Infinite Dimensional Stein Manifolds

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Introduction

This paper aims at studying infinite dimensional Stein manifolds and at deriving an infinite dimensional version of Oka-Cartan's theorem B.

Various investigations have been made in complex analysis in finite dimensions and many kinds of deep and beautiful results were obtained. In this area, Stein manifolds have been playing an important role. The infinite dimensional theory, however, has not yet been fully developed. We shall focus on the case of $\sum \boldsymbol{C}$, which is a countably infinite dimensional space defined as the inductive limit of the finite dimensional spaces $\boldsymbol{C}^n$. In this connection, Dineen showed in [1] that $H^1(U, \mathcal{O}_G)=0$ holds for any pseudo-convex open set U in a topological vector space with the finite open topology, where $\mathcal{O}_G$ denotes the sheaf of germs of Gâteaux holomorphic functions. It was pointed out by Kajiwara [9] that $H^k(U, \mathcal{O}_G)=0$ is valid for every $k \geqq 1$. We proved in [3] that for every $k \geqq 1$, $H^k(U, \mathcal{O})=0$ holds for any pseudo-convex open set U in $\sum \boldsymbol{C}$ by using a soft resolution of the sheaf $\mathcal{O}$. Furthermore, in [4], the notion of analytic subvarieties was introduced and it was shown that for every $k \geqq 1$, $H^k(V, {}_V\mathcal{O})=0$ is valid for an analytic subvariety V of a pseudo-convex open set in $\sum \boldsymbol{C}$.

Following these results, in this article we shall introduce the notion of Stein manifolds in the case of $\sum \boldsymbol{C}$ in the manner quite analogous to that of finite dimensions, assuming only the following three axioms: Separation axiom, Local coordinates axiom and Holomorphic convexity axiom. Together with the above results, we shall obtain the following theorem.

THEOREM. *Let X be a $\sum \boldsymbol{C}$-Stein manifold. Then, we have*

$$H^k(X, \mathcal{O})=0$$

for every $k \geqq 1$.

This corresponds to a $\sum \boldsymbol{C}$-version of Oka-Cartan's Theorem B.

In § 1, the notion of $\sum \boldsymbol{C}$-complex manifolds is defined and its properties are investigated. In § 2, introducing the notion of $\sum \boldsymbol{C}$-Stein manifolds, we shall prove the above mentioned theorem by constructing a sequence of $\sum \boldsymbol{C}$-Oka-Weil domains. We refer to [6, 7, 8] for the theory of analytic functions of several variables, to [2, 10] for the general theory of holomorphic functions on infinite dimensional topological vector spaces and to [5] for the sheaf theory.

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§ 1. $\sum \boldsymbol{C}$-complex manifold

A Hausdorff topological space X is called a $\sum \boldsymbol{R}^m$-manifold if every point in X has a neighborhood which is homeomorphic to an open set in $\sum \boldsymbol{R}^m$, where m is a positive integer.

DEFINITION 1.1. A $\sum \boldsymbol{R}^2$-manifold X is called a *$\sum \boldsymbol{C}$-complex manifold* if an open covering $\mathfrak{U}=\{U_k\}$ of X is given, each element of which is provided with a homeomorphism ϕ_κ of U_κ onto an open set $\tilde{U}_\kappa$ in $\sum \boldsymbol{C}$ such that

(i) For any U_κ, $U_{\kappa'} \in \mathfrak{U}$, the mapping

$$\phi_{\kappa'} \circ \phi_\kappa^{-1}: \ \phi_\kappa(U_\kappa \cap U_{\kappa'}) \longrightarrow \phi_{\kappa'}(U_\kappa \cap U_\kappa)$$

between open sets in $\sum \boldsymbol{C}$ is biholomorphic.

(ii) If ϕ is a homeomorphism of an open set U in X onto an open set in $\sum \boldsymbol{C}$ and the mapping

$$\phi_\kappa \circ \phi^{-1}: \ \phi(U \cap U_\kappa) \longrightarrow \phi_\kappa(U \cap U_\kappa)$$

is biholomorphic for every $U_\kappa \in \mathfrak{U}$, then U belongs to $\mathfrak{U}$.

A triple $(U_\kappa, \phi_\kappa, \tilde{U}_\kappa)$ is called a chart on X.

The set $\mathcal{O}(X)$ of all holomorphic functions on X, which is endowed with the topology of uniform convergence on compact subsets of X, is a Fréchet space.

Now we examine how the intersected two charts are patched together by the transformation of local coordinates. Taking a countably infinite number of functions $\{\phi_j\}$ on $\sum \boldsymbol{C}$, we put $\phi=(\phi_1, \phi_2, \dots)$. Then the values of the mapping ϕ do not generally belong to $\sum \boldsymbol{C}$. In fact, let (z_k) be a system of coordinates in $\sum \boldsymbol{C}$ and put $w_i=\sum_{k=1}^{i} z_k$. The mapping ϕ defined by $\phi(z_1, z_2, \dots)=(w_1, w_2, \dots)$ is not a mapping of $\sum \boldsymbol{C}$ into $\sum \boldsymbol{C}$. Hence the condition that the values of a mapping ϕ belong to $\sum \boldsymbol{C}$ gives restrictions to the functions ϕ_j. Further, if we assume that ϕ is holomorphic, then we have

PROPOSITION 1.1. *Let U and V be open sets in $\sum \boldsymbol{C}$. Suppose that the* *mapping* ϕ: $U \longrightarrow V$ *is holomorphic. Let* n *be a positive integer such that* $U \cap \boldsymbol{C}^n \neq \emptyset$. *Then for each connected component* E *of* $U \cap \boldsymbol{C}^n$, *there exists a positive integer* m *such that*

$$\phi(E) \subset V \cap \boldsymbol{C}^m.$$

[*Proof*] We represent $\phi(x)=(\phi_1(x), \phi_2(x), \dots)$ in the coordinates. Let K be a compact set in E such that $\mathring{K} \neq \emptyset$. Here $\mathring{K}$ denotes the interior of K in the topology of $\boldsymbol{C}^n$. Since $\phi(K)$ is compact, there exists a positive integer m such that $\phi(K) \subset V \cap \boldsymbol{C}^m$. Namely, $\phi_j(z)=0$ holds on K for any $j>m$. Owing to the principle of analytic continuation, we have $\phi_j=0$ on E for any $j>m$. Consequently, $\phi(E) \subset V \cap \boldsymbol{C}^m$. [Q.E.D.]

Applying Proposition 1.1 to the case of $\sum \boldsymbol{C}$-complex manifolds, we have the folloing

PROPOSITION 1.2. *Let* X *be a* $\sum \boldsymbol{C}$*-complex manifold and let* $(U_1, \phi_1=(z_j), \tilde{U}_1)$ *and* $(U_2, \phi_2=(w_j), \tilde{U}_2)$ *be charts on* X. *Suppose that* $U_1 \cap U_2 \neq \emptyset$. *Let* n *be a positive integer such that* $\phi_2(U_1 \cap U_2) \cap \boldsymbol{C}^n \neq \emptyset$. *Then for each connected component* E *of* $\phi_2(U_1 \cap U_2) \cap \boldsymbol{C}^n$, *there exists a positive integer* m *such that*

$$z_j=0 \ \text{on} \ \phi_2^{-1}(E)$$

for any $j>m$.

§ 2. $\sum \boldsymbol{C}$-Stein manifold

First, we shall introduce the notion of $\sum \boldsymbol{C}$-Stien manifolds.

DEFINITION 2.1. A σ-compact $\sum \boldsymbol{C}$-complex manifold X is called a $\sum \boldsymbol{C}$-*Stein manifold* if it satisfies the following conditions:
(i) X is holomorphically convex, that is, for every compact subset K of X, its holomorphically convex hull

$$\hat{K}=\{z \in X;\ |f(z)| \leqq \sup_K |f| \ \text{for every} \ f \in \mathcal{O}(X)\}$$

is a compact subset of X.
(ii) If z_1 and z_2 are different points in X, then there exists $f \in \mathcal{O}(X)$ such that $f(z_1) \neq f(z_2)$.
(iii) For every $z \in X$ there exists a local coordinate system $\phi=(\phi_1, \phi_2, \dots)$ at z such that $\phi_k \in \mathcal{O}(X)$ for every $k>0$.

Now, we shall state a main result.

THEOREM 2.1. *Let* X *be a* $\sum \boldsymbol{C}$*-Stein manifold. Then, we have*

$$H^k(X, \mathcal{O})=0$$

for every $k \geqq 1$.

For the proof, it is sufficient to consider the case where X is connected.
Next, we define the notion of $\sum \boldsymbol{C}$-Oka-Weil domains.

DEFINITION 2.2. Let X be a $\sum \boldsymbol{C}$-Stein manifold. An open set W in X is called a $\sum \boldsymbol{C}$*-Oka-Weil domain* if the following two conditions are satisfied:
(i) There exists an open neighborhood Z of $\overline{W}$ for which there exists a sequence $\{Z_j\}$ of finite dimensional relatively compact analytic subsets of Z such that $Z=\bigcup Z_j$ and that they satisfy the properties:
(1) $W^{(j)}=W \cap Z_j$ is a relatively compact open set in the topology of Z_j.
(2) Z_j is an analytic subvariety of Z_{j+1}.
(ii) There exists a holomorphic mapping Ψ defined on X with values in $\sum \boldsymbol{C}$ such that $\Psi|_W$ is a biholomorphic mapping of W onto an analytic subvariety V of $\Delta(r)=\{(x_i) \in \sum \boldsymbol{C};\ |x_i|<r_i\}$, where $r=(r_i)$ and $0<r_i \leqq \infty$.

Together with the result already obtained, we have the following

PROPOSITION 2.2. *Let X be a $\sum \boldsymbol{C}$-Stein manifold and let W be a $\sum \boldsymbol{C}$-Oka-Weil domain in X. Then we have*

$$H^k(W, \mathcal{O})=0$$

for every $k \geqq 1$.

[*Proof*] Since Ψ: $W \longrightarrow V$ is a biholomorphic mapping, we have the isomorphism

$$H^k(W, \mathcal{O}) \cong H^k(V, {}_V\mathcal{O})$$

for every $k \geqq 0$. On account of Theorem 2.11 in [4], $H^k(V, {}_V\mathcal{O})=0$ holds for every $k \geqq 1$. Thus, it follows that $H^k(W, \mathcal{O})=0$ is valid for every $k \geqq 1$. [Q.E.D.]

Here, we claim that a $\sum \boldsymbol{C}$-Stein manifold is described as a union of $\sum \boldsymbol{C}$-Oka-Weil domains. This will play an essential role in the proof of Theorem 2.1.

PROPOSITION 2.3. *Let X be a $\sum \boldsymbol{C}$-Stein manifold. Suppose that X is connected. Then, there exists an exhausting sequence $\{W_n\}$ of $\sum \boldsymbol{C}$-Oka-Weil domains such that $\overline{W}_n \subset W_{n+1}$.*

[*Proof*] Let $\{K_k\}$ be an increasing sequence of connected holomorphically convex compact sets in X such that $\bigcup_{k=1}^{\infty} K_k = X$.

We prepare the terminology. A connected open set W in X is called, by definition, a finitely relatively compact convex (f.r.c.c. for short) open set if there exists an open set $W' \supset W$ such that $\overline{W}'$ is included in some chart $(U, \phi, \tilde{U})$ and $\phi(W')$ is convex, and if $\phi(W) \cap \boldsymbol{C}^n$ is a relatively compact open set in $\tilde{U} \cap \boldsymbol{C}^n$ for every $n>0$ (The condition that $\phi(W')$ is convex will be used only to guarantee that the intersections of $\phi(W')$ with finite dimensional spaces are connected). Then, we claim the first assertion.

ASSERTION 1. *Every* K_k *has an open neighborhood* X_k *which satisfies the following conditions*:

(1) $X_k \subset X_{k+1}$.

(2) *Every* X_k *has a covering* $\mathfrak{B}_k$ *consisting of finitely many f.r.c.c. open sets;* $(U_\alpha^{(k)}, \phi_\alpha^{(k)}=(\phi_{\alpha,j}^{(k)}), \tilde{U}_\alpha^{(k)})$, $\alpha=1, \dots, s_k$ *such that* $\mathfrak{B}_k \subset \mathfrak{B}_{k+1}$.

(3) *There exist a finite number of functions* $h_i^{(k)} \in \mathcal{O}(X)$, $i=1, \dots, b_k$ *such that for any distinct points* x_1, x_2 *in* X_k, *we can find a function* f *among* $\{h_i^{(k)}, 1 \leqq i \leqq b_k;\ \phi_{\alpha,j}^{(k)}, 1 \leqq \alpha \leqq s_k, 1 \leqq j\}$ *satisfying* $f(x_1) \neq f(x_2)$.

[*Proof* of ASSERTION 1] Since K_k is compact, K_k is covered by a finite number of f.r.c.c. open set $(U_\lambda'^{(k)}, \phi_\lambda'^{(k)}=(\phi_{\lambda,j}'^{(k)}), \tilde{U}_\lambda'^{(k)})$, $\lambda=1, \dots, s_k'$, such that every $U_\lambda'^{(k)} \cap K_k$ is connected. We put $X_k'=\bigcup_\lambda U_\lambda'^{(k)}$. $\tilde{K}_k=K_k \times K_k$ is a compact set in $\tilde{X}_k'=X_k' \times X_k'$ in the Cartesian product topology and

$$N_{k,0}=\{(x, y) \in \tilde{X}_k';\ x, y \in U_\lambda'^{(k)} \text{ for some } \lambda,\ 1 \leqq \lambda \leqq s_k'\}$$

is a neighborhood of the diagonal of $\tilde{K}_k$. For any $(x, y) \in N_{k,0}$ with $x \neq y$, there exists an f.r.c.c. open set $(U_\lambda'^{(k)}, \phi_\lambda'^{(k)}=(\phi_{\lambda,j}'^{(k)}), \tilde{U}_\lambda'^{(k)})$ such that $x, y \in U_\lambda'^{(k)}$ and then, there exists a local coordinate $\phi_{\lambda,p}'^{(k)}$ such that $\phi_{\lambda,p}'^{(k)}(x) \neq \phi_{\lambda,p}'^{(k)}(y)$. If $(x, y) \in \tilde{K}_k - N_{k,0}$, then there exists a function $h \in \mathcal{O}(X)$ such that $h(x) \neq h(y)$. Since h is continuous, there exists a neighborhood $N_{k,(x,y)}$ of (x, y) such that $h(x') \neq h(y')$ for all $(x', y') \in N_{k,(x,y)}$. Since $\tilde{K}_k$ is compact, there exist finitely many such neighborhoods $N_{k,i}=N_{k,(x_i,y_i)}$, $i=1, \dots, b_k$, which cover $\tilde{K}_k - N_{k,0}$. Let $h_i^{(k)}$ be the above chosen function relative to each $N_{k,i}$. Now, we choose an open subset $U_\lambda''^{(k)}$ of $U_\lambda'^{(k)}$ for each λ satisfying the conditions:

(i) $U_\lambda''^{(k)} \cap K_k = U_\lambda'^{(k)}$.

(ii) $U_\lambda''^{(k)} \times U_{\lambda'}''^{(k)} \subset \bigcup_{i=0}^{b_k} N_{k,i}$ for any λ, λ' with $\lambda \neq \lambda'$.

Putting $X_k''=\bigcup_\lambda U_\lambda''^{(k)}$, we define $X_k'''=\bigcap_{j=k}^{\infty} X_j''$. Taking the topology of $\sum C$ into account, we can easily see that X_k''' is open. Further, we choose an f.r.c.c. open subset $U_\lambda'''^{(k)}$ of $U_\lambda''^{(k)} \cap X_k'''$ for each λ such that $U_\lambda'''^{(k)} \cap K_k = U_\lambda''^{(k)}$. We set $U_{\lambda+s_{i-1}}^{(k)}=U_\lambda'''^{(i)}$ for each $i \leqq k$, where $s_i=s_1'+s_2'+\dots+s_i'$ and $s_0=0$, and put $X_k=\bigcup_{\alpha=1}^{s_k} U_\alpha^{(k)}$. Then the condition (1) is satisfied. To verify the condition (2), it is sufficient to consider the covering $\mathfrak{B}_k=\{U_\alpha^{(k)};\ 1 \leqq \alpha \leqq s_k\}$. The way of choosing $\{h_i^{(k)};\ 1 \leqq i \leqq b_k\}$ implies that the condition (3) is fulfilled. [Q.E.D.]

Now, we define m_k inductively with the help of $\mathfrak{B}_k$:

$m_0=0$,

$m_k=\min_n \{n>m_{k-1};\ \phi_\alpha^{(k)}(K_k \cap U_\alpha^{(k)}) \subset \tilde{U}_\alpha^{(k)} \cap C^n \text{ for } 1 \leqq \alpha \leqq s_k\}$, if $k \geqq 1$.

By using m_k we define

$d_0=0$,
$d_k=\min\limits_n \{n>d_{k-1}\,;\ \phi^{(k)}_{\alpha,j}(x)=0$ for all $x\in(\phi^{(k)}_\beta)^{-1}(\tilde{U}^{(k)}_\beta\cap \boldsymbol{C}^{m_k})$, every $j>n$ and every α and β, $1\leqq\alpha,\ \beta\leqq s_k\}$, if $k\geqq1$.

ASSERTION 2. *The integer d_k is determined as a finite number.*

[*Proof* of ASSERTION 2] Since $U^{(k)}_\alpha$ is f.r.c.c., $\phi^{(k)}_\alpha$ is defined on $\overline{U^{(k)}_\alpha}$. Then $\phi^{(k)}_\alpha(K_k\cap\overline{U^{(k)}_\alpha})$ is compact. Thus there exists an integer q such that $\phi^{(k)}_\alpha(K_k\cap\overline{U^{(k)}_\alpha})\subset U^{(k)}_\alpha\cap \boldsymbol{C}^q$. Hence, m_k is determined as a finite number. If $U^{(k)}_\alpha\cap U^{(k)}_\beta\neq\emptyset$, then $\phi_\beta(\overline{U^{(k)}_\alpha}\cap\overline{U^{(k)}_\beta})\cap \boldsymbol{C}^{m_k}$ is compact. Thus $\phi^{(k)}_\alpha\circ(\phi^{(k)}_\beta)^{-1}(\phi_\beta(\overline{U^{(k)}_\alpha}\cap\overline{U^{(k)}_\beta})\cap \boldsymbol{C}^{m_k})$ is also compact and it is therefore contained in $U^{(k)}_\alpha\cap \boldsymbol{C}^n$ for some n. On the other hand, since there exists a convex open set $\tilde{U}\supset\tilde{U}^{(k)}_\beta$, $\tilde{U}\cap \boldsymbol{C}^{m_k}$ is connected. Hence, owing to the principle of analytic continuation, $\phi^{(k)}_{\alpha,j}=0$ on $(\phi^{(k)}_\beta)^{-1}(\tilde{U}\cap \boldsymbol{C}^{m_k})$ for $j>n$. If $U^{(k)}_\alpha\cap U^{(k)}_\beta=\emptyset$, then there exists a chain of f.r.c.c. open sets; $U_\alpha=U_{\alpha_0}$, $U_{\alpha_1},\dots,U_{\alpha_\kappa}=U_\beta$ such that $U_{\alpha_i}\cap U_{\alpha_{i+1}}\neq\emptyset$ for $i=0,1,\dots,\kappa-1$. Therefore by repeating the above argument, $\phi^{(k)}_{\alpha,j}=0$ on $(\phi^{(k)}_\beta)^{-1}(\tilde{U}^{(k)}_\beta\cap \boldsymbol{C}^{m_k})$ for any $j>n'$ and for some n'. This implies that d_k is determined as a finite number. [Q.E.D.]

Then, we define
$Y_k=$The connected component containing K_k of
$\{x\in X_k\,;\ \phi^{(k)}_{\alpha,j}(x)=0$ for every $j>d_k$ and $1\leqq\alpha\leqq s_k\}$.
Hereafter, it is always assumed to take an appropriate connected component of the set on the right hand side in such definitions as the above without mentioning explicitly.

The definition of Y_k implies the following

ASSERTION 3. *The sequence $\{Y_k\}$ is an increasing sequence of finite demensional analytic subsets of X such that $X=\bigcup Y_k$.*

We construct a sequence of $\sum\boldsymbol{C}$-Oka-Weil domains using $\{Y_k\}$. Let an integer c_1 be fixed. First, we can find finitely many holomorphic functions $f^{(1)}_i$, $i=1,\dots,t_1$ on X for which

$$W^{(0)}_1=\{x\in Y_{c_1}\,;\ |f^{(1)}_i(x)|<1 \text{ for } 1\leqq i\leqq t_1\}$$

is a relatively compact open set in Y_{c_1} such that $K_{c_1}\subset W^{(0)}_1$. Choosing K_{c_2} such that $W^{(0)}_1\subset K_{c_2}$, then, we find finitely many holomorphic functions $f^{(2)}_i$, $i=1,\dots,t_2$ on X for which

$$W^{(0)}_2=\{x\in Y_{c_2}\,;\ |f^{(2)}_i(x)|<1 \text{ for } 1\leqq i\leqq t_2\}$$

is a relatively compact open set in Y_{c_2} such that $K_{c_2}\subset W^{(0)}_2$. By repeating this procedure, we have $\{W^{(0)}_k\}$. Hereafter, the sequences $\{K_k\}$, $\{X_k\}$ and $\{Y_k\}$ will newly denote the sequences $\{K_{c_k}\}$, $\{X_{c_k}\}$ and $\{Y_{c_k}\}$, respectively. Now, we choose an open neighborhood $W_1^{(1)}$ of $W_1^{(0)}$ in $Y_2 \cap X_1$ such that $\overline{W_1^{(1)}} \subset W_2^{(0)}$. Namely, there exists a positive number $\delta_{1,\alpha,j}^{(1)}$ such that

$$W_2^{(0)} \supset W_2^{(0)} \cap X_1 \supset$$
$$W_1^{(1)} = \{x \in Y_2 ;\ |f_i^{(1)}(x)| < 1 \text{ for } 1 \leqq i \leqq t_1 ;\ |\phi_{\alpha,j}^{(1)}(x)| < \delta_{1,\alpha,j}^{(1)} \text{ for } d_1 < j \leqq d_2 \text{ and } 1 \leqq \alpha \leqq s_1\}.$$

Similarly, in Y_3, we can find a pair of positive numbers $\delta_{1,\alpha,j}^{(2)} < \delta_{2,\alpha,j}^{(2)}$ such that

$$W_3^{(0)} \supset W_3^{(0)} \cap X_2 \supset$$
$$W_2^{(1)} = \{x \in Y_3 ;\ |f_i^{(2)}(x)| < 1 \text{ for } 1 \leqq i \leqq t_2 ;\ |\phi_{\alpha,j}^{(2)}(x)| < \delta_{2,\alpha,j}^{(2)} \text{ for } d_2 < j \leqq d_3 \text{ and } 1 \leqq \alpha \leqq s_2\} \supset W_2^{(1)} \cap X_1 \supset$$
$$W_1^{(2)} = \{x \in Y_3 ;\ |f_i^{(1)}(x)| < 1 \text{ for } 1 \leqq i \leqq t_1 ;\ |\phi_{\alpha,j}^{(1)}(x)| < \delta_{1,\alpha,j}^{(1)} \text{ for } d_1 < j \leqq d_2 \text{ and } 1 \leqq \alpha \leqq s_1 ;\ |\phi_{\alpha,j}^{(2)}(x)| < \delta_{1,\alpha,j}^{(2)} \text{ for } d_2 < j \leqq d_3 \text{ and } 1 \leqq \alpha \leqq s_2\}.$$

By repeating this procedure, in Y_n, we can find a sequence of positive numbers $\delta_{1,\alpha,j}^{(n-1)} < \delta_{2,\alpha,j}^{(n-1)} < \dots < \delta_{n-1,\alpha,j}^{(n-1)}$ such that putting

$$W_{n-k}^{(k)} = \{x \in Y_n ;\ |f_i^{(n-k)}(x)| < 1 \text{ for } 1 \leqq i \leqq t_{n-k} ;\ |\phi_{\alpha,j}^{(n-k+m)}(x)| < \delta_{n-k,\alpha,j}^{(n-k+m)} \text{ for } d_{n-k+m} < j \leqq d_{n-k+m+1},\ 1 \leqq \alpha \leqq s_{n-k+m} \text{ and } 0 \leqq m \leqq k-1\},$$

we have

$$W_n^{(0)} \supset W_n^{(0)} \cap X_{n-1} \supset W_{n-1}^{(1)} \supset W_{n-1}^{(1)} \cap X_{n-2} \supset W_{n-2}^{(2)} \supset \dots \supset W_2^{(n-2)} \cap X_1 \supset W_1^{(n-1)}.$$

Then, we define $W_n = \lim_{k\to\infty} W_n^{(k)}$.

The construction shows that we get $W_n \subset X_n$ and

$$W_1 \subset \overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset \dots .$$

It is easy to see that W_n is open.

Assertion 4. *W_n is a $\sum C$-Oka-Weil domain.*

[*Proof* of Assertion 4] Taking W_{n+1} as Z and putting $Z_k = W_{n+1} \cap Y_{n+k}$, we can easily see that $W_n \cap Z_k = W_n^{(k)}$ holds and that the condition (i) of the Definition 2.2 is satisfied. Now, we define a mapping Ψ as follows:

$$\Psi = ((h_j^{(n)})_{1 \leqq j \leqq b_n},\ (f_j^{(n)})_{1 \leqq j \leqq t_n},\ (\phi_{\alpha,j}^{(n)})_{1 \leqq \alpha \leqq s_n,\ 1 \leqq j \leqq d_n},$$
$$(\phi_{\alpha,j}^{(n+1)})_{1 \leqq \alpha \leqq s_{n+1},\ d_n+1 \leqq j \leqq d_{n+1}},\ \dots,\ (\phi_{\alpha,j}^{(n+i)})_{1 \leqq \alpha \leqq s_{n+i},\ d_{n+i-1}+1 \leqq j \leqq d_{n+i}},\ \dots).$$

This implies that Ψ is a holomorphic mapping of X into $\sum C$. We can assume that $|\phi_{\alpha,j}^{(k)}| < 1$ on W_n for every $j > d_{k-1}$ and $k > n$. We put

$$M_{\alpha,j}^{(n)} = \sup_{W_n} |\phi_{\alpha,j}^{(n)}(x)| \qquad (0 < M_{\alpha,j}^{(n)} \leqq \infty),$$

$$M_j^{(n)} = \sup_{W_n} |h_j^{(n)}| \qquad (0 < M_j^{(n)} \leqq \infty).$$

Replacing W_n by $W_n^{(k)}$, we define $M_{\alpha,j}^{(n,k)}$ and $M_j^{(n,k)}$ similarly. Since $W_n^{(k)}$ is a relatively compact set in X, $M_{\alpha,j}^{(n,k)}$ and $M_j^{(n,k)}$ are determined as finite numbers for every k, α and j. We set

$$r(n)=((M_j^{(n)})_{1\leqq j\leqq b_n}, (1_j)_{1\leqq j\leqq t_n}, (M_{\alpha,j}^{(n)})_{1\leqq\alpha\leqq s_n,\ 1\leqq j\leqq d_n}, 1, 1, 1, \ldots).$$

$$r(n, k)=((M_j^{(n,k)})_{1\leqq j\leqq b_n}, (1_j)_{1\leqq j\leqq t_n}, (M_{\alpha,j}^{(n,k)})_{1\leqq\alpha\leqq s_n,\ 1\leqq j\leqq d_n},$$

$$(1_{i,j})_{1\leqq i\leqq s_{n+1}+\cdots+s_{n+k},\ d_n+1\leqq j\leqq d_{n+k}}).$$

where 1_j and $1_{i,j}$ denote the number one.

Now, we show that Ψ maps W_n onto an analytic subvariety V_n of a polydisc $\varDelta(r(n))$ in $\sum C$. First, we show that $\Psi|_{Y_{n+k}}$ maps $W_n^{(k)}$ biholomorphically onto a closed subvariety $V_n^{(k)}=\Psi|_{Y_{n+k}}(W_n^{(k)})$ of the polydisc $\varDelta(r(n, k))$ in the finite dimensional subspace of $\sum C$. We can easily see that $\Psi|_{Y_{n+k}}$ is a one-one proper holomorphic mapping. Taking account that $\Psi|_{Y_{n+k}}$ is defined by using the local coordinates at every point in $W_n^{(k)}$, we can conclude that $\Psi|_{Y_{n+k}}$ maps $W_n^{(k)}$ biholomorphically onto $V_n^{(k)}$. Here we need the following result to complete the proof.

ASSERTION 5. *Let ω_k and ν_k be the injections of $W_n^{(k)}$ into W_n and of $V_n^{(k)}$ into V_n, respectively. Then we have*

$$\mathcal{O}_{W_n}\cong\varprojlim_k \omega_{k*}\mathcal{O}_{W_n^{(k)}},$$

$$_{V_n}\mathcal{O}\cong\varprojlim_k \nu_{k*\,V_n^{(k)}}\mathcal{O},$$

where $\mathcal{O}_{W_n^{(k)}}$ and $_{V_n^{(k)}}\mathcal{O}$ denote the sheaves of germs of holomorphic functions on $W_n^{(k)}$ and $V_n^{(k)}$, respectively.

[*Proof* of ASSERTION 5] The former follows from Proposition 2.2 in [3] and latter from Proposition 2.10 in [4]. [Q.E.D.]

Assertion 5, in view of the definition of analytic subvarieties in terms of $\sum C$, implies that Ψ: $W_n\longrightarrow V_n$ is biholomorphic. [Q.E.D.]

The sequence $\{W_n\}$ of $\sum C$-Oka-Weil domains is the required one, which finishes the proof of Proposition 2.3. [Q.E.D.]

Remark. For our case, it is sufficient to treat a connected $\sum C$-Stein manifold X. However, Proposition 2.3 holds without the assumption that X is connected.

Now, in order to prove an approximation theorem, the following lemma is needed.

LEMMA 2.4 [10]. *Let $\varDelta(r)=\{(x_j)\in\sum C;\ |x_j|<r_j\}$ $(0<r_j\leqq\infty)$. Then, the set of all polynomials is dense in $O(\varDelta(r))$.*

Then, we can show the following proposition.

PROPOSITION 2.5. *Let X be a $\sum \boldsymbol{C}$-Stein manifold and let W be a $\sum \boldsymbol{C}$-Oka-Weil domain in X. Then, every holomorphic function defined on W is approximated by holomorphic functions defined on X uniformly on each compact subset of W.*

[*Proof*] The mapping Ψ maps W onto V biholomorphically. Let $\mathcal{J}$ be the sheaf of ideals of V. Since V is an analytic subvariety of $\Delta(r)$, by virtue of Theorem 2.7 in [4], $H^1(\Delta(r), \mathcal{J})=0$ holds. Hence, by Lemma 2.4, we obtain this result. [Q.E.D.]

Under the above consideration, we prove Theorem 2.1.

[*Proof* of THEOREM 2.1.] We shall show this theorem by using the covering cohomology. Let $\mathfrak{M}$ be a covering of X. In view of Theorem 3.6 in [3], we can assume that $\mathfrak{M}$ is a Leray covering. Let $\{W_n\}$ be an increasing sequence of $\sum \boldsymbol{C}$-Oka-Weil domains which exhaust X according to Proposition 2.3. We may assume that for any $U \in \mathfrak{M}$ the set $\{j;\ U \cap (W_{j+1}-W_j) \neq \emptyset\}$ consists of at most two elements. Let α be an arbitrary k-th cocycle with respect to the covering $\mathfrak{M}$. Since $H^k(W_n, \mathcal{O})=0$ holds by Proposition 2.2, there exists a $(k-1)$-th cochain β_n with respect to $\mathfrak{M}|_{W_n}$ such that $\delta\beta_n=\alpha|_{W_n}$. Thus, $\delta(\beta_n-\beta_{n+1}|_{W_n})=0$. Hence, $\beta_n-\beta_{n+1}|_{W_n}$ is a $(k-1)$-th cocycle. The case of $k \geqq 2$ is proved in the same way as in the case of finite dimensions. We shall consider the case of $k=1$. Since $\beta_n-\beta_{n+1}|_{W_n} \in \Gamma(W_n, \mathcal{O})$, owing to Proposition 2.5 there exists $g_n \in \Gamma(X, \mathcal{O})$ such that

$$\|\beta_n-\beta_{n+1}-g_n\|_{\overline{W_{n-1}^{(n)}}} \leqq 2^{-n},$$

where $\|f\|_W=\sup |f(x)|$. Putting $\tilde{\beta}_n=\beta_n+g_1+\ldots+g_{n-1}$, we get

$$\|\tilde{\beta}_n-\tilde{\beta}_{n+1}\|_{\overline{W_{n-1}^{(n)}}} \leqq 2^{-n}.$$

On account of completeness of $\mathcal{O}(W_{n-1})$, $\tilde{\beta}_n-\tilde{\beta}_k$ converges to a function $h_n \in \mathcal{O}(W_{n-1})$ as k tends to infinity. Since $\tilde{\beta}_n-h_n=\tilde{\beta}_{n+1}-h_{n+1}$ on W_{n-1}, putting $\tilde{\beta}=\tilde{\beta}_n-h_n$ on W_{n-1}, we have $\delta\tilde{\beta}=\alpha$. [Q.E.D.]

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