ so that SQ, QP, SR, RP, and SP are all subgroups. As QP and RP are supersolvable and as p is greater than 3 and r, it follows that Q and R normalize P. Lemma 5.1 of [2] now shows that P, Q, and R are contained in some member of $\mathcal{M}(S)$, contrary to our assumption. Therefore, $M\cap C=S$ and the proof is complete.

In the subsequent sections, we shall prove the previously stated theorems keeping parallelism with [1]. The proposition labeled $i.j$ by a pair of positive integers i, j is parallel to the proposition $i.j$ in [1]. (The only exception is the proposition 2.9 which has no counterpart.) In order to prove $i.j$, we often need only duplicate or make obvious changes to the proof in [1], and in that case we shall omit or only touch upon the proof. Less obvious changes are required in Sections 7, 8, and 9. Fortunately, however, our arguments have not become much longer, and the argument for the crucial theorem 9.4 is rather shorter than in [1] because it is a somewhat weaker version of its counterpart and we have made technical progress.

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In this section, we shall study the following situation.

2.1 HYPOTHESIS. G is a $\{2, q\}$-group, q is an odd prime, and an S_q-subgroup K of G is cyclic and nontrivial.

Under this hypothesis, we let $S\in\mathrm{Syl}_2(G)$, $Z=\Omega_1(Z(S))$, $Q=O_2(G)$, and $V=\Omega_1(Z(Q))$.

2.2. The following holds.

(1) QK is a normal subgroup of G.

(2) SK^q is the only maximal subgroup of G that contains S.
(3) If N is a normal subgroup of G with $G\neq SN$ and $Q\leq N$, then $Q\in\mathrm{Syl}_2(N)$ and N contains no S_q-subgroup of G.
(4) $[Q, O^2(G)]=S\cap O^2(G)$.
(5) S/Q is a cyclic group of order dividing $q-1$.
(6) If $S\neq Q$ and tQ is an involution of S/Q, then $x^t\equiv x^{-1}$ (mod Q) for all $x\in K$.
(7) If ($S\neq Q$ and) $g\in G-S$, then $S\cap S^g=Q$.
(8) If $S\neq Q$ and $q=3$, then $QK^q\lhd G$ and $G/QK^q\cong \boldsymbol{D}_6$.

Proof. First of all, G is solvable by Burnside's p^aq^b-theorem. Therefore, if bars denote images in G/Q, then $C_{\bar{G}}(O_q(\bar{G}))\leq O_q(\bar{G})$. As $\bar{K}\in\mathrm{Syl}_q(\bar{G})$ and $\bar{K}$ is cyclic, we have $\bar{K}=O_q(\bar{G})$, proving (1). As $G=SK$, every proper subgroup of $\bar{G}$ containing $\bar{S}$ is of the form $\bar{S}\bar{L}$ with $\bar{L}\leq\bar{K}^q$. As $\bar{S}\bar{K}^q$ is a subgroup by (1), (2) holds. Suppose N is a normal subgroup of G with $G\neq SN$. Then, as $G=SK$, N contains no S_q-subgroup of G. Also, $G=N_G(S\cap N)\cdot SN$ by a Frattini argument and so $N_G(S\cap N)=G$ by (2). Therefore, if $Q\leq N$, then we have $Q=S\cap N\in \mathrm{Syl}_2(N)$, proving (3). For the proof of (4), see 2.2 of [1].

Suppose $S\neq Q$. As $C_{\bar{G}}(\bar{K})=\bar{K}$ and $\bar{K}$ is cyclic, $\bar{x}^{\bar{t}}\neq\bar{x}$ for all nonidentity elements $\bar{t}\in\bar{S}$ and $\bar{x}\in\bar{K}$. This implies that $\bar{G}$ is a Frobenius group with kernel $\bar{K}$ and complement $\bar{S}$. Hence, (6) and (7) follow. Also, $\bar{S}$ acts faithfully on $\bar{K}/\bar{K}^q$. Therefore, $\bar{S}$ is cyclic of order dividing $q-1$, and if $q=3$ then $\bar{G}/\bar{K}^q\cong\boldsymbol{D}_6$.

2.3. If $S\neq Q$, then the following five conditions on elements $g\in G$ are equivalent.
(1) $g\notin SK^q$.
(2) For all $a\in S-Q$ and $b\in S^g-Q$, $\langle a, b\rangle$ contains an S_q-subgroup of G.
(3) $G=\langle S, x\rangle$ for all $x\in S^g-Q$.
(4) $G=\langle S, S^g\rangle$.
(5) $G=\langle S, g\rangle$.

Proof. Assume (1) and write $g=sk$ with $s\in S$ and $k\in K$. Then $k\notin K^q$ and so $K=\langle k^2\rangle$. Let tQ be an involution of S/Q. Then $tkt\equiv k^{-1}$ (mod Q) and $tt^g\equiv tt^k\equiv k^2$ (mod Q) by 2.2. Therefore, $K\leq\langle t, t^g\rangle Q$. If $a\in S-Q$ and $b\in S^g-Q$, then $\langle t, t^g\rangle Q\leq\langle a, b\rangle Q$ by 2.2, and so $\langle a, b\rangle$ contains an S_q-subgroup of G. Therefore, (1) implies (2). Clearly, (2) implies (3), (3) implies (4), (4) implies (5), and (5) implies (1) by 2.2.

The reader is referred to Section 2 of [**1**] for the definitions not given below.

2.4. If $G\neq SC_G(V)$, then the following holds.
(1) If $J(S)\leq Q$, then $K(S)\lhd G$.
(2) If $J(S)\nleq Q$, then $q=3$ and $Z\nleq Z(G)$.

Proof. Argue as in [1], noticing that $C_S(V)=Q$ and $K\nleq C_G(V)$ by 2.2.

2.5. If $C_G(Q)\leq Q$ and $q\neq 3$, then either $Z\leq Z(G)$ or $K(S)\lhd G$.

Proof. As $C_G(Q)\leq Q$, we have $Z\leq V$ and so, if $G=SC_G(V)$, then $Z\leq Z(G)$. If $G\neq SC_G(V)$, then as $q\neq 3$, 2.4 shows that $K(S)\lhd G$.

2.6. If $C_G(Q)\leq Q$ and $q=3$, then $Z\leq Z(G)$ or $Q(K(S))\lhd G$ or $G\in\mathcal{G}'(S)$.

Proof. We may assume that $S\neq Q$. Then $G/QK^q\cong SL_2(2)$ and G is S-irreducible by 2.2. The assertion therefore follows from Theorem D of [3].

The following two lemmas are essentially due to Glauberman (see [1] for the source).

2.7. If $C_G(Q)\leq Q$ and no nonidentity characteristic subgroup of S is normal in G, then $G\cong\Sigma_4\times D\times E$, where E is an elementary abelian 2-group and either $D=1$ or D is the direct product of copies of $\boldsymbol{D}_8$.

Proof. By 2.5 and 2.6, we have $q=3$ and $G\in\mathcal{G}'(S)$. Thus, $G/C_G(V)\cong SL_2(2)\cong\boldsymbol{D}_6$, $[Q, O^2(G)]\leq V$, and $W=[V, O^2(G)]$ has order 4 by the definition of $\mathcal{G}'(S)$. As $O^2(G)\cap C_G(V)\leq C_G(Q/V/1)$ and $C_G(Q)\leq Q$, we have $O^2(G)\cap C_G(V)\leq Q$ and, consequently, $|C_G(V):Q|$ divides $|G:O^2(G)|$. On the other hand, the definition of $\mathcal{G}'(S)$ or 2.2 shows that $Q\in \mathrm{Syl}_2(C_G(V))$. Therefore, $C_G(V)=Q$ and $G/Q\cong\boldsymbol{D}_6$.

Let $A=\mathrm{Aut}\,S$ and assume that $W\leq Q^a$ for all $a\in A$. Then, as $[Q, O^2(G)]\leq W$, we have $Q\cap Q^a\lhd G$ for all $a\in A$ and so $\cap Q^a$ $(a\in A)$ is a nonidentity characteristic subgroup of S and normal in G. Therefore, we can pick $b\in A$ so that $W\nleq Q^b$. Let $R=Q^b$. As $W\nleq Q^b=C_S(V^b)$, we have $V^b\nleq Q$. Let $t\in V^b-Q$ and pick an element g of order 3 inverted by t. Then $g\in\langle t, t^g\rangle$ and so $R\cap R^g\leq C_Q(\langle t, g\rangle)$. As $|Q:R\cap R^g|\leqq 4$ and $C_W(g)=1$, we conclude that $Q=W\times C_Q(\langle t, g\rangle)$. Therefore, $G=W\langle t, g\rangle\times C_Q(\langle t, g\rangle)$ and $W\langle t, g\rangle\cong\Sigma_4$.

Write $C_Q(\langle t, g\rangle)=D\times E$, where E is the direct product of indecomposable groups not isomorphic to $\boldsymbol{D}_8$ and either $D=1$ or D is the direct product of copies of $\boldsymbol{D}_8$. As $S=W\langle t\rangle\times D\times E$ and $W\langle t\rangle\cong\boldsymbol{D}_8$, the Krull-Remak-Schmidt theorem shows that E' is characteristic in S. As it is normal in G, we must have $E'=1$, and then we have $E^2=1$ because $Z(S)^2=E^2\lhd G$. This completes the proof.

2.8. If $C_G(Q)\leq Q$ and α is an automorphism of S of odd prime power order, then some nonidentity α-invariant subgroup of S is normal in G.

Proof. Suppose false. Then by 2.7, $G=\Sigma\times D_2\times\ldots\times D_m\times E$, where $\Sigma\cong\Sigma_4$, $D_i\cong\boldsymbol{D}_8$, and E is an elementary abelian 2-subgroup. Let $D_1=S\cap\Sigma$. Then, as $D_1\cong\boldsymbol{D}_8$, the Krull-Remak-Schmidt theorem shows that α permutes the D_i' and so $\langle\alpha\rangle$ transitively permutes the D_i', $1\leqq i\leqq m$. Therefore, m divides the order of α, which is a power of some odd prime p. Now, S has precisely 2^m maximal elementary abelian subgroups and so, as $2^m\equiv 2 \pmod{p}$, at least two of them, say A and B, are α-invariant. As $[A, B]$ is α-invariant and nontrivial,

$[A, B]$ is not normal in G, and so $A\cap D_1$ and $B\cap D_1$ are distinct maximal elementary abelian subgroups of D_1. Therefore, we may assume $A\cap D_1=O_2(\Sigma)$, and then A is α-invariant and normal in G.

2.9. If $C_G(Q)\leq Q$ and $|K|>q$, then the following holds.
(1) Some chief factor of G within Q has order at least 2^6.
(2) The Frattini factor group of Q has order at least 2^6.

Proof. (1) Suppose false. Then for every chief factor X/Y within Q, we have $|K/C_K(X/Y)|\leqq q$ because $GL_5(2)$ has no cyclic subgroups of order q^2. Hence, we have $1\neq K^q\leq C_G(Q)\leq Q$, a contradiction.

(2) If $|Q/Q^2|\leqq 2^5$, then $|K/C_K(Q/Q^2)|\leqq q$ and $K^q\leq C_G(Q/Q^2)=Q$, a contradiction.

3

We begin the proof of the Main Theorem, which we restate as follows.

MAIN THEOREM. *Let G, H be a pair of groups having a common 2-subgroup S, and assume that $|G:S|$ and $|H:S|$ are powers of odd primes q and r, respectively, and that Sylow q-subgroups of G and Sylow r-subgroups of H are cyclic and nontrivial. Then one of the following holds.*

(1) *Some nonidentity subgroup of S is normal both in G and in H.*
(2) *Either $C_G(O_2(G))\not\leq O_2(G)$ or $C_H(O_2(H))\not\leq O_2(H)$.*
(3) *If $S^*=(S\cap O^2(G))(S\cap O^2(H)))$, $G^*=S^*O^2(G)$, and $H^*=S^*O^2(H)$, then (G^*, H^*) or (H^*, G^*) is of $GL_3(2)$-type or $G_2(2)'$-type or M_{12}-type or ${}^2F_4(2)'$-type.*

As in [1], we shall argue by induction on the order of S, and accordingly we shall assume the following hypothesis throughout the remainder of the paper.

3.1 HYPOTHESIS. G and H are groups having a common 2-subgroup S and the following conditions are satisfied.

(1) $|G:S|$ and $|H:S|$ are powers of odd primes q and r, respectively, and Sylow q-subgroups of G and Sylow r-subgroups of H are cyclic and nontrivial.
(2) No nonidentity subgroup of S is normal both in G and in H.
(3) $C_G(O_2(G))\leq O_2(G)$ and $C_H(O_2(H))\leq O_2(H)$.

Furthermore, if $\bar{G}$ and $\bar{H}$ are groups having a common 2-subgroup $\bar{S}$ with $|\bar{S}|<|S|$ and if $\bar{G}$, $\bar{H}$, and $\bar{S}$ satisfy the conditions (1)-(3) above (with G, H, S replaced by $\bar{G}$, $\bar{H}$, $\bar{S}$), then the conclusion (3) of the Main Theorem holds for $(\bar{G}, \bar{H})$.

Under Hypothesis 3.1, we let $Q=O_2(G)$, $R=O_2(H)$, $V=\Omega_1(Z(Q))$, $W=\Omega_1(Z(R))$, and $Z=\Omega_1(Z(S))$. This notation will be used throughout the remainder of the paper. In this section, we shall make a preliminary study of the pair (G, H), but, as noted in Section 1, the proofs will mostly be omitted.

3.2. If a subgroup X of S is normalized both by an S_q-subgroup of G and by an S_r-subgroup of H, then $X=1$.

3.3. Suppose S^* is a subgroup of S containing $[Q, O^2(G)]$ and $[R, O^2(H)]$. Let $G^*=S^*O^2(G)$ and $H^*=S^*O^2(H)$. Then G^* and H^* satisfy Hypothesis 3.1 with respect to the common 2-subgroup S^*.

3.4. The following holds.

(1) $Q\nleq R$ and, in particular, $R\neq S$.

(2) $[Q, O^2(G)]\nleq Q\cap R$.

(3) $Q\cap R\neq Q\cap R^x$ for every element $x\in G$ such that $G=\langle S, x\rangle$.

(4) The statements (1)-(3) remain true when G, Q, and R are replaced by H, R, and Q, respectively.

Proof. Argue as in [1] noticing that if $S\neq Q$ then $G=\langle S, S^x\rangle$ for some $x\in G$ by 2.3.

3.5. We have $S=QR$.

3.6. If $x\in G$ and $G=\langle S, x\rangle$, then the following holds.

(1) $Q=(R^x\cap Q)(R\cap Q)$.

(2) $G=\langle R^x, R\rangle$.

(3) $Q/R^x\cap Q\cap R$ is the direct product of two cyclic groups each of order $|S:R|$.

(4) The statements (1)-(3) remain true when G, Q, and R are replaced by H, R, and Q, respectively.

3.7. If $\langle V, W\rangle\leq Q\cap R$, then the following holds.

(1) Either $Z\leq Z(G)$ or $Z\leq Z(H)$.

(2) If $Z\leq Z(H)$, then $[Z(R), O^2(H)]=1$.

Proof. Argue as in [1] noticing that if $[W, O^2(H)]\neq 1$ then $H\neq SC_H(W)$.

3.8. If $Z\leq Z(H)$, then the following holds.

(1) $C_V(K)=1$ for each $K\in \mathrm{Syl}_q(G)$.

(2) $Q\in \mathrm{Syl}_2(C_G(V))$ and $C_G(V)$ contains no S_q-subgroup of G.

(3) If x and y are elements of G and H, respectively, such that $G=\langle S, x\rangle$ and $H=\langle S, y\rangle$, then $C_{S^y}(z)\leq Q$ for each nonidentity element z of Z^x.

Proof. (2) As $Z\leq V$ and $Z\nleq Z(G)$, we have $G\neq SC_G(V)$, and so the assertion follows from 2.2.

(3) As $Z^x \leq V$, we have $Q \leq C_S(z)$. If $Q < C_S(z)$, then $G = \langle S^x, C_S(z) \rangle$ by 2.3 and so $z \in Z(G)$, contrary to (1). Therefore, $C_S(z) = Q$ and it will suffice to prove that $C_{S^y}(z) \leq R$. Suppose $C_{S^y}(z) \nleq R$. Then $H = \langle Q, C_{S^y}(z) \rangle$ by 2.3 and (2) of 3.6, and so $z \in Z(H)$. But then we have $C_S(z) \neq Q$ by 3.4, a contradiction.

3.9. Let $S^* = (S \cap O^2(G))(S \cap O^2(H))$, $G^* = S^* O^2(G)$, and $H^* = S^* O^2(H)$. Then the following holds.

(1) If (G, H) is of $GL_3(2)$-type or $G_2(2)'$-type or M_{12}-type or ${}^2F_4(2)'$-type, then $S^* = S$ and hence $G^* = G$ and $H^* = H$.

(2) If $G/Q \cong H/R \cong \boldsymbol{D}_6$ and $Q \cong R \cong \boldsymbol{E}_8$, then (G^*, H^*) is of $GL_3(2)$-type and $G \cong H \cong \Sigma_4 \times \boldsymbol{Z}_2$.

Proof. (2) As in [1], we have that (G^*, H^*) is of $GL_3(2)$-type and so $G^* \cong H^* \cong \Sigma_4$. As $\mathrm{Aut}\,\Sigma_4 \cong \Sigma_4$, we have $G \cong H \cong \Sigma_4 \times \boldsymbol{Z}_2$.

4

4.1 Theorem. (1) *If $V \nleq R$, then $G/Q \cong H/R \cong \boldsymbol{D}_6$ and $Q \cong R \cong \boldsymbol{E}_4$ or $\boldsymbol{E}_8$.*

(2) *If $V \leq R$ but $Z(Q) \nleq R$, then (G, H) is of $G_2(2)'$-type.*

Proof. Pick $x \in G$ and $y \in H$ so that $G = \langle S, x \rangle$ and $H = \langle S, y \rangle$. (This is possible by 2.3.) Arguing as in [1], we immediately get

$$Q^{yx} \cap R^x \cap Q \cap R \cap Q^y = 1$$

and so (5) of 2.2 shows that the Frattini factor groups of Q and R each are of order at most 16. Therefore, $|G:S| = q$ and $|H:S| = r$ by 2.9, and we can proceed as in [1].

4.2 Theorem. *Suppose $Z \leq Z(H)$, $V \leq R$, and $U = \langle V^H \rangle$ is nonabelian. Then one of the following holds.*

(1) *(G, H) is of $G_2(2)'$-type.*

(2) *$G/Q \cong H/R \cong \boldsymbol{D}_6$, $Q \cong \boldsymbol{D}_8 \# \boldsymbol{D}_8$, and $R \cong \boldsymbol{D}_8 * \boldsymbol{D}_8$.*

In the latter case, if $S^* = (S \cap O^2(G))(S \cap O^2(H))$, $G^* = S^* O^2(G)$, and $H^* = S^* O^2(H)$, then (G^*, H^*) is of $G_2(2)'$-type.

Proof. Pick elements $x \in G$ and $y \in H$ so that $G = \langle S, x \rangle$ and $[V, V^y] \neq 1$. (This is possible by 2.3 and our supposition.) Argue as in [1] making the following change: replace "$G/Q \cong \boldsymbol{D}_6$" at the end of the first paragraph by "$G/C_G(V) \cong \boldsymbol{D}_6$". Then we have $q = r = 3$ and $R \cong \boldsymbol{Z}_4 * \boldsymbol{D}_8$ or $\boldsymbol{D}_8 * \boldsymbol{D}_8$ at the sixth paragraph. Consequently, $|Q| = |R| \leqq 2^5$ by 2.2 and 3.4, and so $|G:S| = |H:S| = 3$ by 2.9. Now, proceed as in [1].

5

In the subsequent sections, we shall embed G and H into the amalgamated product F of G and H over S, pick certain elements $g\in G-S$ and $h\in H-S$, and study the sublattice of the subgroup lattice of F generated by $Q^{(gh)^n}$ and $R^{(gh)^n}$ $(n\in\boldsymbol{Z})$. In this section, we shall recall notations and elementary facts concerning this sublattice.

Let $g\in G-S$, $h\in H-S$, and define $f=gh$. For subgroups X of F and integers n, define $X_n=X^{f^n}$. Notice that we do not use this notation for elements of F. For elements $x\in F$, we shall write $x'=x^{-1}$. Changing the notation, we shall denote the sequence

$$\ldots, Q_{-2}, R_{-2}, Q_{-1}, R_{-1}, Q_0, R_0, Q_1, R_1, Q_2, R_2, \ldots$$

also by

$$\ldots, P(-2), P(-1), P(0), P(1), P(2), \ldots$$

Define the bottom B of this sequence to be the intersection of the $P(n)$: $B=\cap P(n)$ $(n\in\boldsymbol{Z})$. The bottom B is determined by the ordered pair (g, h), and so we shall write $B=B(g, h)$ if necessary. Notice that $B(g, h)=B(h', g')$. All the propositions in Section 5 of [1] remain true without changes.

5.1. $S\cap S_n\leq R\cap Q_1$ for all $n\geqq 1$.

5.2. $P(n)\cap P(m)=\cap_{i=n}^{m}P(i)$ if $n<m$.

5.3. $B^f=B$.

5.4. (1) For each integer n, there exists a nonnegative integer k such that $B=P(n)\cap P(n+k)$.

(2) $P(n)\neq B\neq P(n)\cap P(n+1)$ for all n.

5.5. If a subgroup X of S is normalized both by g and by h, then $X\leq B$.

5.6. If g^2 and h^2 are contained in B, then $R_{-n}=(R_{n-1})^g=(R_n)^h$ for all $n\geqq 0$.

5.7. $Q_n\leq H_{n-1}\cap H_n$ and $R_n\leq G_n\cap G_{n+1}$ for all n.

The following definition is implicit in [1]. Let $\mathcal{P}$ be the set of all pairs (g, h) of elements of G and H, respectively, such that $G=\langle S, g\rangle$ and $H=\langle S, h\rangle$, and let $\mathcal{P}'$ be the set of all pairs $(g, h)\in\mathcal{P}$ such that both g^2 and h^2 are contained in $B(g, h)$. We define

$$\mathcal{P}^*=\begin{cases}\mathcal{P}' & \text{if } \mathcal{P}' \text{ is nonempty,}\\ \mathcal{P} & \text{if } \mathcal{P}' \text{ is empty.}\end{cases}$$

Therefore, $\mathcal{P}$ and $\mathcal{P}^*$ are nonempty by 2.3. The subsequent sections will focus on $\mathcal{P}^*$.

6

In this section, we shall work under the following hypothesis.

6.1 HYPOTHESIS. $Z \leq Z(H)$.

Throughout this section (g, h) will denote an arbitrary but fixed pair in $\mathcal{P}$, and we shall retain all the notations defined in Section 5. Thus, $X_n = X^{(gh)^n}$ for subgroups X of F and $n \in \mathbf{Z}$.

6.2. The following holds.
(1) $C_{S^h}(z) \leq Q$ for each nonidentity element z of Z_{-1}.
(2) $C_{S_{-1}}(z) \leq Q$ for each nonidentity element z of Z.

Proof. As $Z_{-1} = Z^{g'}$ and $(g', h) \in \mathcal{P}$, (1) follows from 3.8. 3.8 also shows that if $(x, y) \in \mathcal{P}$ then $C_{S^{yx'}}(z) \leq Q$ for each nonidentity element z of Z. As $S^{h'g'} = S_{-1}$ and $(g, h') \in \mathcal{P}$, (2) holds.

6.3. The following six conditions concerning positive integers n are equivalent.
(1) $Z \leq Q_n$.
(2) $Z \cap Q_n \neq 1$.
(3) $V_{n-1} \leq Q$.
(4) $Z_{n-1} \leq Q$.
(5) $Z_{n-1} \cap Q \neq 1$.
(6) $V \leq Q_{n-1}$.

In the remainder of this section, we shall assume the following hypothesis about (g, h).

6.4 HYPOTHESIS. There is an integer $d > 1$ such that $Z \leq Q_d$ and $Z \nleq Q_{d+1}$.

6.5. The following holds.
(1) $Z \leq Q_{-(d-1)} \cap Q_d \leq H_{-d} \cap H_d$.
(2) $V \leq Q_{-(d-1)} \cap Q_{d-1} \leq H_{-d} \cap H_{d-1}$.
(3) $Z \cap Q_{n+1} = 1 = Z_n \cap Q$ for all $n \geqq d$.
(4) $Q_n \ngeq V \nleq Q_{-n}$ for all $n \geqq d$.

6.6. If $(g, h) \in \mathcal{P}^*$, then $V \cap Q_d = Z$.

Proof. Argue as in [1] making the following change: in the first paragraph, pick y from $L - L^r$. Also, notice that if $I(G) \nleq Q$ and $I(H) \nleq R$ then $\mathcal{P}' \neq \emptyset$ by 2.3 and so $(g, h) \in \mathcal{P}'$. (This remark applies also to 6.7 below.)

6.7. If $(g, h) \in \mathcal{P}^*$, then $Z \cap R_d = 1 = Z \cap R_{-d}$.

6.8. If $(g, h) \in \mathcal{P}^*$, then the following holds.

(1) $R_{-d} \nleq V \nleq R_{d-1}$.
(2) $\langle V_{-(d-1)}, V_d \rangle$ contains an S_r-subgroup of H, which necessarily centralizes $Q_{-(d-1)} \cap Q_d$.

6.9. If $(g, h) \in \mathcal{P}^*$, then the following holds.
(1) $|Z|=2$.
(2) $V=Z_{-1} \times Z$.
(3) $G/C_G(V) \cong \boldsymbol{D}_6$.

6.10. Assume that $(g, h) \in \mathcal{P}^*$ and let $U=\langle V^H \rangle$. Then $U/Z \leq \Omega_1(Z(R/Z))$, $R \in \mathrm{Syl}_2(C_H(U/Z))$, and $C_H(U/Z)$ contains no S_r-subgroup of H.

Proof. As in [1], we have $U/Z \leq \Omega_1(Z(R/Z))$ and so $N=C_H(U/Z)$ is a normal subgroup of H containing R. If $H=SN$, then V is normalized by H as well as G, a contradiction. Therefore, $H \neq SN$. Hence $R \in \mathrm{Syl}_2(N)$ and N contains no S_r-subgroup of H by 2.2.

7

In this section, we shall study the following situation.

7.1 HYPOTHESIS. $Z \leq Z(H)$, $V \leq R$, and $U=\langle V^H \rangle$ is abelian.

7.2. If $(g, h) \in \mathcal{P}$, then the following holds.
(1) $U \leq Q \cap Q^{gh}$.
(2) $Z \leq Q^{(gh)^2}$.

7.3 THEOREM. *Suppose* $(g, h) \in \mathcal{P}^*$, $Z \nleq Q^{(gh)^3}$, *and* $|S:R|=2$. *Let* $S^*=(S \cap O^2(G))(S \cap O^2(H))$, $G^*=S^*O^2(G)$, *and* $H^*=S^*O^2(H)$. *Then* (G^*, H^*) *is of* M_{12}-*type*.

Proof. (We shall use the notations in Section 5. Thus, $X_n=X^{(gh)^n}$ for $X \leq F$ and $n \in \boldsymbol{Z}$.) First of all, 7.2 and our supposition show that (g, h) satisfies Hypothesis 6.4 with $d=2$. Therefore, we can use 6.5–6.10. In particular, $|Z|=2$ and $G/C_G(V) \cong \boldsymbol{D}_6$ by 6.9. Also, $V_{-1} \nleq R$ and $V_1 \nleq R_{-1}$ by 6.8, which together with 7.2 shows that $R_{-1} \cap Q \geq U_{-1} \nleq R$ and that $Q \cap R \geq U \nleq R_{-1}$ because $V_1=V^h \leq U$. As $|S:R|=2$, we conclude that $Q=\langle U_{-1}, U, R_{-1} \cap R \rangle$. Similarly, as $S \geq U_{-1} \nleq R$ and $S^h \geq U_1 \nleq R$, we have $H=\langle U_{-1}, U_1, R \rangle$.

Now, $[R_{-1} \cap R, U_{-1} \cap U] \leq Z_{-1} \cap Z=1$ by 6.10 and 6.9. This shows that $U_{-1} \cap U=V$ as $Q=\langle U_{-1}, U, R_{-1} \cap R \rangle$ and U is elementary abelian. Hence, $[V^h, U_{-1}] \leq [U, U_{-1}] \leq V$ and $[V, U_1] \leq [U, U_1] \leq V_1=V^h$. This shows that $VV^h \lhd H$ as $H=\langle U_{-1}, U_1, R \rangle$. Therefore, $U=VV^h$ and $|U/Z|=4$ by 6.9. 6.10 now shows that $H/C_H(U/Z) \cong \boldsymbol{D}_6$. Furthermore, $C_R(U)=C_R(V) \cap C_R(V^h)=Q \cap Q^h$ by 3.8, and hence $Q \cap Q^h \lhd H$.

Let $Q^*=\langle U^G\rangle$. As $|U/V|=2$, we have $U/V\leq\Omega_1(Z(S/V))$ and hence $Q^*/V\leq\Omega_1(Z(Q/V))$. As a consequence, we have $[Q\cap Q^h, Q^*]\leq U$. Now, as $U_{-1}=U^{g'}\leq Q^*\leq S$, we have $H=\langle Q^*, Q^{*h}, R\rangle$. Therefore, $[Q\cap Q^h, O^2(H)]\leq U$, and U/Z and $R/Q\cap Q^h$ are the only nontrivial chief factors of H within R by 3.6. As both chief factors have order 4, we conclude that $|H:S|=3$ by 2.9.

Let $R^*=(Q^*\cap R)(Q^{*h}\cap R)$. Then $[R, Q^*]\leq Q^*\cap R\leq R^*$ and, similarly, $[R, Q^{*h}]\leq R^*$. As $H=\langle Q^*, Q^{*h}, R\rangle$, this implies that $R^*\lhd H$ and that $[R, O^2(H)]\leq R^*$. Hence, $[R^*, O^2(H)]\nleq U$, and as $[Q\cap Q^h, O^2(H)]\leq U$, we have $Q^*\cap R\nleq Q^h$ and $Q^*\cap Q^h=Q^{*h}\cap Q$. As $|S:Q|=2$, this shows that R^*/U is a product of the two maximal subgroups $Q^*\cap R/U$ and $Q^{*h}\cap R/U$, both of which are elementary abelian as $Q^*/V\leq\Omega_1(Z(Q/V))$. We conclude that R^*/U is elementary abelian. Also, as $Q^*\cap R\nleq Q^h$ and $Q^*\cap Q^h=Q^{*h}\cap Q$, we have $G=\langle R^{*g'}, R^*, Q\rangle$ and $Q\cap R^*=Q^*\cap R$. So, as $[Q, R^*]\leq Q\cap R^*$, we conclude that $[Q, O^2(G)]\leq Q^*$. Now, let $S^*=Q^*R^*$, $G^*=S^*O^2(G)$, and $H^*=S^*O^2(H)$. Then G^* and H^* satisfy Hypothesis 3.1 with respect to S^* by 3.3. Moreover, $O_2(G^*)=Q^*$ and $O_2(H^*)=R^*$ because $Q\cap S^*=Q^*(Q\cap R^*)=Q^*$ and $R\cap S^*=(Q^*\cap R)R^*=R^*$.

As $[Q^*\cap R, R^*]\leq(R^*)'\leq U$, we have $|[Q^*, R^*]U: U|\leqq 2$ and so, as $|U: V|=2$, we have $|[Q^*, R^*]V: V|\leqq 4$. This shows that $|Q^*/V: C_{Q^*/V}(R^*)|\leqq 4$. Therefore, $C_{Q^*/V}(R^{*g'})\cap C_{Q^*/V}(R^*)$ has index at most 16 in Q^*/V and, moreover, it is contained in $Z(G/V)$ because $G=\langle R^{*g'}, R^*, Q\rangle$ and $Q^*/V\leq Z(Q/V)$. We conclude that all chief factors of G within Q have order at most 16, and hence $|G:S|=3$ by 2.9. Consequently, we have $|Q^*/V|=8$ because $Q^*=\langle U^G\rangle$ and $|U/V|=2$. Hence, $Q^*\cap Q^h=U$ and $|R^*/U|=4$.

Now, we have got ready to argue as in the last three paragraphs of 7.3 in [1] to complete the proof.

8

8.1 THEOREM. *Suppose $B(g, h)\neq 1$ for some $(g, h)\in\mathscr{P}$. Let $S^*=(S\cap O^2(G))\cdot(S\cap O^2(H))$, $G^*=S^*O^2(G)$, and $H^*=S^*O^2(H)$. Then one of the following holds.*

(1) *$G/O_2(G)\cong H/O_2(H)\cong \boldsymbol{D}_6$, $O_2(G)\cong \boldsymbol{D}_8\#\boldsymbol{D}_8$, $O_2(H)\cong \boldsymbol{D}_8*\boldsymbol{D}_8$, and (G^*, H^*) is of $G_2(2)'$-type.*

(2) *$G/O_2(G)\cong H/O_2(H)\cong \boldsymbol{D}_6$, $O_2(G)\cong \boldsymbol{D}_8*\boldsymbol{D}_8$, $O_2(H)\cong \boldsymbol{D}_8\#\boldsymbol{D}_8$, and (H^*, G^*) is of $G_2(2)'$-type.*

(3) *(G^*, H^*) or (H^*, G^*) is of M_{12}-type and $S^*\neq S$.*

As in [1], we shall derive 8.1 from 8.2 below.

8.2. If $B(g, h)\neq 1$ for some $(g, h)\in\mathscr{P}$, then $S^*\neq S$ and one of the following holds.

(1) (G, H) or (H, G) satisfies the hypothesis of 4.2.

(2) (G, H) or (H, G) satisfies the hypothesis of 7.3 with respect to some $(g, h)\in\mathscr{P}$.

The proof of 8.2 will be divided into fifteen parts each having their counterpart in [1]. Pick $(g, h)\in\mathcal{P}$ so that $|B(g, h)|$ is maximal and assume $B(g, h)\neq 1$. 5.4 shows that there exist integers n and m with $n\leqq m$ such that $B(g, h)=P(n)\cap P(m)$. Pick such n and m so that $k=m-n$ is minimal. The k is uniquely determined by the pair (g, h), so we shall write $k=k(g, h)$ if necessary. Now, let $B=B(g, h)$ and define $\mathcal{P}_B=\{(x, y)\in\mathcal{P}|\ x$ and y normalize $B\}$. Arguing as in [1], we first get the following.

(a) There exists a pair $(x, y)\in\mathcal{P}$ such that $|B(x, y)|=|B|$ and $k(x, y)$ is even.

Therefore, we may assume the following.

(b) k is even.

Let $k=2\ell$, $\ell\geqq 0$. Then either $B=Q\cap Q_\ell$ or $B=R\cap R_\ell$, and we may assume that the latter holds by the symmetry between G and H. This assumption enables us to prove the following.

(c) The following holds.
(1) $|N_H(B)|=|N_S(B)||H:S|$.
(2) $O_2(N_H(B))=N_R(B)\neq N_S(B)$.
(3) $N_S(B)\leq Q$.

In proving (3) above, notice that $N_H(S)=S$ by 2.2. As a consequence of (c), we have the following.

(d) If $(x, y)\in\mathcal{P}_B$, then the following holds.
(1) $B(x, y)=B$.
(2) $k(x, y)$ is even.
(3) If $k(x, y)=2m$, then $B(x, y)\neq Q\cap Q^{(xy)^m}$.

Now, we shall prove the following by means of (c).

(e) There exists a pair $(x, y)\in\mathcal{P}_B$ such that $x^2\in N_Q(B)$ and $y^2\in N_R(B)$.

Proof. By (c), there exists an element $w\in N_H(B)$ such that $S^h=S^w$. As $hw'\in N_H(S)=S\leq G$, $v=ghw'$ is contained in G. Moreover, $v\in N_G(B)$ by 5.3 and $G=\langle S, v\rangle$. Suppose $\langle v\rangle$ is not a 2-group. Then $\langle v, Q\rangle\leq KQ$ for $K\in\mathrm{Syl}_q(G)$ by 2.2 and so, as $G=\langle S, v\rangle$, we have $\langle v, Q\rangle=KQ$. This shows that $\langle v\rangle$ contains an S_q-subgroup of G, which necessarily normalizes B. But also some S_r-subgroup of H normalizes B by (c), which is impossible by 3.2. Therefore, $\langle v\rangle$ is a 2-group and so contained in S^z for some $z\in G$. As $G=\langle S, v\rangle$, we have $G=\langle S, z\rangle$ and so if we pick $x\in\langle v\rangle-Q$ so that $x^2\in Q$, then $x\in N_G(B)$, $x^2\in N_Q(B)$, and $G=\langle S, x\rangle$ by 2.3. Also, as $S^h=S^w$, we can pick an element $y\in N_{S^h}(B)-N_R(B)$ so that $y^2\in N_R(B)$ by (c), and for such y, we have $H=\langle S, y\rangle$ by 2.3. Therefore, (x, y) meets all the requirements of (e).

By virtue of (d) and (e), we may assume the following.

(f) $(g, h)\in\mathcal{P}_B$, $g^2\in N_Q(B)$, and $h^2\in N_R(B)$.

Next, we have the following also from (c).

(g) $\ell\geqq 2$.

Now, for each subgroup X of F, we define $X^*=N_{<X,B>}(B)/B$. In the following two propositions, we shall slightly depart from [1].

(h) The pair $(H^*, (H^{g'})^*)$ satisfies the conditions (a), (b''), and (c) in Section 1 with respect to the common 2-subgroup Q^*.

Proof. First of all, (f) shows that g acts by conjugation on F^* and interchanges H^* and $(H^{g'})^*$. Consequently, $H^*\cong(H^{g'})^*$ and, as $Q^*\in\mathrm{Syl}_2(H^*)$ by (c), $Q^*\in\mathrm{Syl}_2((H^{g'})^*)$ also. Let $L\in\mathrm{Syl}_r(N_H(B))$ and notice that $L\in\mathrm{Syl}_r(H)$ by (c). Now, suppose $H^*=Q^*O(H^*)$. Then L centralizes $N_R(B)/B$. Also, g interchanges $N_R(B)$ and $N_{R^{g'}}(B)$ by (f) and so g normalizes $N_{R^{g'}}(B)\cap N_R(B)$. Therefore, if $y\in L-L^r$, then $B(g, y)\geq N_{R^{g'}}(B)\cap N_R(B)\geq B$ by 5.5 and, as $(g, y)\in\mathcal{P}$, the maximality of $|B|$ yields that $N_{R^{g'}}(B)\cap N_R(B)=B$. This, however, implies that $B=R^{g'}\cap R=R_{-1}\cap R$, contrary to (g). Therefore, $H^*\neq Q^*O(H^*)$ and we have shown that $(H^*, (H^{g'})^*)$ satisfies (a) and (b'') in Section 1. In order to verify (c), let X/B be maximal among all subgroups of Q^* that are normal both in H^* and in $(H^{g'})^*$. Then g normalizes X as g normalizes Q^* and interchanges H^* and $(H^{g'})^*$. Therefore, $B(g, h)\geq X\geq B$, and hence $X=B$.

(i) $H^*/O(H^*)\cong\Sigma_4$ or $\Sigma_4\times\boldsymbol{Z}_2$ and $O_{2',2}(H^*)=O(H^*)\times R^*$.

Proof. As H^* is 2-irreducible by 2.2, (h) and 1.2 show that there exists a pair $(\bar{G}, \bar{H})$ satisfying the conditions (a), (b'), (c), and (d) in Section 1 with respect to a common 2-subgroup $\bar{S}$ such that $\bar{G}\cong H^*/O(H^*)$ and $\bar{H}\cong(H^{g'})^*/O((H^{g'})^*)$. Also, we have $O_{2',2}(H^*)=O(H^*)\times R^*$ by (c) and 1.1. Notice that $\bar{G}$, $\bar{H}$, and $\bar{S}$ satisfy Hypothesis 3.1 as $|\bar{S}|<|S|$. Now, suppose $\Omega_1(Z(O_2(\bar{G})))$ and $\Omega_1(Z(O_2(\bar{H})))$ are contained in $O_2(\bar{G})\cap O_2(\bar{H})$. Then either $\Omega_1(Z(\bar{S}))\leq Z(\bar{G})$ or $\Omega_1(Z(\bar{S}))\leq Z(\bar{H})$ by 3.7. However, as $H^*\cong(H^{g'})^*$, we have $\bar{G}\cong\bar{H}$ and so $\Omega_1(Z(\bar{S}))\leq Z(\bar{G})\cap Z(\bar{H})$ as in the proof of 1.3, a contradiction. Therefore, either $\Omega_1(Z(O_2(\bar{G})))\nleq O_2(\bar{H})$ or $\Omega_1(Z(O_2(\bar{H})))\nleq O_2(\bar{G})$, and 4.1 shows that $\bar{G}/O_2(\bar{G})\cong\bar{H}/O_2(\bar{H})\cong\boldsymbol{D}_6$ and $O_2(\bar{G})\cong O_2(\bar{H})\cong\boldsymbol{E}_4$ or $\boldsymbol{E}_8$. If $O_2(\bar{G})\cong\boldsymbol{E}_4$ then $\bar{G}\cong\Sigma_4$, while if $O_2(\bar{G})\cong\boldsymbol{E}_8$ then $\bar{G}\cong\Sigma_4\times\boldsymbol{Z}_2$ by 3.9.

From (h) and (i), we get the following.

(j) There exists a pair $(x, y)\in\mathcal{P}_B$ such that $x^2\in N_{R^{x'}\cap R}(B)$ and $y^2\in B$.

Proof. As $h\in N_H(B)$ by (f), (h) and (i) show that there exists an element $y\in N_{Q^h}(B)-N_R(B)$ such that $y^2\in B$, and for such y, we have $H=\langle S, y\rangle$ by 2.3. Also, (h) and (i) show that R^* and $(R^{g'})^*$ are distinct maximal subgroups of Q^*.

As g interchanges them, we have $\langle g\rangle^\sharp Q^\sharp/R^\sharp\cap(R^{g'})^\sharp\cong \boldsymbol{D}_8$ and so the coset $gN_Q(B)$ contains an element x such that $x^2\in N_{R^{g'}}(B)\cap N_R(B)$. As $G=\langle S, x\rangle$ and $R^{x'}=R^{g'}$, the pair (x, y) meets all our requirements.

In view of (d) and (j), we shall assume the following.

(k) $g^2\in N_{R^{g'}\cap R}(B)$ and $h^2\in B$.

Also, we get the following from (h) and (i).

(l) $|N_{R^{g'}\cap R}(B):B|\leqq 4$, and if equality holds then there exists an element $x\in N_G(B)$ such that $G=\langle S, x\rangle$ and $x^2\in B$.

Proof. (h) and (i) show that $Q^\sharp\cong \boldsymbol{D}_8$ or $\boldsymbol{D}_8\times \boldsymbol{Z}_2$ and that $R^\sharp$ and $(R^{g'})^\sharp$ are the two distinct elementary abelian maximal subgroups of $Q^\sharp$. Hence, $R^\sharp\cap(R^{g'})^\sharp\cong \boldsymbol{Z}_2$ or $\boldsymbol{E}_4$. Suppose $R^\sharp\cap(R^{g'})^\sharp\cong \boldsymbol{E}_4$. Then $R^\sharp\cap(R^{g'})^\sharp=\boldsymbol{Z}(H^\sharp)\times \boldsymbol{Z}((H^{g'})^\sharp)$ by (h) and (i). Therefore, $\langle g\rangle^\sharp(R^\sharp\cap(R^{g'})^\sharp)\cong \boldsymbol{D}_8$ by (k), and the coset $g(N_R(B)\cap N_{R^{g'}}(B))$ contains an element x such that $x^2\in B$. The x meets all our requirements.

From (l), we have

(m) $\ell\leqq 3$, and if equality holds then $|N_{R^{g'}\cap R}(B):B|=4$.

Now, we shall prove

(n) $[B, O^2(N_H(B))]=1$.

Proof. Let C be a nonidentity subgroup of Q such that i) $N_H(B)\leq N_H(C)$, ii) $C^g=C$, and iii) $N_Q(C)$ is maximal subject to i) and ii). Arguing as in [**1**], we have $N_Q(C)\in \mathrm{Syl}_2(N_H(C))$ and that no nonidentity characteristic subgroup of $N_Q(C)$ is normal in $N_H(C)$. Furthermore, we have $N_H(C)\neq N_Q(C)O(N_H(C))$ by (i) and so, by 1.1, no nonidentity characteristic subgroup of $N_Q(C)O(N_H(C))/O(N_H(C))$ is normal in $N_H(C)/O(N_H(C))$. Therefore, 2.6 applied to $N_H(C)/O(N_H(C))$ shows that if $L\in \mathrm{Syl}_3(N_H(B))$ then $|[N_R(B), L]|\leqq 4$. As $[N_R(B), L]\nleq B$ by (i), we conclude that $[B, L]=1$. (In fact, we have $O(N_H(C))=1$ by the $A\times B$-lemma.)

From (n), we have

(o) $V\cap B=1$.

Now, we can complete the proof of 8.2. We have $\ell=2$ or 3 by (g) and (m). If $\ell=2$, then the arguments of [**1**] show that (H, G) satisfies the hypothesis of 4.2 with $S^*\neq S$. Assume $\ell=3$. Then we may assume that $g^2\in B$ as well as $h^2\in B$ by (m), (l), and (d), and hence $(g, h)\in \mathscr{P}^*$. Also, we have $|Q|\geqq 64$ by (o) and (m). Then the arguments of [**1**] show that $Z\leq Z(H)$, $V\leq R$, and $Z\nleq Q_8$. If $\langle V^H\rangle$ is nonabelian, then (G, H) satisfies the hypothesis of 4.2, whereas $|Q|\geqq 64$, a contradiction. Therefore, $\langle V^H\rangle$ is abelian, and (G, H) satisfies the hypothesis of 7.3 with $S^*\neq S$.

9

In this section, we shall work under the following hypothesis.

9.1 HYPOTHESIS. $Z \leq Z(H)$, $V \leq R$, $U = \langle V^H \rangle$ is abelian, and $B(x, y) = 1$ for all $(x, y) \in \mathcal{P}$.

Because of the last condition in 9.1, $\mathcal{P}'$ consists of the pairs of involutions g, h such that $G = \langle S, g \rangle$ and $H = \langle S, h \rangle$. Therefore, if $(g, h) \in \mathcal{P}^*$, then the pairs (g', h), (g', h'), and (g, h') are also contained in $\mathcal{P}^*$. As in [1], 9.1 has the following two immediate consequences.

9.2. Every element $(g, h) \in \mathcal{P}$ satisfies Hypothesis 6.4 for some integer $d > 1$.

The integer d in 9.2 is uniquely determined by the pair (g, h), and so we shall write $d = d(g, h)$ if necessary.

9.3. If $(g, h) \in \mathcal{P}$, then the following holds.
(1) $C_S(g) \cap C_S(h) = 1$.
(2) If $x \in S^g - Q$ and $y \in S^h - R$, then $C_S(x) \cap C_S(y) = 1$.

Proof. As $C_S(g) \cap C_S(h) \leq B(g, h)$ by 5.5, (1) follows from 9.1. As $(x, y) \in \mathcal{P}$ by 2.3, (2) follows from (1).

The remainder of this section is devoted to the proof of the following theorem.

9.4 THEOREM. *We have $d(g, h) = 2$ for some $(g, h) \in \mathcal{P}^*$.*

We shall assume that $d(g, h) > 2$ for all $(g, h) \in \mathcal{P}^*$, and argue for a contradiction. In the following propositions 9.5–9.12, (g, h) will denote an arbitrary but fixed pair in $\mathcal{P}^*$, and we shall use the notations defined in Section 5. Thus, $X_n = X^{(gh)^n}$ for $X \leq F$ and $n \in \boldsymbol{Z}$. Also, we set $d = d(g, h)$.

9.5. We have $Z \cap U_n = 1 = Z_n \cap U$ for all $n \geqq 2$.

Proof. We shall first prove $Z \cap U_n = 1$ for $n \geqq 2$. As $U_n \leq Q_{n+1}$ by 7.2, 6.5 shows that it suffices to consider the case $2 \leqq n < d$. 6.5 and 5.2 show that $Z_{-(d-n+1)} \leq Q_{-(2d-n)} \cap Q_{n-1} \leq R \cap Q_{n-1}$. Also, 6.7 shows that $Z_{-(d-n+1)} \cap R_{n-1} = 1$. Hence, $S_{n-1} \geq R \cap Q_{n-1} \nleq R_{n-1}$. Similarly, we have $Z_{d+1} \leq Q_2 \cap Q_{2d+1} \leq Q_2 \cap R_n$ and $Z_{d+1} \cap R_1 = 1$, and hence $(S^h)^{gh} \geq Q_2 \cap R_n \nleq R^{gh}$. Now, assume that $1 \neq z \in Z \cap U_n$. Then $C_{S_{n-1}}(z) \geq R \cap Q_{n-1} \nleq R_{n-1}$ and so $C_{S_n}(z) \leq Q_n$ by 9.3. Hence, $C_{R_n}(z) \leq Q_n \cap R_n$. Now, $[R_n, z] \leq Z_n$ by 6.10 and so, as $|Z| = 2$ by 6.9, we have $|R_n : C_{R_n}(z)| \leqq 2$. We conclude that $C_{R_n}(z) = Q_n \cap R_n$. Now, as $n \geqq 2$, we have $Q_2 \cap R_n \leq Q_n \cap R_n$ and so $C_{(S^h)^{gh}}(z) \nleq R^{gh}$. Also, $(S^{g'})^{gh} = S^h$ centralizes z. This contradicts 9.3 and, therefore, we have proved $Z \cap U_n = 1$. As (g, h) is an arbitrary (though fixed) pair in $\mathcal{P}^*$, we may argue with (g', h') in place of (g, h) to conclude that $Z \cap$

$U^{(g'h')^n}=1$. Hence, $Z^{(hg)^n}\cap U=1$ and, as h normalizes Z and U, it follows that $Z^{(gh)^n}\cap U=1$; that is, $Z_n\cap U=1$.

9.6. If n is a nonnegative integer and $[U, U_j]=1$ for all j such that $0\leqq j\leqq n$, then $U\leq Q_{n+1}$ and $U_n\leq Q$.

Proof. We argue by induction on n. We may assume $n>0$ by 7.2, and we have $U\leq Q_n\leq S_n$ and $U_n\leq Q_1\leq S^h$ by the induction hypothesis. Then as $Z_{n+1}\leq U_n$, we have $U\leq C_{S_n}(Z_{n+1})\leq Q_{n+1}$ by 6.2. Also, as $Z_{-1}\leq U$, we have $U_n\leq C_{S^h}(Z_{-1})\leq Q$ by 6.2.

9.7. We have $[U, U_1]=1$.

Proof. As $d\geqq 3$, 6.5 and 5.2 show that $V\leq Q_{-(d-1)}\cap Q_{d-1}\leq R_{-2}\cap R_1$, and hence $V_1\leq R_{-1}$. Also, $V\leq U^{g'}=U_{-1}$. Therefore, $[VV_1, U_{-1}]\leq Z_{-1}=Z^{g'}\leq V$ by 6.10, which implies that U_{-1} normalizes VV_1. That is, $U^{g'}$ normalizes VV^h, and as (g, h) is arbitrary, we may replace (g, h) by (g, h') to conclude that $U^{g'}$ normalizes $VV^{h'}$, and hence $U^{g'h}$ mormalizes VV^h. Now, assume that $U^{g'}\not\leq R$. Then $U^{g'h}\not\leq R$ and so, by 2.3, $\langle U^{g'}, U^{g'h}\rangle$ contains an S_r-subgroup L of H, which necessarily normalizes VV^h. As L does not centralize VV^h/Z by 3.2 and $|VV^h/Z|=4$ by 6.9, we conclude that $r=3$. Consequently, $|S:R|=2$ and so $VV^h\lhd\langle U^{g'}, R, L\rangle=H$. However, this shows that $U=VV^h=VV_1$ and so $U^{g'}=U_{-1}=V_{-1}V\leq R$, contrary to our assumption. Therefore, $U^{g'}\leq R$, and replacing (g, h) by (g', h), we have $U^g\leq R$ also. Therefore, $U_{-1}\leq R$ and $U\leq R_{-1}$, and hence $[U_{-1}, U]\leq Z_{-1}\cap Z=1$ by 6.10 and 6.9. This completes the proof of 9.7.

At this point, we can prove 9.8 below as in [1]. However, unlike [1], we shall not use it in 9.9.

9.8. We have $r=3$.

Now, we shall prove the following.

9.9. We have $[U, U_n]=1$ for all n such that $0\leqq n\leqq d-2$.

Proof. Suppose $[U, U_j]=1$ for all j such that $0\leqq j\leqq n$, where $0\leqq n<d-2$. (This is the case for $n=0$ by 9.1.) We wish to prove $[U, U_{n+1}]=1$. By 9.7, we may assume $1\leqq n<d-2$, and by 9.6, we have $U_{n+1}\leq Q_1\leq S^h\leq H$ and $U\leq Q_{n+1}\leq H_{n+1}$. Now, as $n+2<d$, we have $V\leq Q\cap Q_{n+2}\leq R_{n+1}$ by 6.5 and 5.2, and so $[V, U_{n+1}]\leq U\cap Z_{n+1}=1$ by 6.10 and 9.5 as $n+1\geqq 2$. Therefore, $U_{n+1}\leq C_{S^h}(Z_{-1})\leq Q$ by 6.2 and then $U_{n+1}\leq Q\cap R_{n+1}\leq R$ by 5.2. Therefore, $[U, U_{n+1}]\leq Z\cap U_{n+1}=1$ by 6.10 and 9.5. The proof of 9.9 is complete by induction.

Arguing as in [1], we can derive the following from 9.8 and 9.9.

9.10. We have $d=3$ and $[Z_{-2}, Z_2]=Z$.

9.11. We have $U=VV_1$.

Now, we shall prove the following.

9.12. We have $[U^{g'}, U^{g'h}]=Z$.

Proof. Replacing (g, h) by (g', h), we shall prove instead that $[U^g, U^{gh}]=Z$. By 9.11 and 6.9, we have $U_{-1}=Z_{-2}(U_{-1}\cap U)$ and $U_1=(U\cap U_1)Z_2$, and so $[U_{-1}, U_1]=Z$ by 9.7 and 9.10. Therefore, $[U^{g'}, U^{gh}]=Z$. Now, let $K\in \mathrm{Syl}_q(G)$ and write $g=kx$ with $k\in K$ and $x\in S$. Let yQ be an involution of S/Q. Then, we have $U^{g'}=U^{k'}=U^{yk'}=U^{ky}=U^{gx'y}$ by 2.2, and $x'y\in S=Q(R\cap Q^h)$ by 3.5 and 3.6. Therefore, $U^{g'}$ and U^g are conjugate under some element of $R\cap Q^h$, and as $R\cap Q^h$ normalizes both U^{gh} and Z, we conclude that $[U^g, U^{gh}]=Z$, as required.

Now, we complete the proof of 9.4. Let $1\neq z\in Z_3$ and $1\neq z^*\in Z_{-2}$. Then $z\in S^h-R$ by 6.5 and 6.7, and so $[U^{g'}, U^{g'z}]=Z$ by 9.12 applied to $(g, z)\in \mathcal{P}^*$. Now, $U^{g'}=\langle z^*, V\rangle$ by 9.11 and 6.9, and $V\leq U^{g'}\cap U$. Hence, $U^{g'z}=\langle z^{*z}, V^z\rangle$ and $V^z\leq U^{g'z}\cap U$. Also, $[U^{g'}, U]=1$ by 9.7, and so $[U, U^{g'z}]=1$ as z normalizes U. Therefore, $[U^{g'}, U^{g'z}]=\langle [z^*, z^{*z}]\rangle=\langle [z^{*z}, z^*]\rangle$, and we conclude that $Z=\langle (z^*z)^4\rangle=\langle (zz^*)^4\rangle$ as $|Z|=2$. Now, as $z^*\in Z_{-2}=Z^{g'h'g'}$, we have $z^{*h'g'h'}\in Z^{(g'h')^3}$. Also, as $z\in Z_3=Z^{(gh)^3}$, we have $z^{h'g'h}\in Z^{ghg}=Z^{(hg)^2}$. Therefore, replacing (g, h) by (g', h') and (z, z^*) by $(z^{*h'g'h'}, z^{h'g'h'})$, we have

$$Z=\langle (z^{h'g'h'}z^{*h'g'h'})^4\rangle=\langle (zz^*)^4\rangle^{h'g'h'}=Z^{h'g'h'}.$$

However, this implies that $Z=Z_{-1}$. With this contradiction, we have established 9.4.

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10.1 Theorem. *Under Hypothesis* 9.1 *with* $|S:R|>2$, *let* $S^*=(S\cap O^2(G))(S\cap O^2(H))$, $G^*=S^*O^2(G)$, *and* $H^*=S^*O^2(H)$. *Then* $|S:S^*|\leqq 2$ *and* (G^*, H^*) *is of* ${}^2\mathrm{F}_4(2)'$*-type.*

Proof. By 9.2 and 9.4, there is a pair $(g, h)\in \mathcal{P}^*$ which satisfies Hypothesis 6.4 with $d=2$. Argue as in [1] with this pair, making the following changes. In the first paragraph, replace "$C_H(U/Z)=R$" by "$R\in \mathrm{Syl}_2(C_H(U/Z))$" and "$H/R\cong \boldsymbol{F}_{20}$" by "$H/C_H(U/Z)\cong \boldsymbol{F}_{20}$". In the second paragraph, pick an element $y\in H$ of order $|H:S|$ contained in $\langle V_{-1}, V_2\rangle$. Then, at the eighth paragraph, we can construct a normal series $1<T<P<Q$ of G such that $|T|=32$, $|P:T|=8$ or 16, and $|Q:P|=4$. Also, we have $|U|=32$ and $|R:U|=16$ or 32. Therefore, $|G:S|=q$ and $|H:S|=r$ by 2.9, and we can proceed as in [1].

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We shall conclude the proof of the Main Theorem by summarizing the results in Sections 3–10. Let $S^*=(S\cap O^2(G))(S\cap O^2(H))$, $G^*=S^*O^2(G)$, and $H^*=S^*O^2(H)$. Then G^* and H^* satisfy Hypothesis 3.1 with respect to the common 2-subgroup S^* by 3.3. Assume that $Z(G)\not\geq Z\not\leq Z(H)$. Then $\langle V, W\rangle \not\leq Q\cap R$ by 3.7, and hence $G/Q\cong H/R\cong \boldsymbol{D}_6$ and $Q\cong R\cong \boldsymbol{E}_4$ or $\boldsymbol{E}_8$ by 4.1. Therefore, (G^*, H^*) is of $GL_3(2)$-type by 3.9. Assume, therefore, that $Z\leq Z(H)$. If $V\not\leq R$, then $Z=Q\cap R$ by 4.1 and 3.5, which contradicts 3.4. Therefore, $V\leq R$. If $U=\langle V^H\rangle$ is nonabelian, then (G^*, H^*) is of $G_2(2)'$-type by 4.2 and 3.9. Assume, therefore, that U is abelian. If $B(g, h)\neq 1$ for some $(g, h)\in\mathcal{P}$, then (G^*, H^*) is of $G_2(2)'$-type or M_{12}-type by 8.1 and 8.2. Assume, therefore, that $B(x, y)=1$ for all $(x, y)\in\mathcal{P}$. Then by 9.2 and 9.4, there is an element $(g, h)\in\mathcal{P}^*$ which satisfies Hypothesis 6.4 with $d=2$. Therefore, if $|S:R|=2$, then (G^*, H^*) is of M_{12}-type by 7.3. If $|S:R|>2$, then (G^*, H^*) is of ${}^2F_4(2)'$-type by 10.1. We have proved the Main Theorem by induction on $|S|$ and also established Theorems A and B.

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