Examples of Multiplicative η-products

Dedicated to Professor Yukiyoshi Kawada on his 70th birthday

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Introduction

Let $\eta(z)$ be the Dedekind eta-function:

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q=e^{2\pi iz}.$$

Then we consider a product of the following form

$$\prod_{t=1}^{\infty}\eta(tz)^{r_t}$$

where

(0.1) r_t is an integer and nonzero for only a finite number of t,

(0.2) $\sum_t r_t$ is positive and even,

(0.3) $\sum_t tr_t=0$ or 24.

We denote by $\eta_\pi(z)$ the above product. Let

$$\eta_\pi(z)=\sum_{n=0}^{\infty}a_nq^n$$

be a q-expansion of $\eta_\pi(z)$. $\eta_\pi(z)$ is called *multiplicative* (η-product) if the coefficients of a q-expansion of $\eta_\pi(z)$ satisfy the conditions:

$$a_1\neq 0 \tag{0.4}$$

$$a_1a_{mn}=a_ma_n \text{ if } (m,n)=1. \tag{0.5}$$

A remarkable fact is that many examples of multiplicative η-products are derived from Frame shapes (characteristic polynomials) of elements of $\cdot 0$, the automorphism group of the Leech lattice (Koike [10] and see § 1.3 of this paper). We note that, if the condition (0.2) is replaced by the one that $r_t \geqq 0$ for any t ($\sum r_t$ may be odd), such multiplicative η-products have been completely classified by Dummit-Kisilevsky-Mckay [4] and are closely related to M_{24}, the Mathieu group of degree 24.

The purpose of this paper is to find other examples of multiplicative η-products. We will give such examples in the following two ways:

(i) Applications of some transformations to Frame shapes of elements of $\cdot 0$ (§ 2),

(ii) Search by a machine (a personal computer) (§ 3).

In § 4, we give some examples of primitive cusp forms (and Eisenstein series) which are expressed as η-products (Table 4, Th. 4.3, Table 5).

For Leech lattice and its automorphism group $\cdot 0$, we refer the readers to [2], but what we need about them in this paper is just Table I of [12]. In particular, the notations of conjugate classes of $\cdot 0$ such as 2A, -3C, ... are the ones in the Table.

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§ 1. η-products associated with conjugate classes of $\cdot 0$.

1.1 A symbol

$$\prod_t t^{r_t} \qquad (t,\ r_t \in \boldsymbol{Z},\quad t>0)$$

is called a *generalized permutaton* if $r_t \neq 0$ for only a finite number of t. The degree and the weight of a generalized permutation $\pi = \prod_t t^{r_t}$ are defined as

$$\deg(\pi) = \sum_t t r_t$$

$$wt(\pi) = \frac{1}{2}\sum_t r_t$$

respectively.

Let $\eta(z)$ be the Dedekind η-function:

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{2\pi i z}.$$

For a generalized permutation $\pi = \prod t^{r_t}$, put

$$\eta_\pi(z) = \prod_t \eta(tz)^{r_t}. \tag{1.1}$$

If $\deg(\pi) \equiv 0 \bmod 24$, we see from Th. 2.1 that $\eta_\pi(z)$ is a modular form (possibly meromorphic at some cusp) of weight $wt(\pi)$ and some level N with some character.

1.2. Let $\pi(z)$ be a polynomial of degree $d \geqq 1$ satisfying the following conditions:

(1.2) *$\pi(z)$ has rational integral coefficients,*
(1.3) *the leading coefficient of $\pi(z)$ is* 1,
(1.4) *all roots of the equation $\pi(z)=0$ are roots of unity.*

It is easy to see that such a polynomial can be written uniquely in the form

$$\prod_t (z^t-1)^{r_t} \qquad (r_t \in \boldsymbol{Z})$$

where t ranges over all positive integers $\leqq d$. Thus we can assign to a polynomial $\pi(z)$ a generalized permutation $\prod_t t^{r_t}$ of degree d which is called a *Frame shape* of $\pi(z)$.

REMARK. It is easy to see that $\sum r_t$ is the multiplicity of the root $z=1$ of the equation $\pi(z)=0$. In particular, we have $\sum r_t \geqq 0$.

1.3. $\cdot 0$, the automorphism group of the Leech lattice, has a natural 24-dimensional representation ρ_0 over the rational number field. For every element $\sigma \in \cdot 0$, the characteristic polynomial $\pi_\sigma(z)=\det(zI-\rho_0(\sigma))$ of a linear transformation $\rho_0(\sigma)$ is a polynomial of degree 24 satisfying the conditions (1.2)-(1.4). Thus a Frame shape can be assingned to $\pi_\sigma(z)$. This is called a Frame shape of an element σ of $\cdot 0$. Obviously, as two conjugate elements have the same Frame shapes, we also refer to a Frame shape of a conjugate class of $\cdot 0$. The list of Frame shapes of conjugate classes of $\cdot 0$ is given in Table I of Appendix of [12]. In the present paper, this table will be used freely.

Let π be a Frame shape of an element of $\cdot 0$. Then we see from Table I of [12] that

(1.5) *$wt(\pi)$ is a non-negative integer.*

Furthermore, the associated η-products $\eta_\pi(z)$ have some interesting properties:

(1.6) *If $wt(\pi)=0$, $\eta_\pi(z)$ is a modular function w.r.t $\Gamma_0(N)$ for some N. Furthermore, there exists a discrete subgroup Γ of $SL(2,\boldsymbol{Z})$ containing $\Gamma_0(N)$ such that the genus of Γ is zero and $\eta_\pi(z)$ is a generator of a function field for Γ.* (cf. [12; Th. 3.1])

(1.7) *Let $wt(\pi)>0$. Then $\eta_\pi(z)$ is a cusp form or an Eisenstein series of $wt(\pi)$ (and of some level with some character), according as $r_t \geqq 0$ for any t or not. A further property of $\eta_\pi(z)$ is as follows: Let $\eta_\pi(z)=\sum_{n=1}^{\infty} a_n q^n$. Then we have*

(1.8) *if $(m, n)=1$, then $a_{mn}=a_m a_n$.*

Thus $\eta_\pi(z)$ is multiplicative. (cf. Koike [10])

Following Koike [10], we say that a Frame shape π of $\cdot 0$ is

of type F if $wt(\pi)=0$,
of type C if $wt(\pi)>0$ and $r_t \geqq 0$ for any t,
of type E if $wt(\pi)>0$ and $r_t<0$ for some t.

§ 2. Some transformations of $\eta_\pi(z)$.

2.1. The following theorem is a slight generalization of Honda-Miyawaki [7: Th. 1], from which we can know the level, weight, character of $\eta_\pi(z)$. The proof can be done quite similarly as in [7] (cf. Harada [5]).

THEOREM 2.1. *Let π be a generalized permutation Πt^{r_t}. Suppose the following conditions are satisfied:*

(2.1) $wt(\pi)=\frac{1}{2}\sum r_t$ *is a non-negative integer.*

(2.2) $\deg(\pi)=\sum tr_t\equiv 0 \ mod\ 24.$

Let N and f be positive integers satisfying

(2.3) $t|N$ *if* $r_t\neq 0$

(2.4) $\sum\frac{Nr_t}{t}\equiv 0 \, mod\ 24,$

(2.5) *f is square free and the number $f^{-1}\Pi t^{r_t}$ is a rational square.*

(2.6) $N\equiv 0\ mod\ 4$ *if* $(-1)^{wt(\pi)}f\equiv -1\ mod\ 4,$
$N\equiv 0\ mod\ 8$ *if* $f\equiv 2\ mod\ 4.$

Then we have

$$\eta_\pi(Az)=\chi(d)(cNz+d)^{wt(\pi)}\eta_\pi(z)$$

where $A=\begin{pmatrix} a & b \\ cN & d\end{pmatrix}\in\Gamma_0(N)$ *and $\chi(d)$ is the quadratic character of the field* $Q(\sqrt{\varepsilon f})$ $(\varepsilon=(-1)^{wt(\pi)})$ *defined mod N.*

Suppose furtheremore

(2.7) $\sum_t (t, e)^2 r_t/t\geqq 0$ *for all positive integers e.*

Then $\eta_\pi(z)$ is holomorphic at each cusp of $\Gamma_0(N)$ and if, in (2.7), $\geqq$ *is replaced by* $>$, *then $\eta_\pi(z)$ is a cusp form.*

2.2 For a generalized permutation $\pi=\Pi t^{r_t}$ and a positive integer Q, a generalized permutation $\pi * Q$ is defined as follows:

$$\pi * Q=\Pi\,(Qt/(Q,t)^2)^{r_t}. \tag{2.8}$$

Let N be a positive integer and Q be a divisor of N with $\left(Q, \frac{N}{Q}\right)=1$. Then a matrix

$$W_Q=W_{Q,N}=\begin{pmatrix} aQ & b \\ cN & dQ\end{pmatrix} a,\ b,\ c,\ d\in \boldsymbol{Z},\ \det(W_Q)=Q$$

is called an Atkin-Lehner's involution of $\Gamma_0(N)$. Then we see from the well known transformation formula of the Dedekind eta-function

(2.9) $\eta_{\pi}(z)|W_Q = c\eta_{\pi*Q}(z)$ (c: some constant)

We say that $\eta_{\pi*Q}(z)$ (resp. $\pi*Q$) is a W-*transformation* of $\eta_{\pi}(z)$ (resp. π), and $\pi*Q$ is also denoted by $\pi*W_{Q,N}$ or $\pi*W_Q$. Also we say that π is multiplicative if $\eta_{\pi}(z)$ is so.
Now we have the following

THEOREM 2.2. *Let π be a generalized permutation. Suppose*
(1) *$wt(\pi)$ is a positive integer*
(2) $\deg(\pi)=0$ *or* 24
(3) *π is multiplicative.*
(4) *the associated $\eta_{\pi}(z)$ is holomorphic at all cusps.*
Then there exists a positive integer N such that, for any divisor $Q>1$ of N with $\left(Q, \frac{N}{Q}\right)=1$, $\deg(\pi*Q)=0$ *or* 24, *and then the* W-*transformation $\pi*W_{Q,N}$ of π is also multiplicative.*

This theorem is proved by Koike as a corollary of a more general theorem on multiplicative modular forms (cf. [11])

REMARK. It seems very likely that Th. 2.2 holds without assuming (1) and (4). If so, this will be very useful for the works of a machine (cf. § 3.3).

REMARK. Let π be a Frame shape of an element of $\cdot 0$ with $wt(\pi)>0$. Then we see from Table I of [12] that any W-transformation of π is of degree 0 or again a Frame shape of some element of $\cdot 0$. Thus no "new" multiplicative η-products of positive degree can be obtained from W-transformations of Frame shapes of $\cdot 0$.

2.3. We will define another transformation of a generalized permutation. Let $\pi=\Pi t^{r_t}$ be a generalized permutation and e be a non-negative integer such that

(2.10)
$$2^e | t \text{ for any } t, \quad 2^{e+1} \nmid t \text{ for some } t.$$

Let

$$U=\{t | r_t \neq 0 \text{ and } 2^{e+1} | t\}$$

$$V=\{t | r_t \neq 0 \text{ and } 2^{e+1} \nmid t\}.$$

Then put

(2.11)
$$\pi*T_e = (\prod_{t\in U} t^{r_t})(\prod_{t\in V} \{(2t)^{3r_t}.\ t^{-r_t}.\ (4t)^{-r_t}\})$$

$\pi*T_e$ has the same degree as π. The transformation (2.11) is called a T-*transformation* of π.

LEMMA 2.3.

$$\eta_{\pi*T_e}(z)=c\eta_\pi\left(z+\frac{1}{2^{e+1}}\right) \tag{2.12}$$

where $c=\exp(\pi di/2^e)$ *and* $d=\deg(\pi)$.

Proof. It is easy to see that

$$\eta\left(z+\frac{1}{2}\right)=\eta(2z)^3/\eta(z)\eta(4z). \tag{$*$}$$

Clearly, if $t\in U$, we have

$$\eta\left(t\left(z+\frac{1}{2^e}\right)\right)=\eta(tz).$$

If $t\in V$, we have

$$\begin{aligned}\eta(t(z+1/2^{e+1}))&=\eta\left(tz+\frac{t'}{2}\right) \qquad (t':\ \text{odd})\\ &=\eta\left(tz+\frac{1}{2}\right)\\ &=\eta(2tz)^3/\eta(tz)\eta(4tz) \qquad \text{by } (*).\end{aligned}$$

From this we get (2.12), q.e.d.

THEOREM 2.4. *Let* π *be a generalized permutation with* $\deg(\pi)=0$ *or* 24 *and* e *be a non-negative integer defined by* (2.10). *If* π *is multiplicative, so is* $\pi*T_e$.

Proof. Case (1): $\deg(\pi)=0$. Then we must have $e=0$, as π is multiplicative. Let

$$\eta_\pi(z)=1+\sum_{n=1}^{\infty}a_nq^n.$$

Then we have

$$a_1a_{mn}=a_ma_n \text{ if } (m,\ n)=1 \tag{2.13}$$

and

$$\eta_\pi\left(z+\frac{1}{2}\right)=1+\sum_{n=1}^{\infty}(-1)^na_nq^n.$$

If $(m, n)=1$ and so m or n is odd, we have, by (2.13),

$$(-1)a_1\cdot(-1)^{mn}a_{mn}=(-1)^ma_m\cdot(-1)^na_n.$$

Thus $\eta_\pi\left(z+\frac{1}{2}\right)$ is multiplicative. Then Lemma 2.3 yield that $\pi*T_e$ is multiplicative.

Case (2): $\deg(\pi)=24$. we have

$$\eta_\pi(z)=\sum_{n=0}^{\infty} a_n q^{2^e n+1}$$

(2.14) $$a_m a_n = a_k \ (k=2^e mn+m+n) \text{ if } (2^e m+1, 2^e n+1)=1,$$

$$\eta_\pi(z+1/2^{e+1})=\sum (-1)^n a_n q^{2^e n+1}$$

Suppose $(2^e m+1, 2^e n+1)=1$. Then we have

(2.15) $$(-1)^m(-1)^n=(-1)^{2^e mn+m+n}.$$

In fact, if $e>0$, this is obvious. If $e=0$, we have $(m+1, n+1)=1$ and so m or n is even. Thus $(-1)^{m+n}=(-1)^{mn+m+n}$, which yield (2.15). Then it follows from (2.14) and (2.15) that $\eta_\pi(z+1/2^{e+1})$ is multiplicative and so is $\eta_{\pi*T_e}(z)$ by Lemma 2.3, q.e.d.

2.4. Let $\boldsymbol{P}$ be the set of all multiplicative η-products. It is convenient to define an equivalence relation on $\boldsymbol{P}$. Namely, for $\pi_1, \pi_2 \in \boldsymbol{P}$, $\pi_1 \sim \pi_2$ if π_2 is obtained by applying a finite number of W- and T-transformations to π_1.

EXAMPLE 2.5. (1) Let $\pi_1=1^{24}$, which is a Frame shape of the identity element of $\cdot 0$. Then we have $\pi_1 * T_0 = 2^{72}/1^{24}4^{24}$ and two generalized permutations π_1 and $\pi_1 * T_0$ produce an equivalence class. (2) Let $\pi_1=2^{16}/1^8$ which is a Frame shape of a conjugate class $-2A$ of $\cdot 0$. Then $\{\pi_1, \pi_1 * T_0, \pi_1 * W_2, \pi_1 * W_2 T_0\}$ is an equivalence class. We note that $\pi_1 * T_0 = 1^8 4^8/2^8$ is a Frame shape of a conjugate class $-4A$ of $\cdot 0$ and other generalized permutations are of degree 0.

We observe from Table I of [12] that there are 25 Frame shapes of type C and 45 Frame shapes of type E. It is not difficult to show that any two of Frame shapes of type C are not equivalent, while those of type E are divided into 29 equivalence classes. Thus there exist 54 equivalence classes containing Frame shapes of $\cdot 0$.

§ 3. Multiplicative η-products not associated with $\cdot 0$.

3.1. Let π be one of the following generalized permutations:

(3.1) $$3^2 9^2, \qquad 6.18, \qquad 8.16.$$

It is known [9] that the associated η-products $\eta_\pi(z)$ are primitive cusp forms (i.e. new forms and eigenfunctions of all Hecke operators), and, in particular, multiplicative.

Equivalence classes containing these generalized permutations are

$$\{3^2 9^2,\ (3^2 9^2)*T_0\},\qquad \{6.18,\ (6.18)*T_1\},\qquad \{8.16,\ (8.16)*T_1\}$$

and any generalized permutations in these classes are not equivalent to Frame shapes of $\cdot 0$.
We note that $(6.18)*T_1$ and $(8.16)*T_1$ are primitive cusp forms (cf. Th. 4.3), while $(3^2 9^2)*T_0$ is not (cf. Table 3 in § 4.3).

3.2. Now we search for other examples which are not equivalent to any Frame shapes of $\cdot 0$. For that purpose, we consider the following problem:

Let Π_{24} *be the set of Frame shapes of polynomials of degree* 24 *satisfying the conditions* (1.2), (1.3) *and* (1.4). *Then, for which* $\pi \in \Pi_{24}$ *is* $\eta_\pi(z)$ *multiplicative ?*

We have $|\Pi_{24}| = 115{,}966$. This was computed by H. Yamada.

REMARK 3.1. Let $\pi = \Pi t^{r_t} \in \Pi_{24}$. Then $\sum r_t$ may be odd and so the associated η-product $\eta_\pi(z)$ may be a modular form of half integral weight.

Now we have the following

THEOREM 3.2. *Let* $\pi \in \Pi_{24}$. *If* $\eta_\pi(z)$ *is multiplicative, then we have the following* (I) *and* (II):
(I). *If* $wt(\pi)$ *is an integer, one of the followings holds*:
(1) π *is equivalent to a Frame shape of* $\cdot 0$,
(2) π *is one of three generalized permutations in* (3.1),
(3) π *is one of the following generalized permutations*:

$1^3 6^5/2^2 3 \sim 2^5 12^3/4.6^3$,
$2^2 3^2 18^2/1.6^2 9$
$3^3 18^2/6^2 9 \sim 2^2 12^3/4.6^2$
$1.3.4^4 24^2/2^2 8^2 12^2$,

where $\sim$ *denotes the equivalence relation defined in* § 2.4.
(II). *If* $wt(\pi)$ *is a half integer,* π *is one of the following generalized permutations*:

$6^5/3^2 \sim 3^2 12^2/6$, 8^3,
$16^2/8 \sim 8.32/16$, 24
$3.18^2/6.9 \sim 6^2 9.36/3.12.18 \sim 8^2 48/16.24$

Originally the author proved this theorem under an additional assumption that

(#) the order of π (=L.C.M of $\{t|r_t \neq 0\}$) is less than 100

by using a personal computer (NEC-PC9801F$_2$), but H. Yamada verified by using a bigger machine at Hitotsubashi University that the theorem holds without the assumption (#). For the outline of the proof, see § 3.3. Here we will make some remarks and explain how to see that Frame shapes given in I-(3) and II of Th. 3.2 are multiplicative.

Remarks. (1) Among 115,966 Frame shapes of Π_{24}, 60,914 Frame shapes are of order<100. It should be also noted that the order of elements of Π_{24} given in (I) and (II) of Th. 3.2 is at most 48. (2) The number of Frame shapes belonging to (1) of Th. 3.2 is 83. Besides 70 Frame shapes of $\cdot 0$, the following thirteen Frame shapes are such ones:

$2^{14}/1^4$ $(4C*T)$,	$1^4 4^8/2^6$ $(-4C*T)$
$1^2 4^3 8^2/2^3$ $(-8E*T)$,	$2^7 8^2/1^2 4$ $(8E*T)$
$1.10^3/2.5$ $(-20C*T)$,	$2^8 3.12^4/1^3 4^4 6^4$ $(-12K*T)$,
$1.3.12^3/4.6^2$ $(12H*T)$,	$4^2 16^2/8^2$ $(8D*T_2)$
$2^4 5.20^2/1.4^2 10^2$ $(20C*T)$,	$2^3 8.24/4^2 6$ $(12H*T_1 W_{24})$,
$1.4^2 6^3 24/2^2 3.8.12$ $(-24F*T)$,	$2^3 3.12.24/1.6^2 8$ $(24F*T)$,
$4^3 40/8.20$ $(-20C*TW_{40})$,	

among which just $2^{14}/1^4$ and $2^7 8^2/1^2 4$ are (imprimitive) cusp forms (cf. Th. 4.4). For the notations like $4C*T$, see the paragraph after Table 5 in § 4.3.

Let π be a Frame shape in Π_{24} listed in (3) of Th. 3.2. Then $\eta_\pi(z)$ is an Eisenstein series. More explicitly, $\eta_\pi(z)$ is identified with the following Eisentein series: (Note that the idetification yields the mutiplicativity of $\eta_\pi(z)$):

Table 1

π	$\eta_\pi(z)$	$E(z)$
$1^3 6^5/2^3 3$	$E(z)-4E(2z)-E(3z)+4E(6z)$	$E_{1,2}^{(2)}(z)$
$2^5 12^3/4.6^3$	$E(z)-E(2z)-9E(3z)+9E(6z)$	$E_{1,2}^{(2)}(z)$
$2^2 3^2 18^2/1.6^2 9$	$E(z)+2E(2z)$	$E_{3,12}^{(1)}(z)$
$3^3 18^2/6^2 9$	$E(z)-E(3z)-4E(4z)+4E(12z)$	$E_{1,3}^{(1)}(z)$
$2^2 12^3/4.6^2$	$E(z)-3E(3z)-E(4z)+3E(12z)$	$E_{1,3}^{(1)}(z)$
$1.3.4^4 24^2/2^2 8^2 12^2$	$E(z)-E(2z)-4E(4z)+4E(8z)$	$E_{1,3}^{(1)}(z)$

For the notations of "primitive" Eisenstein series in the third column, see § 4.3. The identifications of $\eta_\pi(z)$ with the Eisenstein series in the second column are done as follows:

(3.2) *Firstly, by using* Th. 2.1, *observe the coincidence of level, weight, character between both modular forms and then observe the coincidence of the first several coefficients of* q*-expansions of both modular forms* (cf. [6: 811]).

For Frame shapes listed in (4) of Th. 3.2, we have the following Table:

Table 2

π	$\eta_\pi(z)$
$6^5/3^2$	$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n}{3}\right)nq^{n^2}$
$3^2 12^2/6$	$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)nq^{(3n+1)^2}$

8^3	$\sum_0^\infty (-1)^n(2n+1)q^{(2n+1)^2}$.
$16^2/8$	$q\theta(q^8,\ q^{16})$
$8.32/16$	$q\theta(-q^8,\ q^{16})$
24	$q\theta(-q^{12},\ q^{36})$
$3.18^2/6.9$	$q\theta(-q^6,\ q^9)$
$6^29.36/3.12.18$	$q\theta(q^6,\ q^9)$
$8^248/16.24$	$-\omega^{-1}q\theta(\omega q^4,\ q^4)$ $(\omega=\exp(2\pi i/3))$

Here we used the notations:

$$\theta(x,\ q)=\sum_{-\infty}^{\infty} x^n q^{n^2}.$$

The identifications of $\eta_\pi(z)$ with these θ-functions follow from the product formulas:

$$\theta(x,\ q)=\prod_{n=1}^{\infty}(1-q^{2n})(1+xq^{2n-1})(1+x^{-1}q^{2n-1}),$$

(cf. [13; Appendix] or [8: p188]). These identifications were noted by T. Tasaka.

3.3 The proof of Th. 3.2 is done with the help of a machine. The works of a machine consist of the following two steps:

First step. We can list up all elements of Π_{24} as follows:

Let n be a positive integer $\leqq 90$ and π_n be a Frame shape of cyclotomic polynomial of primitive n-th roots of unity. As is well known, we have

$$\deg(\pi_n)=\varphi(n),$$

$$\pi_n=\prod_{t|n} t^{\mu(n/t)}$$

where φ and μ are the Euler and Möbius functions respectively.
We note that, if $\varphi(n)\leqq 24$, we have $n\leqq 90$.
Now consider a Diophantine equation

$$\sum_{n=1}^{90}\varphi(n)u_n=24 \qquad (u_n\geqq 0). \tag{3.3}$$

For each (non-negative) solution $\{u_1,\dots,u_{90}\}$, put

$$\pi=\pi(u_i,\cdots,u_{90})=\pi_1^{u_1}\pi_2^{u_2}\cdots\pi_{90}^{u_{90}}. \tag{3.4}$$

Then π is of degree 24 and so $\pi\in\Pi_{24}$. Obviously all Frame shapes of Π_{24} can be obtained in this way. In particular, $|\Pi_{24}|=115,966$ is equal to the number of solutions of (3.3).

EXAMPLE 3.3. Let m be a positive integer. If, in the equation (3.3), n are ranged only over divisors of m and we have m=L.C.M of $\{n|u_n\neq 0\}$, then a Frame shape $\pi=\prod\pi_n^{u_n}$ is of order m. Let $m=6$. Then we have

n	1	2	3	6
$\varphi(n)$	1	1	2	2
π_n	1	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{1.6}{2.3}$

and the Diophantine equation (3.3) becomes

$$u_1+u_2+2u_3+2u_6=24. \tag{3.5}$$

For a solution {4, 2, 6, 3} of (3.5), we have

$$\pi=\pi(4,\ 2,\ 6,\ 3)=1^4.\ \left(\frac{2}{1}\right)^2\left(\frac{3}{1}\right)^6\left(\frac{1.6}{2.3}\right)^3=\frac{3^3 6^3}{1.2}$$

which is a Frame shape of a conjugate class $6F$ of $\cdot 0$.
For a solution {4, 2, 2, 7}, we have

$$\pi(4,\ 2,\ 2,\ 7)=1^4.\ \left(\frac{2}{1}\right)^2\left(\frac{3}{1}\right)^2\left(\frac{1.6}{2.3}\right)^7=\frac{1^7 6^7}{2^5 3^5}$$

which is not a Frame shape of $\cdot 0$. We have 650 solutions of (3.5) with $u_4\neq 0$ and 132 solutions with $u_2\neq 0$, $u_3\neq 0$ and $u_4\neq 0$. Thus there exist 782 Frame shapes of degree 24 and order 6.

Second step. For a solution $\{u_1,\cdots,u_{90}\}$ of (3.3), let $\pi=\pi(u_1,\cdots,u_{90})$ be a Frame shape as in (3.4) and

$$\eta_\pi(z)=\sum_{n=1}^{\infty} a_n q^n$$

be a q-expansion of $\eta_\pi(z)$. Computing the first several coefficients $a_n (1\leqq n\leqq 21)$ of $\eta_\pi(z)$, we pick up all $\pi(u_1,\cdots,u_{90})$ such that the multiplicativity $a_{mn}=a_m a_n$ $((m,\ n)=1)$ holds for $1\leqq m,\ n\leqq 21$. Of course, all of these do not necessarily yield multiplicative η-products. Here, in order to see whether these actually yield multiplicative η-products, it is very useful to check the following conditions:

(*) *There exists a positive integer N such that the order of π divides N and, for any divisor $Q>1$ of N with* $\left(Q,\ \frac{N}{Q}\right)=1$, $\deg(\pi * Q)=0$ *or* 24. (cf. a Remark below Th. 2.2)

If a picked up π does not satisfy the condition (*), it turns out that the multiplicativity breaks down by examining larger coefficients a_n $(n>21)$. Otherwise, we see that π must be one of Frame shapes listed in Th. 3.2.

Remark. The condition (*) is not sufficient for $\eta_\pi(z)$ to be multiplicative. For instance, a Frame shape $\pi=1^7 6^7/2^5 3^5$ mentioned in Example 3.3 satisfies (*) but is not multiplicative. In fact, $a_2 a_3\neq a_6$, as is easily seen.

Example 3.4. Among 782 Frame shapes of order 6, the ones for which the "multiplicativity" $a_{mn}=a_m a_n$ $((m,\ n)=1)$ hold for $1\leqq m,\ n\leqq 21$, are as follows:

$1.6^6/2^2 3^3$, $1^4 2.6^5/3^4 \sim 1^5 3.6^4/2^4 \sim 2^5 3^4 6/1^4$
6^4, $2^3 6^3$, $1^2 2^2 3^2 6^2$
$2^4 6^4/1^2 3^2$, $3^3 6^3/1.2$
$1^3 6^5/2^3 3$, $6^5/3^2$.

All of these satisfies (*) and actually yield multiplicative η-products. Namely, in the case of the order 6, the multiplicativity $a_{mn}=a_m a_n$ for $1 \leqq m, n \leqq 21$ yields the real multiplicativity. According to the computations of H. Yamada, in general, the partial multiplicativity $a_{mn}=a_m a_n$ $(1 \leqq m, n \leqq 21)$ are satisfied by 189 Frame shapes of Π_{24}, among which 101 ones yield multiplicative η-products.

REMARK 3.5. We made use of the condition (*) after the several coefficients of $\eta_\pi(z)$ were computed. However, if Th. 2.2 is true without the assumptions (1) and (4), the condition (*) will be able to be examined before the computations of coefficients of $\eta_\pi(z)$. This will make the working time of a machine extremely short.

§4. "Primitive" modular forms expressed as a η-product

4.1. As in §2.4, let $\boldsymbol{P}$ be the set of all generalized permutations of degree 24 such that the associated η-products are multiplicative and $\boldsymbol{P}_0$ be the subset of elements of $\boldsymbol{P}$ which are equivalent to some element of Π_{24} given in (1)~(3) of Th. 3.2.

REMARK 4.1. The author knows no examples of multiplicative η-products other than $\eta_\pi(z)$ for $\pi \in \boldsymbol{P}_0$. We have $|\boldsymbol{P}_0|=260$ (60 cusp forms and 200 Eisenstein series) and $|\boldsymbol{P}_0/\sim|=61=54+3+4$.
In this section, we study which element in $\boldsymbol{P}_0$ is a primitive modular form.

4.2. A cusp form is called *primitive* if it is a new form and an eigenfunction of all Hecke operators. We know (cf. [9]) that, if π is a Frame shape of $\cdot 0$ of C-type or one of three Frame shapes listed in (3.1), then $\eta_\pi(z)$ is a primitive cusp form. Let C be the set of these Frame shapes.
Then we easily see

LEMMA 4.2. *Let $\pi \in C$. If $\pi \neq 1^4 2^2 4^4$ or $1^2 2.4.8^2$, an equivalence class containing π consists of two elements π and $\pi * T$, where T is a transformation defined in* §2.3.

THEOREM 4.3. *Let $\pi \in C$ and let e and T be an integer and a transformation defined for π by* (2.10) *and* (2.11) *respectively. Then if $e>0$, $\eta_{\pi * T}(z)$ is also a primitive cusp form and, if $e=0$, $\eta_{\pi * T}(z)$ is not primitive.*

Proof. Firstly it should be noted that, if $e>0$, $\eta_{\pi * T}(z)$ is just a χ-transform of $\eta_\pi(z)$ in the terminology of [1] where χ is some Dirichlet character of conductor 2^{e+1} and order 2. Then direct applications of [1; Th. 3.1 and Th. 3.2] yield the first statement. It is easy to see that, if $e=0$, $\eta_{\pi*T}(z)$ is a linear combination of $\eta_\pi(kz)$ $(k=1,2,4)$ as in the following table. This shows that $\eta_{\pi*T}(z)$ is not a new form, q.e.d.

Table 3

π	$\eta_{\pi*T}(z)$
1^{24}	$\eta_\pi(z)+48\eta_\pi(2z)+4096\eta_\pi(4z)$
$1^8 2^8$	$\eta_\pi(z)+16\eta_\pi(2z)$
$1^6 3^6$	$\eta_\pi(z)+12\eta_\pi(2z)+64\eta_\pi(4z)$
3^8	$\eta_\pi(z)+16\eta_\pi(4z)$
$1^4 2^2 4^4$	$\eta_\pi(z)+8\eta_\pi(2z)$
$1^4 5^4$	$\eta_\pi(z)+8\eta_\pi(2z)+16\eta_\pi(4z)$
$1^2 2^2 3^2 6^2$	$\eta_\pi(z)+4\eta_\pi(2z)$
$1^3 7^3$	$\eta_\pi(z)+6\eta_\pi(2z)+8\eta_\pi(4z)$
$1^2 2.48^2$	$\eta_\pi(z)+4\eta_\pi(2z)$
$1^2 11^2$	$\eta_\pi(z)+4\eta_\pi(2z)+4\eta_\pi(4z)$
$1.2.7.14$	$\eta_\pi(z)+2\eta_\pi(2z)$
$1.3.5.15$	$\eta_\pi(z)+2\eta_\pi(2z)+4\eta_\pi(4z)$
3.21	$\eta_\pi(z)+2\eta_\pi(4z)$
1.23	$\eta_\pi(z)+2\eta_\pi(2z)+\eta_\pi(4z)$
$3^2 9^2$	$\eta_\pi(z)+4\eta_\pi(4z)$

For convenience of the readers, we give the following Table 4 of π $(\in C)$ and $\pi*T$ with $e>0$. These yield primitive cusp forms by Th. 4.3.

Table 4

π	$\pi*T$	π	$\pi*T$
2^{12}	$4^{36}/2^{12}8^{12}$	$2^4 4^4$	$4^{16}/2^4 8^4$
4^6	$8^{18}/4^6 16^6$	6^4	$12^{12}/6^4 24^4$
$2^3 6^3$	$4^{12}12^{12}/2^3 6^3 8^3 24^3$	$4^2 8^2$	$8^8/4^2 16^2$
$2^2 10^2$	$4^6 20^6/2^2 8^2 10^2 40^2$	$2.4.6.12$	$4^4 12^4/2.6.8.24$
12^2	$24^6/12^2 48^2$	4.20	$8^3 40^3/4.16.20.80$
2.22	$4^3 44^3/2.8.22.88$	6.18	$12^3 36^3/6.18.24.72$
8.16	$16^4/8.64$		

THEOREM 4.4 (1) *If $\pi=1^4 2^2 4^4$, an equivalence class containing π consists of four elements*

$$\pi=\pi_1=1^4 2^2 4^4,\ \pi_2=2^{14}/1^4,\ \pi_3=4^{14}/8^4,\ \pi_4=8^{38}/4^{14}16^{14}.$$

$\eta_{\pi_4}(z)$ is primitive, while $\eta_\pi(z)$ and $\eta_{\pi_3}(z)$ are not.
(2) *Let $\pi=1^2 2.4.8^2$. Then an equivalence class containing π consists of*

$$\pi_1=\pi,\ \pi_2=2^7 8^2/1^2 4,\ \pi_3=2^2 8^7/4.16^2,\ \pi_4=4^5 8^5/2^2 16^2.$$

$\eta_{\pi_4}(z)$ is primitive, while $\eta_{\pi_2}(z)$ and $\eta_{\pi_3}(z)$ are not.

Proof. (1) Let $\pi=1^4 2^2 4^4$. Then we have

$$\pi_2=\pi*T_0,\ \pi_3=\pi*T_0*W_8,\ \pi_4=\pi*T_0*W_8*T_2.$$

From this, it follows that an equivalence class containing π consists of four elements π_i $(1\leqq i\leqq 4)$. Also we have

$$\eta_{\pi_2}(z)=\eta_\pi(z)+8\eta_\pi(2z)$$

$$\eta_{\pi_3}(z)=\eta_\pi(z)+4\eta_\pi(2z)$$

which can be seen by following the procedure of (3.2). Thus $\eta_{\pi_2}(z)$ and $\eta_{\pi_3}(z)$ are not primitive. Now we will prove that $\eta_{\pi_4}(z)$ is primitive.
Let ξ be a (primitive) Grössencharacter of the imaginary quadratic field $Q(\sqrt{-1})$ defined as follows:

$$\xi((\alpha))=\chi(\alpha)\alpha^4$$

where α is an integer of $Q(\sqrt{-1})$ and χ is a character of $Q(\sqrt{-1})$ of conductor 4 with $\chi(\sqrt{-1})=1$ and $\chi(2+\sqrt{-1})=-1$. Then ξ is a primitive Grössencharacter of $Q(\sqrt{-1})$ of conductor 4. Let $L(s, \xi)$ be a L-function with the character ξ. Then, by comparing the coefficients of $L(s, \xi)$ and $\eta_{\pi_4}(z)$, we see that $\eta_{\pi_4}(z)$ is a cusp form corresponding to $L(s, \xi)$. Since ξ is a primitive character, $\eta_{\pi_4}(z)$ is a primitive cusp form [3: § 4.8].
(2) Let $\pi=1^2 2.4.8^2$. Then we have

$$\pi_1=\pi,\ \pi_2=\pi*T_0,\ \pi_3=\pi*T_0*W_{16},\ \pi_4=\pi*T_0*W_{16}*T_1$$

and

$$\eta_{\pi_2}(z)=\eta_\pi(z)+4\eta_\pi(2z)$$

$$\eta_{\pi_3}(z)=\eta_\pi(z)+2\eta_\pi(2z).$$

Furthermore, if ξ is a Grössencharacter of $Q(\sqrt{-2})$ defined by

$$\xi((\alpha))=\chi(\alpha)\alpha^2$$

where α is an integer of $Q(\sqrt{-2})$ and χ is a character of $Q(\sqrt{-2})$ of conductor 2 with $\chi(1+\sqrt{-2})=-1$. Then $\eta_{\pi_4}(z)$ is seen to be a cusp from corresponding to a L-function with the character ξ. Then (2) follows from these facts, q.e.d.

REMARK 4.4. In the above proof of Th. 4.3, the proofs that $\eta_{\pi_4}(z)$ for $\pi=1^4 2^2 4^4$ or $1^2 2.4.8^2$ is a primitive cusp form are due to M. Koike.

REMARK 4.5. We have $|C|=28$. and, among 28 elements of C, there exist 15 elements with $e=0$ (cf. Table 3) and 13 elements with $e>0$ (cf. Table 4). Thus the number of primitive cusp forms we have found in this paragraph § 4.2 is $43=15+13\times 2+2$.

REMARK 4.6. $\eta_\pi(z)$ ($\pi=1^4 2^2 4^4$ resp. $1^2 2.4.8^2$) is a primitive cusp form which corresponds to a L-function with a Grössencharacter $\xi((\alpha))=\alpha^4$ (resp. α^2) of conductor 1 of a quadratic field $Q(\sqrt{-1})$ (resp. $Q(\sqrt{-2})$). Furthermore it $\pi=1^3 7^3$, $2^3 6^3$, 4^6, 3^2, 9^2, 6^4, 4^2, 8^2 or 3^8, $\eta_\pi(z)$ yield other examples of η-products which correspond to L-functions with some Grössencharacter of quadratic fields.

4.3 In this paragraph, we deal with "primitive" Eisenstein series. Let k be a positive integer and χ and ψ be primitive Dirichlet characters of conductor A and B with $\chi\psi(-1)=(-1)^k$ respectively. Put, for a positive integer n,

$$a_n=a_n(\chi,\ \psi,\ k)=\sum_{d|n}\chi(d)\psi\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^{k-1}.$$

Then a function

$$E_{\chi,\psi}^{(k)}(z)=c_{\chi,\psi,k}+\sum_{n=1}^{\infty}a_n q^n \qquad (q=e^{2\pi iz})$$

is a modular form of level AB, weight k and character $\chi\psi$, where $c_{\chi,\psi,k}$ is a suitable constant, unless $A=B=1$ and $k=2$ (cf. [3: § 4.7] or [6: p690]). Such a modular form is called a primitive Eisenstein series. If χ and ψ are of order 2, $E_{\chi,\psi}^{(k)}$ is denoted also by $E_{A,B}^{(k)}$ although, for a given integer N, a primitive Dirichlet character of conductor N and order 2 is not necessarily unique. For example, if $N=8$ (resp. 24), we have two primitive Dirichlet characters of conductor 8 (resp. 24) and order 2. But it is not difficult to see which character of conductor A (resp. B) should be taken as χ (resp. ψ) for $E_{A,B}^{(k)}$ in the following Table 5. For example, for $E_{24,24}^{(1)}$, two distinct primitive characters of conductor 24 should be taken as χ and ψ.

REMARK 4.7. Also when χ and ψ are imprimitive Dirichlet characters defined mod A and B respectively, $E_{\chi,\psi}^{(k)}$ is a modular form of level AB, weight k and character $\chi\psi$. Such a modular form is, however, a linear combination of $E(nz)$ ($n=1,2,\dots$) where $E(z)$ is a primitive Eisenstein series except for the case $k=2$, $A=1$ and ψ is a trivial Dirichlet character defined mod $B>1$.
In the following table, we list up primitive Eisenstein series which is equal to $\eta_\pi(z)$ for some element π of $\boldsymbol{P}_0$. The identifications are done by following the procedure of (3.2).

Table 5

$E_{1,4}^{(1)}$	$2^{10}/1^4 4^4$	$8D*W_8T$
$E_{1,8}^{(1)}$	$2^3 4^3/1^2 8^2$	$16A*TW_{16}T$

$E^{(1)}_{3,5}$	$1^2 15^2/3.5$	$15E$
$E^{(1)}_{1,15}$	$3^2 5^2/1.15$	$15E*W_5=15E*W_3$
$E^{(1)}_{4,5}$	$1.2.10.20/4.5$	$20C$
$E^{(1)}_{1,20}$	$2.4.5.10/1.20$	$20C*W_4=20C*W_5$
$E^{(1)}_{3,8}$	$1.4.6.24/3.8$	$24F$
$E^{(1)}_{1,24}$	$2.3.8.12/1.24$	$24F*W_3=24F*W_8$
$E^{(1)}_{4,8}$	$2^2 16^2/4.8$	$16A$
$E^{(1)}_{4,12}$	$2.6.8.24/4.12$	$24E=12G*T_1$
$E^{(1)}_{8,6}$	$4^2 16^2/8^2$	$8D*T_2$
$E^{(1)}_{4,24}$	$2.8^4 12^4 48/4^3 6.16.24^3$	$24F*TW_{16}T_1$
$E^{(1)}_{8,12}$	$4^4 6.16.24^4/2.8^3 12^3 48$	$24F*TW_{48}T_1$
$E^{(1)}_{3,12}$	$1.4.6^4 9.36/2.3^2 12^2 18$	?
$E^{(1)}_{24,24}$	$8^5 12^2 48^2 72^5/4^2 16^2 24^4 36^2 144^2$	?
$E^{(2)}_{5,1}$	$5^5/1$	$5C$
$E^{(2)}_{1,5}$	$1^5/5$	$5C*W_5$
$E^{(2)}_{3,3}$	$1^3 9^3/3^2$	$9C$
$E^{(2)}_{4,4}$	$2^4 8^4/4^4$	$8B$
$E^{(2)}_{8,1}$	$2^3 4.8^2/1^2$	$-8E$
$E^{(2)}_{1,8}$	$1^2 2.4^3/8^2$	$-8E*W_8$
$E^{(2)}_{3,4}$	$1.2^2 3.12^2/4^2$	$-12H=12I*W_{12}$
$E^{(2)}_{4,3}$	$1^2 4.6^2 12/3^2$	$12I=-12H*W_{12}$
$E^{(2)}_{12,1}$	$2^2 3^2 4.12/1^2$	$-12I$
$E^{(2)}_{1,12}$	$1.3.4^2 6^2/12^2$	$-12I*W_{12}$
$E^{(2)}_{4,8}$	$4^9 16^2/2^2 8^5$	$-8E*TW_{16}T_1$
$E^{(2)}_{8,4}$	$2^2 8^9/4^5 16^2$	$-8E*TW_{16}T_1W_{32}$
$E^{(3)}_{3,1}$	$3^9/1^3$	$3C$
$E^{(3)}_{1,3}$	$1^9/3^3$	$3C*W_3$
$E^{(3)}_{4,1}$	$2^6 4^4/1^4$	$-4C$
$E^{(3)}_{1,4}$	$1^4 2^6/4^4$	$-4C*W_4$

In the above table, the third column shows Frame shapes of $\cdot 0$ and transformations from which we get generalized permutations in the second column. For instance, in the first row of the table, the notation $8D*W_8T$ implies that $2^{10}/1^4 2^4$ can be obtained by applying transformations W_8 and T to a Frame shape of conjugate class of $\cdot 0$ named $8D$. The name of conjugate class of $\cdot 0$ is the one used in Table I of [12]. Furthermore, the symbol "?" in the third column implies that a generalized permutation in the second column is not equivalent to any Frame snapes of $\cdot 0$. In fact, let $\pi=2^2 3^2 18^2/1.6^2 9$ be a generalized permutation which can be seen in (3) of Th. 3.2. Then we have

$$1.4.6^4 9.36/2.3^2 12^2 18=\pi*T,$$

$$8^5 12^2 48^2 72^5/4^2 16^2 24^4 36^2 144^2=\pi*W_{72}T_2.$$

The equivalence class containing π consists of π, $\pi*T$, $\pi*W_{72}$, $\pi*W_{72}T_2$.

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