Two Remarks on Harada Conjecture

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0. Introduction

Let G be a finite group and p be a prime number. Let $\{\chi_1, \cdots, \chi_s\}$ be the set of all irreducible complex characters of G. For a subset J of the index set $\{1, \cdots, s\}$, we put $\{\chi_J\} = \{\chi_j \mid j \in J\}$ and $\rho_J = \sum_{j \in J} \chi_j(1)\chi_j$.

In [3], K. Harada stated the following;

CONJECTURE A. If $\rho_J(x)=0$ for any p-singular element x of G, then $\{\chi_J\}$ is a union of p-blocks of G.

He proved that if a Sylow p-subgroup of G is cyclic, then Conjecture A holds. Also, in [5], we showed that if G is p-solvable, then Conjecture A holds. In this note, we prove the following two results.

THEOREM 1. If a Sylow 2-subgroup of G is dihedral, semidihedral or quaternion, then Conjecture A holds for $p=2$.

THEOREM 2. If a Sylow 3-subgroup of G is elementary abelian of order 9, then Conjecture A hold for $p=3$.

As noted in [3], the proof of Conjecture A is reduced to that of the following;

CONJECTURE A'. If $\rho_J(x)=0$ for any p-singular element x of G and $\{\chi_J\} \subseteqq B$ for a p-block B of G, then $\{\chi_J\} = \phi$ or B.

We actually prove the following "block-version" of Theorem 1 and 2.

THEOREM 1'. If a defect group of B is dihedral, semidihedral or quaternion, then Conjecture A′ holds for $p=2$.

THEOREM 2′. If a defect group of B is elementary abelian of order 9, then Conjecture A′ holds for $p=3$.

It is immediate that Theorem 1′ or 2′ together with Theorem A′ in [3] (cyclic defect case) implies Theorem 1 or 2, respectively.

1. Preliminary lemmas

Let B be a p-block of a finite group G. Suppose that B consists of $k(B)$ irreducible complex characters $\chi_1, \cdots, \chi_{k(B)}$ and that $l(B)$ characters $\varphi_1, \cdots, \varphi_{l(B)}$ are all irreducible Brauer characters associated with B. For $x \in G$, we define $\boldsymbol{\chi}_B(x)$ to be the column vector of size $k(B)$ whose i-th component is $\chi_i(x)$. For $1 \leqq m \leqq l(B)$, let $\boldsymbol{d}_m$ be the column of size $k(B)$ whose i-th component d_{im} is the decomposition number of χ_i with respect to φ_m. Then we have

$$\boldsymbol{\chi}_B(x)=\sum_{m=1}^{l(B)} \boldsymbol{d}_m \varphi_m(x) \quad \text{for any } p\text{-regular element } x \text{ of } G.$$

In particular,

$$\boldsymbol{\chi}_B(1)=\sum_{m=1}^{l(B)} \boldsymbol{d}_m \varphi_m(1) .$$

For $J \subseteqq \{1, \cdots, k(B)\}$, let $\boldsymbol{\chi}_J$ be the column of size $k(B)$ whose i-th component c_i is defined as follows; if $i \in J$, then $c_i=\chi_i(1)$ and $c_i=0$ otherwise.

LEMMA 1. The following are equivalent.

(1) $\rho_J(x)=0$ for any p-singular element x of G.
(2) $\boldsymbol{\chi}_J$ is an integral linear combination of $\boldsymbol{d}_m$, $m=1, \cdots, l(B)$.

Proof. (1)⇒(2). Let Φ_m be the principal indecomposable character of G which corresponds to φ_m. Since ρ_J vanishes on all p-singular elements of G, ρ_J is an integral linear combination of Φ_m, $m=1, \cdots, l(B)$;

$$\rho_J=\sum_m \alpha_m \Phi_m=\sum_m \alpha_m \sum_i d_{im}\chi_i=\sum_i \Big(\sum_m \alpha_m d_{im}\Big)\chi_i . \tag{1.1}$$

By the linear independence of $\{\chi_i\}$, we obtain

$$\boldsymbol{\chi}_J=\sum_m \alpha_m \boldsymbol{d}_m , \tag{1.2}$$

as desired.

(2)⇒(1). Suppose that (1.2) holds, then by using equation (1.1) again we have $\rho_J=\sum_m \alpha_m \Phi_m$. So (1) holds. This completes the proof.

In the following three lemmas we will give some conditions on blocks under which Conjecture A′ holds.

LEMMA 2. If $l(B)=1$, then Conjecture A′ holds.

Proof. Suppose that ρ_J satisfies the condition of Conjecture A′. Then by Lemma 1 we have $\boldsymbol{\chi}_J=\alpha_1\boldsymbol{d}_1$ for some integer α_1. Comparing the components, we get $\alpha_1=0$ or $\varphi_1(1)$. So $\boldsymbol{\chi}_J=0$ or $\boldsymbol{\chi}_B(1)$, as desired.

LEMMA 3. If $k(B)-l(B)=1$, then Conjecture A′ holds.

Proof. For simplicity, we set $k=k(B)$ and $l=l(B)$. Since $k-l=1$, every non-identity element in a defect group of B is conjugate to fixed one, say x, and $C_G(x)$ has the unique block b_x with $b_x^G=B$. Furthermore, $l(b_x)=1$ and so B has the unique column $\boldsymbol{d}^x={}^t(d_1^x, \cdots, d_k^x)$ of higher decomposition numbers. Note that every d_i^x is an integer.

We may assume that χ_k is of height 0. For $1\leqq i\leqq l$, we put $\tau_i=d_k^x\chi_i-d_i^x\chi_k$. Because τ_i vanishes on all p-singular elements of G, τ_i is an integral linear combination of principal indecomposable characters of G in B. Then we have

$$(1.3)\qquad d_k^x x_i-d_i^x x_k\equiv 0 \pmod{p^a} \quad (i=1, \cdots, 1),$$

where $x_i=\chi_i(1)$ and $p^a=|G|_p$.

Now suppose that ρ_J satisfies the condition of Conjecture A′. By Lemma 1, $\boldsymbol{\chi}_J$ is orthogonal to $\boldsymbol{d}^x$;

$$\sum_{j\in J} d_j^x x_j=0\,.$$

From (1.3), we have

$$\sum_{j\in J} (d_j^x)^2 x_k\equiv 0 \pmod{p^a}\,.$$

As χ_k is of height 0, we get

$$\sum_{j\in J} (d_j^x)^2\equiv 0 \pmod{p^d}\,,$$

where d is the defect of B. $\sum_{i=1}^{k} (d_i^x)^2=p^d$ and $d_i^x\neq 0$ for all i, we have $J=\phi$ or $\{1, \cdots, k\}$. Hence Conjecture A′ holds.

LEMMA 4. Suppose that $l(B)=2$ and that B has a basic set consisting of two irreducible complex characters of height 0, Then Conjecture A′ holds for B.

Proof. We may assume that χ_1 and χ_2 have height 0 and $\{\chi_1, \chi_2\}$ forms a basic set for B. Let D be the decomposition matrix for B with respect to this basic set. Suppose that D is decomposable, i.e. by suitable rearranging the rows, D has the form

$$\begin{array}{c} \\ \chi_{\pi(1)} \\ \vdots \\ \chi_{\pi(t)} \\ \chi_{\pi(t+1)} \\ \vdots \\ \chi_{\pi(k)} \end{array} \begin{array}{c} \begin{array}{cc} \chi_1 & \chi_2 \end{array} \\ \left(\begin{array}{c|c} a_1 & \\ \vdots & 0 \\ a_t & \\ \hline & b_{t+1} \\ 0 & \vdots \\ & b_k \end{array}\right) \end{array},$$

where π is a permutation on $\{1, \cdots, k\}$. Then the Cartan matrix C for B has the form $C=\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ where $c_1=\sum a_i^2$, $c_2=\sum b_j^2$. Since det C is a power of p, c_1 and c_2 are also powers of p. And so $\{c_1, c_2\}$ are the elementary divisors of C. By [2] p157 Theorem 4.16, we may assume that $c_1<p^d$ and $c_2=p^d$ where d is the defect of B. As $\Phi=\sum_{i=1}^{t} a_i\chi_{\pi(i)}$ is an integral linear combination of principal indecomposable characters of G in B, we have $\Phi(1)\equiv 0 \pmod{|G|_p}$. On the other hand,

$$\Phi(1)=\sum_{i=1}^{t} a_i\chi_{\pi(i)}(1)=\sum_{i=1}^{t} a_i^2\chi_1(1)=c_1\chi_1(1).$$

Since χ_1 is of height 0, we get $c_1\equiv 0 \pmod{p^d}$ but this is a contradiction.

Hence D is indecomposable, and so we may assume that D has the form

$$\begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cc} \chi_1 & \chi_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u & v \\ \vdots & \vdots \end{pmatrix} \end{array},$$

where $u\neq 0$ and $v\neq 0$. As before we set $x_i=\chi_i(1)$. Then we have

$$x_3=ux_1+vx_2. \tag{1.4}$$

Let $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ be the first and second column vectors of D.

Now suppose that ρ_J satisfies the condition of Conjecture A′. By Lemma 1,

$$\boldsymbol{\chi}_J=\boldsymbol{d}_1\alpha+\boldsymbol{d}_2\beta \quad \text{for some integer } \alpha,\ \beta. \tag{1.5}$$

Comparing the first and second components of (1.5), we have $\alpha=0$ or x_1, $\beta=0$ or x_2. Interchanging $\boldsymbol{\chi}_J$ and $\boldsymbol{\chi}_B(1)-\boldsymbol{\chi}_J$ if necessary, we may assume that $\alpha=0$. If $\beta=x_2$ holds, then we get $x_3=vx_2$ by comparing the third component of (1.5). Since $u\neq 0$, this contradicts (1.4). Therefore $\beta=0$ and so $\boldsymbol{\chi}_J=0$ as desired. This completes the proof.

2. Proof of Theorem 1′

At first suppose that B is a 2-block of G with dihedral defect group of order 2^n. The structure of such a block has been determined in Brauer [1]. By [1] Theorem 1 and 5, we have the following.

B consists of $2^{n-2}+3$ irreducible complex characters; $\chi_1, \chi_2, \chi_3, \chi_4, \chi^{(j)}$ ($j=1, \cdots, 2^{n-2}-1$). All $\chi^{(j)}$ have the same degree x^*. And if we put $x_i=\chi_i(1)$, then we have

$$\delta_1 x_1+\delta_2 x_2=-\delta_3 x_3-\delta_4 x_4=x^*, \tag{2.1}$$

where $\delta_i=\pm 1$.

Now suppose that B and ρ_J satisfy the condition of Conjecture A′ for $p=2$. We need to show that $\{\chi_J\}=\phi$ or B.

By Lemma 1, we see that $\boldsymbol{\chi}_J={}^t(c_1, c_2, c_3, c_4, c^{(1)}, \cdots, c^{(2^{n-2}-1)})$ is orthogonal to all columns of higher decomposition numbers for B. So we can apply Proposition (6G) in [1] for $\boldsymbol{\chi}_J$ to get

$$\delta_1 c_1+\delta_2 c_2=-\delta_3 c_3-\delta_4 c_4=c^{(1)}=\cdots=c^{(2^{n-2}-1)}. \tag{2.2}$$

Interchanging $\boldsymbol{\chi}_J$ and $\boldsymbol{\chi}_B(1)-\boldsymbol{\chi}_J$ if necessary, we may assume that $c^{(1)}=\cdots=c^{(2^{n-2}-1)}=0$. If $c_1=x_1$ holds, then $c_2=x_2$ by (2.2). But this contradicts (2.1). Thus $c_1=c_2=0$, and similarly $c_3=c_4=0$. Therefore $\boldsymbol{\chi}_J=0$, and so Conjecture A′ holds for B.

When B has a quaternion or semidihedral defect group, we can use the results of Olsson [6]. We omit the proof since it is similar to that of the dihedral case.

3. Proof of Theorem 2′

Suppose that B is a 3-block of G with an elementary abelien defect group of order 9. The author studied the structure of such a block in [4]. The proof in this section depends heavily on the results in [4], and therefore it is necessary to expect the reader to have some familiarity with [4]. In particular we use the notations and results in Section 2 of [4] freely.

Suppose that B and ρ_J satisfy the condition of Conjecture A′ for $p=3$. We want to show $\{\chi_J\}=\phi$ or B.

In [4], in order to determine $k(B)$ and $l(B)$ we divide into eleven cases: B is of type 1, $\boldsymbol{Z}_2$ (two cases), $\boldsymbol{Z}_4$, $E_4(a)$, $E_4(b)$, $\boldsymbol{Z}_8$, $D_8(a)$, $D_8(b)$, Q_8 or SD_{16} (see Table 1 in Section 0 of [4]).

If B is of type 1 or $E_4(b)$, then we have $l(B)=1$ and so the result follows from Lemma 2.

If B is of type $\boldsymbol{Z}_8$ or Q_8, then we have $k(B)-l(B)=1$ by the same arguments as in the proof of $(2H)$ in [4]. Thus, we get the result by using Lemma 3.

Therefore it remains seven cases, namely, type $\boldsymbol{Z}_2$ (two cases), $\boldsymbol{Z}_4$, $E_4(a)$, $D_8(a)$, $D_8(b)$ or SD_{16}. In this seven cases, we will determine the decomposition matrix for B to get the result. Let $B=\{\chi_1, \cdots, \chi_{k(B)}\}$ and set $x_i=\chi_i(1)$, $3^a=|G|_3$. Note that every χ_i has height 0 by [2] p158 Theorem 4.18.

1° B is of type $\boldsymbol{Z}_4$.

We have $k(B)=6$, $l(B)=4$ in this case. B has two columns $\boldsymbol{d}^x$, $\boldsymbol{d}^{xy}$ of higher decomposition numbers which satisfy the conditions that

$$(\boldsymbol{d}^x, \boldsymbol{d}^x)=(\boldsymbol{d}^{xy}, \boldsymbol{d}^{xy})=9\,, \qquad (\boldsymbol{d}^x, \boldsymbol{d}^{xy})=0\,,$$
$$\boldsymbol{d}^x \text{ and } \boldsymbol{d}^{xy} \text{ are integral columns,}$$

where (,) denotes the usual Hermitian inner product (see the proof of $(2D)$ in [4]).

Then we have the following two possibilities for $\boldsymbol{d}^x$, $\boldsymbol{d}^{xy}$;

$$(3.1) \qquad \begin{cases} \boldsymbol{d}^x = {}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, -2\varepsilon_6) \\ \boldsymbol{d}^{xy} = {}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, -2\varepsilon_5, \varepsilon_6) \end{cases} \text{ or } \begin{cases} \boldsymbol{d}^x = {}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, -2\varepsilon_6) \\ \boldsymbol{d}^{xy} = {}^t(\varepsilon_1, \varepsilon_2, -\varepsilon_3, -\varepsilon_4, 2\varepsilon_5, \varepsilon_6) \end{cases},$$

where $\varepsilon_i=\pm 1$.

Assume that the latter case of (3.1) occurs. We will derive a contradiction from this assumption. Let G_0 be the set of all 3-regular elements of G. For $s \in G_0$, we get

$$(\boldsymbol{\chi}_B(s), \boldsymbol{d}^x)=(\boldsymbol{\chi}_B(s), \boldsymbol{d}^{xy})=0\,,$$

by the orthogonality relations. So we obtain that the following relations hold on G_0;

$$\varepsilon_1\chi_1+\varepsilon_2\chi_2+\varepsilon_3\chi_3+\varepsilon_4\chi_4+\varepsilon_5\chi_5-2\varepsilon_6\chi_6=0\,,$$
$$\varepsilon_1\chi_1+\varepsilon_2\chi_2-\varepsilon_3\chi_3-\varepsilon_4\chi_4+2\varepsilon_5\chi_5+\varepsilon_6\chi_6=0\,.$$

The decomposision matrix D and the Cartan matrix C for B with respect to the basic set $\{\varepsilon_1\chi_1, \varepsilon_3\chi_3, \varepsilon_4\chi_4, \varepsilon_6\chi_6\}$ are as follows;

$$D=\begin{pmatrix} \varepsilon_1 & & & \\ -\varepsilon_2 & -3\varepsilon_2 & -3\varepsilon_2 & 5\varepsilon_2 \\ & \varepsilon_3 & & \\ & & \varepsilon_4 & \\ & 2\varepsilon_5 & 2\varepsilon_5 & -3\varepsilon_5 \\ & & & \varepsilon_6 \end{pmatrix}, \qquad C=\begin{pmatrix} 2 & 3 & 3 & -5 \\ 3 & 14 & 13 & -21 \\ 3 & 13 & 14 & -21 \\ -5 & -21 & -21 & 35 \end{pmatrix}.$$

So the elementary divisors of C is $\{1, 1, 9, 9\}$, but this contradicts the well known property of C ([2] p157 Theorem 4.16).

Therefore the former case of (3.1) occurs. As $(\boldsymbol{\chi}_B(1), \boldsymbol{d}^x-\boldsymbol{d}^{xy})=0$, we have

$\varepsilon_5=\varepsilon_6$. By the orthogonality relations, we get that the following relations hold on G_0;

$$\varepsilon_1\chi_1+\varepsilon_2\chi_2+\varepsilon_3\chi_3+\varepsilon_4\chi_5=\varepsilon_5\chi_5\,,\qquad \chi_5=\chi_6\,.$$

The decomposition matrix for B with respect to the basic set $\{\varepsilon_1\chi_1, \varepsilon_2\chi_2, \varepsilon_3\chi_3, \varepsilon_4\chi_4\}$ is as follows;

$$\begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \varepsilon_3 & \\ & & & \varepsilon_4 \\ \varepsilon_5 & \varepsilon_5 & \varepsilon_5 & \varepsilon_5 \\ \varepsilon_5 & \varepsilon_5 & \varepsilon_5 & \varepsilon_5 \end{pmatrix}.$$

Since $\Phi_i=\varepsilon_i\chi_i+\varepsilon_5\chi_5+\varepsilon_5\chi_6$ $(i=1, 2, 3, 4)$ is an integral linear combination of principal indecomposable characters of G in B, we have $\varepsilon_i x_i+2\varepsilon_5 x_5\equiv 0 \pmod{3^a}$. So,

(3.2) $$\varepsilon_1 x_1\equiv\varepsilon_2 x_2\equiv\varepsilon_3 x_3\equiv\varepsilon_4 x_4 \pmod{3^a}\,.$$

By Lemma 1, $\boldsymbol{\chi}_J={}^t(c_1, \cdots, c_6)$ is orthogonal to $\boldsymbol{d}^x$ and $\boldsymbol{d}^{xy}$, and so we have

$$\varepsilon_1 c_1+\varepsilon_2 c_2+\varepsilon_3 c_3+\varepsilon_4 c_4=\varepsilon_5 c_5\,,\qquad c_5=c_6\,.$$

As before we may assume that $c_5=c_6=0$. Thus,

(3.3) $$\sum_{i=1}^{4}\varepsilon_i c_i=0\,.$$

Because $3^{a-2}\|\varepsilon_i x_i$ and $c_i=0$ or x_i, it follows from (3.2) and (3.3) that $c_i=0$ for $1\leqq i\leqq 4$. Hence $\boldsymbol{\chi}_J=0$, as desired.

2° B is of type $D_8(a)$.

By [4] (2F), we have $k(B)=9$, $l(B)=5$. B has four columns $\boldsymbol{d}_1^x$, $\boldsymbol{d}_2^x$, $\boldsymbol{d}_1^y$, $\boldsymbol{d}_2^y$ of higher decomposition numbers which satisfy the following conditions (see the proof of (2E) in [4]);

(3.4) $$\begin{cases} (\boldsymbol{d}_1^x, \boldsymbol{d}_1^x)=(\boldsymbol{d}_2^x, \boldsymbol{d}_2^x)=6\,, & (\boldsymbol{d}_1^x, \boldsymbol{d}_2^x)=3\,, \\ (\boldsymbol{d}_1^y, \boldsymbol{d}_1^y)=(\boldsymbol{d}_2^y, \boldsymbol{d}_2^y)=6\,, & (\boldsymbol{d}_1^y, \boldsymbol{d}_2^y)=3\,, \\ (\boldsymbol{d}_i^x, \boldsymbol{d}_j^y)=0 \quad \text{for any } i, j\in\{1, 2\}, & \\ \boldsymbol{d}_1^x,\ \boldsymbol{d}_2^x,\ \boldsymbol{d}_1^y \text{ and } \boldsymbol{d}_2^y \text{ are integral columns.} & \end{cases}$$

From (3.4), we have the following solutions for $\boldsymbol{d}_1^x$, $\boldsymbol{p}_2^x$, $\boldsymbol{d}_1^y$, $\boldsymbol{d}_2^y$. (Interchange $\{\boldsymbol{d}_1^x, \boldsymbol{d}_2^x\}$ and $\{\boldsymbol{d}_1^y, \boldsymbol{d}_2^y\}$ if necessary.)

$$(3.5)\qquad \begin{cases} \boldsymbol{d}_1^x={}^t(\varepsilon_1, & \varepsilon_2, & \varepsilon_3, & \varepsilon_4, & \varepsilon_5, & \varepsilon_6, & 0, & 0, & 0) \\ \boldsymbol{d}_2^x={}^t(0, & 0, & 0, & \varepsilon_4, & \varepsilon_5, & \varepsilon_6, & \varepsilon_7, & \varepsilon_8, & \varepsilon_9) \\ \boldsymbol{d}_1^y={}^t(\lambda\varepsilon_1, & -\lambda\varepsilon_2, & 0, & \mu\varepsilon_4, & -\mu\varepsilon_5, & 0, & \nu\varepsilon_7, & -\nu\varepsilon_8, & 0) \\ \boldsymbol{d}_2^y={}^t(\lambda\varepsilon_1, & 0, & -\lambda\varepsilon_3, & \mu\varepsilon_4, & 0, & -\mu\varepsilon_6, & \nu\varepsilon_7, & 0, & -\nu\varepsilon_9), \end{cases}$$

where $\varepsilon_i, \lambda, \mu, \nu=\pm 1$.

We may assume that $\mu=1$. Then we get $(\lambda, \nu)=(-1, -1)$, because otherwise we have a contradiction by computing the elementary divisors of the Cartan matrix. The decomposition matrix for B with respect to the basic set $\{\varepsilon_1\chi_1, \varepsilon_2\chi_2, \varepsilon_3\chi_3, \varepsilon_4\chi_4, \varepsilon_5\chi_5\}$ is as follows;

$$\begin{pmatrix} \varepsilon_1 & & & & \\ & \varepsilon_2 & & & \\ & & \varepsilon_3 & & \\ & & & \varepsilon_4 & \\ & & & & \varepsilon_5 \\ -\varepsilon_6 & -\varepsilon_6 & -\varepsilon_6 & -\varepsilon_6 & -\varepsilon_6 \\ & \varepsilon_7 & \varepsilon_7 & \varepsilon_7 & \\ \varepsilon_8 & & \varepsilon_8 & & \varepsilon_8 \\ & & -\varepsilon_9 & -\varepsilon_9 & -\varepsilon_9 \end{pmatrix}.$$

Using the same arguments as in 1°, we have

$$(3.6)\qquad \begin{aligned} \varepsilon_1x_1-\varepsilon_6x_6+\varepsilon_8x_8 &\equiv \varepsilon_2x_2-\varepsilon_6x_6+\varepsilon_7x_7\equiv\varepsilon_3x_3-\varepsilon_4x_4+\varepsilon_8x_8 \\ &\equiv \varepsilon_2x_2-\varepsilon_4x_4+\varepsilon_9x_9\equiv\varepsilon_3x_3-\varepsilon_5x_5+\varepsilon_7x_7\equiv 0 \pmod{3^a}. \end{aligned}$$

Put $y_i=\varepsilon_ix_i/3^{a-2}$. Then $y_i\not\equiv 0 \pmod 3$. By (3.6), we get

$$(3.7)\qquad \begin{cases} y_1+y_8\equiv y_6, \quad y_2+y_7\equiv y_6, \quad y_3+y_8\equiv y_4, \\ y_2+y_9\equiv y_4, \quad y_3+y_7\equiv y_5, \quad y_1+y_9\equiv y_5 \pmod 9. \end{cases}$$

From (3.7), we have

$$(3.8)\qquad y_1\equiv y_2\equiv y_3\equiv -y_4\equiv -y_5\equiv -y_6\equiv y_7\equiv y_8\equiv y_9\not\equiv 0 \pmod 3.$$

Now let $\boldsymbol{\chi}_J={}^t(c_1, \cdots, c_9)$ and $d_i=\varepsilon_ic_i/3^{a-2}$. So $d_i=0$ or y_i, and it follows from the orthogonality relations that

$$(3.9)\qquad \begin{cases} d_1+d_2+d_3=-(d_4+d_5+d_6)=d_7+d_8+d_9, \\ d_1-d_4+d_7=d_2-d_5+d_8=d_3-d_6+d_9. \end{cases}$$

The six terms in (3.9) are all equal, so we denote by d the common value. Suppose that $d\equiv 0 \pmod 3$. Then using (3.8) we see that three d_i's in each term of equations (3.9) are simultaneously zero or non-zero. Therefore we may assume that $d_1=d_2=d_3=0$, and so we have $d_4=d_7=d_5=d_8=d_6=d_9=0$ by using the second equation of (3.9). Hence $\chi_J=0$, the desired result.

Next suppose that $d\not\equiv 0 \pmod 3$. We may assume that $d\equiv y_1 \pmod 3$. Then we conclude that exactly two of three d_i's in each term of equations (3.9) are zero. So,

$$(d_1, d_2, d_3)=(y_1, 0, 0),\ (0, y_2, 0) \text{ or } (0, 0, y_3).$$

Assume that the first case occurs, then the non-zero d_i's in (3.9) are d_1, d_5, d_9 or d_1, d_6, d_8. If d_1, d_5 and d_9 are non-zero, then $y_1=-y_5=y_9$ and so $y_1\equiv 0$ (mod 3) by (3.7). The same conclusion holds if d_1, d_6 and d_8 are non-zero. But this is a contradiction. In the second and third case, we have a contradiction by the same argument. Hence $d\not\equiv 0 \pmod 3$ does not occur and so the proof in case B is of type $D_8(a)$ is complete.

3° B is of type $E_4(a)$

By [4] (2E), we have $k(B)=9$, $l(B)=4$. B has five columns $\boldsymbol{d}_1^x$, $\boldsymbol{d}_2^x$, $\boldsymbol{d}_1^y$, $\boldsymbol{d}_2^y$, $\boldsymbol{d}^{xy}$ of higher decomposition numbers which satisfy the conditions that

$$(3.10)\quad \begin{cases} \boldsymbol{d}_1^x,\ \boldsymbol{d}_2^x,\ \boldsymbol{d}_1^y \text{ and } \boldsymbol{d}_2^y \text{ satisfy (3.4)}, \\ (\boldsymbol{d}^{xy}, \boldsymbol{d}^{xy})=9, \text{ and } \boldsymbol{d}^{xy} \text{ is an integral column which is orthogonal to the} \\ \quad \text{above four columns.} \end{cases}$$

For $\boldsymbol{d}_1^x$, $\boldsymbol{d}_2^x$, $\boldsymbol{d}_1^y$, $\boldsymbol{d}_2^y$, we have the solution (3.5). Thus, we determine $\boldsymbol{d}^{xy}$ only. As in 2° we may assume that $\mu=1$ in (3.5). Since λ and ν in (3.5) are signs, we have four possibilities for (λ, ν).

(a) The case $(\lambda, \nu)=(1, 1)$.

In this case $\boldsymbol{d}^{xy}$ is uniquely determined (up to sign) as follows;

$$\boldsymbol{d}^{xy}=\pm{}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4, -\varepsilon_5, -\varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9).$$

Since $\chi_B(1)$ is orthogonal to $\boldsymbol{d}_1^x$, $\boldsymbol{d}_2^x$, $\boldsymbol{d}_1^y$, $\boldsymbol{d}_2^y$, $\boldsymbol{d}^{xy}$, we have

$$(3.11)\quad \begin{cases} \varepsilon_1x_1+\varepsilon_2x_2+\varepsilon_3x_3=\varepsilon_4x_4+\varepsilon_5x_5+\varepsilon_6x_6=\varepsilon_7x_7+\varepsilon_8x_8+\varepsilon_9x_9=0, \\ \varepsilon_1x_1+\varepsilon_4x_4+\varepsilon_7x_7=\varepsilon_2x_2+\varepsilon_5x_5+\varepsilon_8x_8=\varepsilon_3x_3+\varepsilon_6x_6+\varepsilon_9x_9(=0). \end{cases}$$

Let $\chi_J={}^t(c_1, \cdots, c_9)$, then similarly we have

$$(3.12)\quad \begin{cases} \varepsilon_1c_1+\varepsilon_2c_2+\varepsilon_3c_3=\varepsilon_4c_4+\varepsilon_5c_5+\varepsilon_6c_6=\varepsilon_7c_7+\varepsilon_8c_8+\varepsilon_9c_9=0, \\ \varepsilon_1c_1+\varepsilon_4c_4+\varepsilon_7c_7=\varepsilon_2c_2+\varepsilon_5c_5+\varepsilon_8c_8=\varepsilon_3c_3+\varepsilon_6c_6+\varepsilon_9c_9(=0). \end{cases}$$

We may assume that $c_1=0$, and so $\varepsilon_2c_2+\varepsilon_3c_3=0$ by (3.12). Using (3.11) $c_2=c_3=0$ and so all c_i are zero. Therefore $\chi_J=0$, as desired.

(b) The case $(\lambda, \nu)=(1, -1)$ or $(-1, 1)$.

As in (a), $\boldsymbol{d}^{xy}$ is uniquely determined (up to sign);

$$d^{xy} = \pm {}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4, -\varepsilon_5, -\varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9,) .$$

The remaining proof is similar to that of (a).

(c) The case $(\lambda, \nu) = (-1, -1)$.

In this case, we have the following three possibilities for d^{xy};

$$\begin{aligned} d^{xy} = & \pm {}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4, -\varepsilon_5, -\varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9) , \\ & \pm {}^t(\varepsilon_1, \varepsilon_2, -\varepsilon_3, \delta\varepsilon_4, -\delta\varepsilon_5, -\varepsilon_6, \delta\varepsilon_7, -\delta\varepsilon_8, \varepsilon_9) , \\ & \pm {}^t(-\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4, \delta\varepsilon_5, -\delta\varepsilon_6, \varepsilon_7, \delta\varepsilon_8, -\delta\varepsilon_9) , \quad (\delta = \pm 1) . \end{aligned}$$

In the first case, the remaining proof is similar to that of (a). In the second and third cases, we get a contradiction by computing the elementary divisors of the Cartan matrix for B.

Hence, the proof in case B is of type $E_4(a)$ is complete.

4° B is of type SD_{16}.

In this case, we have $k(B) - l(B) = 2$. B has two columns d_1^x, d_2^x of higher decomposition numbers which satisfy the following conditions (see the proof of $(2E)$ and $(2G)$ in [4]);

$$(3.13) \qquad (d_1^x, d_1^x) = (d_2^x, d_2^x) = 6 , \qquad (d_1^x, d_2^x) = 3 .$$

We have two possibilities; (a) d_1^x and d_2^x are integral or (b) $\overline{d_1^x} = d_2^x$. We argue these two cases separately.

(a) d_1^x and d_2^x are integral.

By (3.13), d_1^x and d_2^x are determined as follows;

$$\begin{aligned} d_1^x &= {}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, 0, 0, 0) , \\ d_2^x &= {}^t(0, 0, 0, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9) , \quad (\varepsilon_i = \pm 1) . \end{aligned}$$

So $k(B) = 9$, and the decomposition matrix for B with respect to the basic set $\{\varepsilon_1\chi_1, \varepsilon_2\chi_2, \varepsilon_3\chi_3, \varepsilon_4\chi_4, \varepsilon_5\chi_5, \varepsilon_7\chi_7, \varepsilon_8\chi_8\}$ is the following;

$$\begin{pmatrix} \varepsilon_1 & & & & & & \\ & \varepsilon_2 & & & & & \\ & & \varepsilon_3 & & & & \\ & & & \varepsilon_4 & & & \\ & & & & \varepsilon_5 & & \\ \varepsilon_6 & -\varepsilon_6 & -\varepsilon_6 & -\varepsilon_6 & -\varepsilon_6 & & \\ & & & & & \varepsilon_7 & \\ & & & & & & \varepsilon_8 \\ \varepsilon_9 & \varepsilon_9 & \varepsilon_9 & & & -\varepsilon_9 & -\varepsilon_9 \end{pmatrix} .$$

Set $y_i=\varepsilon_i x_i/3^{a-2}$. Then by the same arguments as in 1°, we get

$$y_1\equiv y_2\equiv y_3,\ y_4\equiv y_5\equiv y_6,\ y_7\equiv y_8\equiv y_9,\ y_1+y_7\equiv y_4 \pmod 9\,. \tag{3.14}$$

If $y_1+y_4\equiv 0$ and $y_4+y_7\equiv 0 \pmod 9$ hold, then by (3.14) we have $3y_4\equiv 0 \pmod 9$ but this is absurd. So we may assume that

$$y_1+y_4\not\equiv 0 \pmod 9\,. \tag{3.15}$$

Now let $\chi_J={}^t(c_1,\cdots,c_9)$ and set $d_i=\varepsilon_i c_i/3^{a-2}$. Then $d_i=0$ or y_i, and it follows from the orthogonality relations that

$$\sum_{i=1}^{6} d_i=0\,,\qquad \sum_{i=4}^{9} d_i=0\,. \tag{3.16}$$

We may assume that $d_1=0$. Thus,

$$d_2+d_3+d_4+d_5+d_6=0\,.$$

Using (3.14) and (3.15), we conclude that $d_i=0$ for $1\leqq i\leqq 6$. Now the second equation of (3.16) implies that

$$d_7+d_8+d_9=0\,.$$

Since $y_7\not\equiv 0$ (mod 3), we have $d_7=d_8=d_9=0$. Therefore $\chi_J=\mathbf{0}$, the desired result.

(b) $\overline{\boldsymbol{d}_1^x}=\boldsymbol{d}_2^x$.

In this case, we set $\boldsymbol{d}_1^x=\boldsymbol{a}+\boldsymbol{b}\omega$ where ω is a cubic root of unity and $\boldsymbol{a}$, $\boldsymbol{b}$ are integral columns. From (3.13), $\boldsymbol{a}$ and $\boldsymbol{b}$ are determined as follows (see the proof of $(2E)$ in [4]);

$$\boldsymbol{a}={}^t(0,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6)\,,$$
$$\boldsymbol{b}={}^t(\varepsilon_1,\varepsilon_2,0,0,0,0)\,,\qquad (\varepsilon_i=\pm 1)\,.$$

At this stage the result is easily proved by the same methods as in (a), so we omit the remaining proof.

Remark. A. Watanabe of Kumamoto University proved that $k(B)=9$ holds in case B is of type SD_{16}. Therefore the case (b) of 4° in the above proof can be omitted.

5° B is of type $\boldsymbol{Z}_2$ (fixed point free action) or of type $D_8(b)$.

In this cases, we have $k(B)=6$, $l(B)=2$ ([4] $(2B)$, $(2F)$). By the same arguments as in 1°, we get that the following relations hold on G_0;

$$\chi_1=\chi_2+\varepsilon\chi_3,\ \chi_4=\chi_5=\chi_6=\chi_1\,,$$

where $\varepsilon=\pm 1$. So $\{\chi_2,\chi_3\}$ is a basic set for B. Hence the result follows from Lemma 4.

6° B is of type $\boldsymbol{Z}_2$ (not fixed point free action).

In this case, we have $k(B)=9$, $l(B)=2$ ([4] (2C)). By the same arguments as in 1°, we get that the following relations hold on G_0;

$$\chi_1=\chi_2+\chi_3,\ \chi_4=\chi_5=\chi_1,\ \chi_6=\chi_7=\chi_2,\ \chi_8=\chi_9=\chi_3\ .$$

So $\{\chi_2, \chi_3\}$ is a basic set for B. Hence the result follows from Lemma 4.

Thus, we have shown that $\chi_J=0$ or $\chi_B(1)$ for B of all types. This completes the proof of Theorem 2′ and so Theorem 2 also holds.

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