A Remark on the Stability Condition for the Raviart-Thomas Mixed Finite Elements

By Fumio KIKUCHI

Department of Mathematics, College of Arts and Sciences,
University of Tokyo, Komaba, Meguro-ku, Tokyo 153

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1. Introduction

Raviart and Thomas [10] presented a family of mixed finite element models for the Dirichlet problem of the Poisson equation

$$\operatorname{div}(\operatorname{grad}\lambda)=g \quad \text{in } \Omega, \qquad \lambda=0 \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a bounded polygonal domain in $\boldsymbol{R}^2$ with boundary $\partial\Omega$, λ is an unknown function defined over $\bar{\Omega}$ (=closure of Ω), and g is a given function defined in Ω. They first decomposed the differential equation in (1) as

$$u-\operatorname{grad}\lambda=0, \qquad \operatorname{div} u=g, \tag{2}$$

and then introduced a weak form to this system of equations. It is to be noted that they used a Hilbert space other than Sobolev spaces in the weak formulation as will be shown later. As usual, we must construct suitable finite-dimensional subspaces of this Hilbert space to solve the present problem by the finite element method, and we can find examples of such subspaces in [10]. The arising finite element models are of mixed type, since the weak formulation may be related to a saddle-point type variational principle, see Brezzi [4]. Their idea has drawn much attention in various fields such as plasticity, fluid mechanics, and electromagnetism [1], although it is seldom used to solve the original problem (1). However, a non-conforming model equivalent to the simplest mixed model is now known, see Arnold-Brezzi [2] and Marini [9].

In establishing the validity of mixed finite element models, it is essential to check the stability condition of Babuska-Brezzi [3, 4]. Raviart and Thomas [10] showed this in a fairly complicated fashion. Later, Fortin [6] simplified the proof by using a special operator, but, unfortunately, his original idea is valid only when Ω is a *convex* bounded polygonal domain.

In this note, we will generalize the idea of Fortin to the case where Ω is a general bounded polygonal domain. To this end, we will use another domain Ω_0 such that $\Omega\subset\Omega_0$. Then using the operator introduced by Fortin, we can show the stability condition for the Raviart-Thomas finite elements. Such an idea has been used in some other problems in numerical analysis of the finite element method: see e. g. Kikuchi [8] and Suzuki [11].

2. Preliminaries

Let Ω be a bounded domain in $\boldsymbol{R}^2$, and let $L_2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$ and $H^2(\Omega)$ be the usual real Sobolev spaces related to Ω. For convenience, the inner product of $L_2(\Omega)$ and $\{L_2(\Omega)\}^2$ are both denoted by $(\cdot,\cdot)$. Similarly, the norm of $L_2(\Omega)$ and $\{L_2(\Omega)\}^2$ are both denoted by $\|\cdot\|$. Moreover, the norm of $\{H^1(\Omega)\}^2$ is designated by $\|\cdot\|_1$. If it is necessary to specify the domain in the notations of inner products or norms, we will use notations such as $(\cdot,\cdot)_\Omega$, $\|\cdot\|_\Omega$ and $\|\cdot\|_{1,\Omega}$. We also define a Hilbert space of real vector functions by

$$H(\mathrm{div}, \Omega)=\{v\in\{L_2(\Omega)\}^2\,;\ \mathrm{div}\, v\in L_2(\Omega)\} \tag{3}$$

equipped with the norm

$$\|v\|_{H(\mathrm{div},\Omega)}=\{\|v\|^2+\|\mathrm{div}\, v\|^2\}^{1/2}\,. \tag{4}$$

Clearly, this is not a usual Sobolev space. The independent variable for functions defined in Ω is denoted by $x=\{x_1, x_2\}$.

A weak formulation appropriate for (2) is: given $g\in L_2(\Omega)$, find $\{u, \lambda\}\in H(\mathrm{div}, \Omega)\times L_2(\Omega)$ such that

$$(u, v)+(\lambda, \mathrm{div}\, v)=0\,, \qquad (\mathrm{div}\, u, \mu)=(g, \mu) \tag{5}$$

for all $\{v, \mu\}\in H(\mathrm{div}, \Omega)\times L_2(\Omega)$. This is a mixed formulation since two types of unknown functions u and λ appear, and may be related to a saddle-point type variational principle unlike the original problem (1): see Raviart-Thomas [10].

Hereafter, we assume that Ω is a bounded *polygonal* domain in $\boldsymbol{R}^2$. To solve (5) by the finite element method, we first consider a regular family of triangulations $\{T^h\}_{h\in\Lambda}$ of Ω, where the index set Λ is a bounded subset of $]0,\infty[$ which has zero as an accumulation point. In the finite element method, h is the maximum side length of all triangles in each triangulation T^h, and the terminology "regular" roughly means that the triangles in T^h are not too flat when $h\to 0$. For precise meaning of "regular", see for example Ciarlet [5].

The next step is to construct a suitable finite-dimensional subspace of $H(\mathrm{div}, \Omega)\times L_2(\Omega)$ over each T^h. Raviart and Thomas introduced a variety of such finite element spaces in [10]. A typical (and the simplest) example is given by the following $V_h\times W_h$ for each $h\in\lambda$.

(i) Each $v_h\in V_h$ is a linear polynomial vector function of the form $\{\alpha_1+\alpha_2 x_1, \alpha_3+\alpha_2 x_2\}$ in each triangle $T\in T^h$, where α_1, α_2 and α_3 are coefficients, and the normal component of v_h is continuous across the interelement boundaries. Note that the coefficient of x_1 in the first component is common to that of x_2 in the second component.

(ii) Each $\mu_h \in W_h$ is a constant function in each $T \in T^h$ (and not necessarily continuous in Ω). In other words, μ_h is a step function in Ω.

It is fairly easy to see that the present $V_h \times W_h$ is actually a finite-dimensional subspace of $H(\mathrm{div}, \Omega) \times L_2(\Omega)$. Raviart and Thomas proposed a variety of triangular finite elements as well as some quadrilateral ones. The results in this note will be proved for the afore-mentioned finite element model, but are actually valid for any other models that Raviart and Thomas presented.

Now the finite element analog of (5) for each $h \in \Lambda$ is: given $g \in L_2(\Omega)$, find $\{u_h, \lambda_h\} \in V_h \times W_h$ such that

$$(u_h, v_h) + (\lambda_h, \mathrm{div}\, v_h) = 0\,, \qquad (\mathrm{div}\, u_h, \mu_h) = (g, \mu_h) \tag{6}$$

for all $\{v_h, \mu_h\} \in V_h \times W_h$. In numerical analysis of this type of mixed finite element methods, it is essential to establish the following Babuska-Brezzi stability condition [3, 4]:

$$\sup_{v_h \in V_h \setminus \{0\}} (\mathrm{div}\, v_h, \mu_h)/\|v_h\|_{H(\mathrm{div}, \Omega)} \geqq k\|\mu_h\|\,;\ \forall \mu_h \in W_h\,, \tag{7}$$

where k is a positive constant independent of $h \in \Lambda$. Raviart and Thomas showed the above condition for the afore-mentioned family of finite element spaces in a fairly complicated fashion. Later, Fortin [6] considerably simplified the proof when Ω has a sufficiently smooth boundary. Unfortunately, polygonal domains do not have sufficiently smooth boundaries. Nevertheless, his method remains valid if Ω is convex, as we will see later. His proof is based on the following lemma, which holds even when Ω is not convex.

LEMMA 1 (Fortin [6]) *Let $\{V_h \times W_h\}_{h \in \Lambda}$ be the Raviart-Thomas family of finite element spaces over a regular family of triangulations $\{T^h\}_{h \in \Lambda}$ of a bounded polygonal domain Ω in $\boldsymbol{R}^2$. Then, there exists a family of mappings $\{\Pi_h\}_{h \in \Lambda}$ such that Π_h for each $h \in \Lambda$ is a mapping from $\{H^1(\Omega)\}^2$ into V_h and satisfies that*

$$(\mathrm{div}\, \Pi_h v, \mu_h) = (\mathrm{div}\, v, \mu_h)\,;\ \forall \{v, \mu_h\} \in \{H^1(\Omega)\}^2 \times W_h\,, \tag{8}$$

$$\|\Pi_h v\|_{H(\mathrm{div}, \Omega)} \leqq C_1 \|v\|_1\,;\ \forall v \in \{H^1(\Omega)\}^2\,, \tag{9}$$

where C_1 is a positive constant independent of $h \in \Lambda$.

3. Main Results

We will show in this section that the family of finite element spaces $\{V_h \times W_h\}_{h \in \Lambda}$ explained in the preceding section satisfies the Babuska-Brezzi stability condition when Ω is a general bounded polygonal domain. To this end, we consider a bounded domain Ω_0 in $\boldsymbol{R}^2$ such that $\Omega \subset \Omega_0$.

First, let us recall the following wellknown results given for example in Grisvard [7].

LEMMA 2 *Let Ω_0 be a bounded domain in $\boldsymbol{R}^2$ which is convex or has a sufficiently smooth boundary. Then, for each $\mu^* \in L_2(\Omega_0)$, there exists a unique function $\phi^* \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$ such that*

$$\operatorname{div}(\operatorname{grad} \phi^*) = \mu^*, \qquad \|\operatorname{grad} \phi^*\|_{1,\Omega_0} \leqq C_2 \|\mu^*\|_{\Omega_0}, \tag{10}$$

where C_2 is a positive constant dependent only on Ω_0.

By using the lemma above and the operator Π_h introduced in the preceding section, we can show the following main results.

THEOREM *Let $\{V_h \times W_h\}_{h \in \Lambda}$ be the Raviart-Thomas family of finite element spaces over a regular family of triangulations $\{T^h\}_{h \in \Lambda}$ of a bounded polygonal domain Ω in $\boldsymbol{R}^2$. Then there exists a positive constant k (independent of $h \in \Lambda$) such that*

$$\sup_{v_h \in V_h \setminus \{0\}} (\operatorname{div} v_h, \mu_h)/\|v_h\|_{H(\operatorname{div},\Omega)} \geqq k \|\mu_h\|_{\Omega}\,; \ \forall \mu_h \in W_h. \tag{11}$$

Proof Fix a bounded convex domain Ω_0 such that $\Omega \subset \Omega_0$ (actually, Ω_0 need not be convex if its boundary is smooth enough). For each $\mu_h \in W_h \subset L_2(\Omega)$, let us define its extension μ_h^* to Ω_0 as follows:

$$\mu_h^*(x) = \mu_h(x) \ \text{ if } \ x \in \Omega\,; \ \mu_h^*(x) = 0 \ \text{ if } \ x \in \Omega_0 \setminus \Omega.$$

Clearly, μ_h^* belongs to $L_2(\Omega_0)$ with $\|\mu_h^*\|_{\Omega_0} = \|\mu_h\|_{\Omega}$. Then, by Lemma 2, there exists a unique $\phi^* \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$ such that

$$\operatorname{div}(\operatorname{grad} \phi^*) = \mu_h^*, \qquad \|\operatorname{grad} \phi^*\|_{1,\Omega_0} \leqq C_2 \|\mu_h^*\|_{\Omega_0} = C_2 \|\mu_h\|_{\Omega}.$$

Let us consider the restriction ϕ of ϕ^* to Ω, and put $v = \operatorname{grad} \phi$. Then $v \in \{H^1(\Omega)\}^2 \subset H(\operatorname{div}, \Omega)$, and

$$\operatorname{div} v = \operatorname{div}(\operatorname{grad} \phi) = \mu_h, \qquad \|v\|_{1,\Omega} = \|\operatorname{grad} \phi\|_{1,\Omega} \leqq C_2 \|\mu_h\|_{\Omega}.$$

We now consider $\Pi_h v \in V_h$ for this v, where Π_h is the operator given in Lemma 1. Then

$$(\operatorname{div} \Pi_h v, \mu_h)_{\Omega} = (\operatorname{div} v, \mu_h)_{\Omega} = \|\mu_h\|_{\Omega}^2,$$
$$\|\Pi_h v\|_{H(\operatorname{div},\Omega)} \leqq C_1 \|v\|_{1,\Omega} \leqq C_1 C_2 \|\mu_h\|_{\Omega}.$$

From these two relations, we have, if $\mu_h \neq 0$,

$$(\operatorname{div} \Pi_h v, \mu_h)_{\Omega}/\|\Pi_h v\|_{H(\operatorname{div},\Omega)} \geqq (C_1 C_2)^{-1} \|\mu_h\|_{\Omega}.$$

The desired relation immediately follows from the above, and the proof is complete.

Remark If Ω itself is convex, we can take Ω_0 as Ω in the present proof. This is exactly what Fortin did in [6], and our proof is an extension of his proof.

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