Relative Cohomology on a Countably Infinite Dimentional Space

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Introduction

Various investigations in complex analysis in locally convex topological vector spaces have been made by many authors (see for example [2], [12]).

We have been concerned with the case of a countably infinite dimensional topological vector space, which is denoted by $\sum \boldsymbol{C}$ ([3], [4], [5]). It was shown that a result similar to Oka-Cartan's theorem B is valid for our space $\sum \boldsymbol{C}$ ([5]). On this basis we will now investigate the relative cohomology groups toward the hyperfunction theory on infinite dimensional spaces.

In this paper we will show the vanishing of the relative cohomology groups corresponding to those in the theory of hyperfunctions and analytic functionals with compact carrier in the case of finite dimensions. As an application this result gives a negative answer to the following problem: If an open set U in $\sum \boldsymbol{C}$ satisfies the condition that $H^k(U, \mathcal{O})=0$ for every $k \geqq 1$, then is it pseudo-convex? In fact, we can show that there exists an open set U which is not pseudo-convex but satisfies that $H^k(U, \mathcal{O})=0$ for every $k \geqq 1$.

In §0 we review the space $\sum \boldsymbol{C}$ and a result used later. In §1 we will give a flabby resolution of the sheaf $\mathcal{O}$ of germs of holomorphic functions on $\sum \boldsymbol{C}$, which is not the canonical flabby resolution. By using this resolution, we will show the vanishing of the relative cohomology groups with support in $\sum \boldsymbol{R}$. Let K be a compact set in $\sum \boldsymbol{C}$ such that $K \subset \boldsymbol{C}^m$ satisfying that $H^k(K, \mathcal{O}_m)=0$ for every $k \geqq 1$. In §2 it is shown that the relative cohomology groups with support in K vanish and that the space of analytic functionals with compact carrier is represented in terms of relative cohomology groups on finite dimensional spaces. As a corollary we prove the existence of open sets which are not pseudo-convex but whose cohomology groups vanish. In §3 we will introduce the hyperfunctions with an infinite number of holomorphic parameters by means of relative cohomology groups.

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§ 0. Preliminaries

Hereafter we refer to [6, 8, 9] for the general results on the theory of holomorphic functions of several variables, to [10, 13] for the results on hyperfunctions and to [2, 12] for the general theory of holomorphic functions on infinite dimensional topological vector spaces.

Now we introduce the space $\sum \boldsymbol{C}$ and state the vanishing of cohomology groups of a pseudo-convex open set in $\sum \boldsymbol{C}$.

We denote by $\sum \boldsymbol{C}$ the direct sum of complex planes $\boldsymbol{C}$ endowed with the inductive limit topology of the spaces $\{\boldsymbol{C}^n; u_{n+1}^n\}$ were $u_{n+1}^n: \boldsymbol{C}^n \to \boldsymbol{C}^{n+1}$ is defined by $u_{n+1}^n((z_1, \cdots, z_n)) = (z_1, \cdots, z_n, 0)$. For its topological properties see [3]. We denote by $\mathcal{O}$ the sheaf of germs of holomorphic functions on $\sum \boldsymbol{C}$. Then we have the following theorem ([1], [3]):

THEOREM 0.1. *Let U be a pseudo-convex open set in $\sum \boldsymbol{C}$. Then we have*

$$H^k(U, \mathcal{O}) = 0 \qquad \textit{for } k \geqq 1 .$$

We remark that this result was extended to the case of analytic subvarieties of a pseudo-convex open set in $\sum \boldsymbol{C}$ and further to the case of Stein manifolds in the sense of $\sum \boldsymbol{C}$ (for the detail see [4], [5]).

§ 1. Vanishing of the relative cohomology with support in $\sum \boldsymbol{R}$.

In this section we will investigate the relative cohomology groups with support in $\sum \boldsymbol{R}$. First we introduce a flabby resolution of the sheaf $\mathcal{O}$ of germs of holomorphic functions on $\sum \boldsymbol{C}$. We denote the canonical flabby resolution of the sheaf $\mathcal{O}_n$ of germs of holomorphic functions on $\boldsymbol{C}^n$ as follows:

$$0 \longrightarrow \mathcal{O}_n \longrightarrow \mathcal{F}_n^0 \longrightarrow \mathcal{F}_n^1 \longrightarrow \mathcal{F}_n^2 \longrightarrow \cdots .$$

Hereafter we use the notation:

$$U_n = U \cap \boldsymbol{C}^n .$$

Let us recall the following isomorphism ([3]):

$$\mathcal{O}(U) \cong \varprojlim_n \mathcal{O}_n(U_n) , \tag{1.0}$$

where the projective limit is taken with respect to the restriction mappings. The restriction mapping of $\mathcal{F}_{n+1}^k(U_{n+1})$ to $\mathcal{F}_n^k(U_n)$ is well defined and denoted by

$r_n^{n+1}(k)$. Now we define $\mathscr{F}_\infty^k(U)$ as follows:

$$\mathscr{F}_\infty^k(U)=\varprojlim_n \{\mathscr{F}_n^k(U)\,;\ r_n^{n+1}(k)\}\,.$$

The presheaf $\{\mathscr{F}_\infty^k(U)\}$ defines a sheaf, which is denoted by $\mathscr{F}_\infty^k$. We denote by $r_n(k)$ the restriction mapping of $\mathscr{F}_\infty^k(U)$ to $\mathscr{F}_n^k(U_n)$. Let us recall the well known result (see for example Theorem 8 in Chap. IV, D of [8]).

LEMMA 1.1. *Let U be a pseudo-convex open set in $\boldsymbol{C}^{n+1}$. Then the restriction mapping of $\mathcal{O}_{n+1}(U)$ to $\mathcal{O}_n(U\cap \boldsymbol{C}^n)$ is surjective.*

Now we have the following

LEMMA 1.2. *Let U be an open set in $\boldsymbol{C}^{n+1}$ and let S be a locally closed subset of U. Then the restriction mapping*

$$r_n^{n+1}(k): \Gamma_S(U, \mathscr{F}_{n+1}^k) \longrightarrow \Gamma_{S\cap \boldsymbol{C}^n}(U\cap \boldsymbol{C}^n, \mathscr{F}_n^k)$$

is surjective.

Proof 1. It is obvious that the restriction mapping of the stalk $\mathcal{O}_{n+1,z}$ to the stalk $\mathcal{O}_{n,z}$ is surjective. Then in view of the definition of $\mathscr{F}_n^0$ the mapping $r_n^{n+1}(0)$ is surjective.

2. Let the sheaf $\mathscr{Z}_n^0$ be denoted by the sequence:

$$0 \longrightarrow \mathcal{O}_n \longrightarrow \mathscr{F}_n^0 \longrightarrow \mathscr{Z}_n^0 \longrightarrow 0\,. \tag{1.1}$$

Now we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{n+1}(V) & \longrightarrow & \mathscr{F}_{n+1}^0(V) & \longrightarrow & \mathscr{Z}_{n+1}^0(V) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_n(V\cap \boldsymbol{C}^n) & \longrightarrow & \mathscr{F}_n^0(V\cap \boldsymbol{C}^n) & \longrightarrow & \mathscr{Z}_n^0(V\cap \boldsymbol{C}^n) & \longrightarrow & 0
\end{array}$$

If V is pseudo-convex, both rows are exact. It is shown that the second column is exact. Therefore the restriction mapping of $\mathscr{Z}_{n+1}^0(V)$ to $\mathscr{Z}_n^0(V\cap \boldsymbol{C}^n)$ is surjective. Hence, the induced mapping of the stalk $\mathscr{Z}_{n+1,z}^0$ to the stalk $\mathscr{Z}_{n,z}^0$ is surjective. In view of the definition of the sheaf $\mathscr{F}_n^1$, the restriction mapping $r_n^{n+1}(1)$ of $\Gamma_S(U, \mathscr{F}_{n+1}^1)$ to $\Gamma_{S\cap \boldsymbol{C}^n}(U\cap \boldsymbol{C}^n, \mathscr{F}_n^1)$ is surjective for an arbitrary open set U in $\boldsymbol{C}^{n+1}$. Let the sheaf $\mathscr{Z}_n^i$ be defined by the following exact sequence:

$$0 \longrightarrow \mathscr{Z}_n^{i-1} \longrightarrow \mathscr{F}_n^i \longrightarrow \mathscr{Z}_n^i \longrightarrow 0\,. \tag{1.2}$$

We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{Z}^0_{n+1}(V) & \longrightarrow & \mathcal{F}^1_{n+1}(V) & \longrightarrow & \mathcal{Z}^1_{n+1}(V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{Z}^0_n(V\cap \boldsymbol{C}^n) & \longrightarrow & \mathcal{F}^1_n(V\cap \boldsymbol{C}^n) & \longrightarrow & \mathcal{Z}^1_n(V\cap \boldsymbol{C}^n) & \longrightarrow & 0
\end{array}$$

Let V be a pseudo-convex open set in $\boldsymbol{C}^{n+1}$. Then, taking account of the exactness of both rows of the diagram, the restriction mapping of $\mathcal{Z}^1_{n+1}(V)$ to $\mathcal{Z}^1_n(V\cap \boldsymbol{C}^n)$ is surjective. Therefore, $r^{n+1}_n(2)$ is surjective. Consequently, we can show inductively that the restriction mapping $r^{n+1}_n(k)$ of $\Gamma_S(U, \mathcal{F}^k_{n+1})$ to $\Gamma_{S\cap \boldsymbol{C}^n}(U\cap \boldsymbol{C}^n, \mathcal{F}^k_n)$ is surjective in the similar way. [Q. E. D.]

Corollary 1.3. *Let U be an open set in $\sum \boldsymbol{C}$ and let S be an arbitrary subset of U. Then the restriction mapping*

$$r_n(k): \Gamma_S(U, \mathcal{F}^k_\infty) \longrightarrow \Gamma_{S\cap \boldsymbol{C}^n}(U_n, \mathcal{F}^k_n)$$

is surjective.

We prepare another lemma.

Lemma 1.4. *The sheaf $\mathcal{F}^k_\infty$ is flabby.*

Proof. Let U be an arbitrary open set in $\sum \boldsymbol{C}$. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_{\boldsymbol{C}^{n+1}\setminus U_{n+1}}(\boldsymbol{C}^{n+1}, \mathcal{F}^k_{n+1}) & \longrightarrow & \mathcal{F}^k_{n+1}(\boldsymbol{C}^{n+1}) & \longrightarrow & \mathcal{F}^k_{n+1}(U_{n+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{\boldsymbol{C}^n\setminus U_n}(\boldsymbol{C}^n, \mathcal{F}^k_n) & \longrightarrow & \mathcal{F}^k_n(\boldsymbol{C}^n) & \longrightarrow & \mathcal{F}^k_n(U_n) & \longrightarrow & 0
\end{array}$$

The flabbiness of $\mathcal{F}^k_n$ implies that the rows are exact. Owing to Lemma 1.2, the columns are exact. It follows from Proposition 13.2.2 in [7] (Mittag-Leffler's lemma) that $\mathcal{F}^k_\infty$ is flabby. [Q. E. D.]

Remark. It it easy to show that the sheaf is acyclic in stead of flabby. In fact, we can prove it by applying Proposition 13.3.1 in [7].

Combining the above lemmas, we have the following

Proposition 1.5. *The following is a flabby resolution of $\mathcal{O}$ on $\sum \boldsymbol{C}$:*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{F}^0_\infty \longrightarrow \mathcal{F}^1_\infty \longrightarrow \mathcal{F}^2_\infty \longrightarrow \cdots.$$

Proof. Let U be a pseudo-convex open set in $\sum \boldsymbol{C}$. Then, we will show that the following sequence is exact:

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{F}^0_\infty(U) \xrightarrow{\rho} \mathcal{F}^1_\infty(U) \xrightarrow{\rho} \mathcal{F}^2_\infty(U) \longrightarrow \cdots.$$

Let g be an arbitrary element of $\mathcal{F}^{k+1}_\infty(U)$ for $k \geqq 1$ such that $\rho(g)=0$. Put $g_n = r_n(k+1)(g)$. Consider the diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{F}^{k-1}_{n+1}(U_{n+1}) & \longrightarrow & \mathcal{F}^{k}_{n+1}(U_{n+1}) & \longrightarrow & \mathcal{F}^{k+1}_{n+1}(U_{n+1}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{F}^{k-1}_{n}(U_{n}) & \longrightarrow & \mathcal{F}^{k}_{n}(U_{n}) & \longrightarrow & \mathcal{F}^{k+1}_{n}(U_{n}) & \longrightarrow & \cdots . \end{array}$$

Every row is exact (see for example Theorem 7.4.3 in [9]), and therefore for each g_n we can find $f_n \in \mathcal{F}^k_n(U_n)$ such that $\rho(f_n)=g_n$. Since $\rho(r^{n+1}_n(k)(f_{n+1})-f_n)=0$ holds, there exists $h_n \in \mathcal{F}^{k-1}_n(U_n)$ such that

$$\rho(h_n)=r^{n+1}_n(k)(f_{n+1})-f_n .$$

By virtue of Lemma 1.2, there exists $h_{n+1} \in \mathcal{F}^{k-1}_{n+1}(U_n)$ such that $r^{n+1}_n(k-1)(h_{n+1})=h_n$. Put $f'_{n+1}=f_{n+1}-\rho(h_{n+1})$. Then

$$\rho(f'_{n+1})=g \quad \text{and} \quad r^{n+1}_n(k)(f'_{n+1})=f_n .$$

Thus, taking f'_{n+1} as f_{n+1} from the beginning, we obtain the sequence $\{f_n\}$ satisfying the following conditions:

(i) $\rho(f_n)=g_n$,
(ii) $r^{n+1}_n(k)(f_{n+1})=f_n$.

This sequence determines an element $f \in \mathcal{F}^k_\infty(U)$. The way of choosing $\{f_n\}$ implies that $\rho(f)=g$. In the remaining cases, owing to Lemma 1.1, we can prove the exactness similarly. Together with Lemma 1.4, we have the required result. [Q. E. D.]

Remark. The flabby resolution obtained in Proposition 1.5 does not coinside with the canonical flabby resolution of the sheaf $\mathcal{O}$. We write the canonical flabby resolution as follows:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{L}^0_\infty \longrightarrow \mathcal{L}^1_\infty \longrightarrow \mathcal{L}^2_\infty \longrightarrow \cdots.$$

We compare them with each other. Let U be an open set in $\sum \boldsymbol{C}$. Then,

$$(\mathcal{L}^0_\infty)(U)=\bigcup_{z\in U} \varinjlim_{W_z} \varprojlim_{n} \mathcal{O}_n(W_{z,n})$$
$$(\mathcal{F}^0_\infty)(U)=\bigcup_{z\in U} \varprojlim_{n} \varinjlim_{W_z} \mathcal{O}_n(W_{z,n}),$$

where W_z runs over open neighborhoods of z. Now we consider the following function.

$$f(z)=\sum_{k=1}^{\infty}\frac{z_k}{1-k\left(\sum_{i=1}^{\infty} z_i\right)}.$$

Then, the germ f at the origin belongs to $\varprojlim_{n}\varinjlim_{W_0}\mathcal{O}_n(W_{0,n})$ but does not to $\varinjlim_{W_0}\varprojlim_{n}\mathcal{O}_n(W_{0,n})$. Thus the sheaf $\mathcal{F}^0_\infty$ is different from the sheaf $\mathcal{L}^0_\infty$.

The following theorem is fundamental in the theory of hyperfunctions.

THEOREM 1.6 [13]. *Let U be an open set in $\boldsymbol{C}^n$. Then we have*

$$H^k_{\boldsymbol{R}^n\cap U}(U,\mathcal{O}_n)=0 \qquad \textit{for } k\neq n\,.$$

Now, we go on to the main result in this section.

THEOREM 1.7. *Let U be an arbitrary open set in $\sum \boldsymbol{C}$. Then we have*

$$H^k_{\Sigma\boldsymbol{R}\cap U}(U,\mathcal{O})=0$$

for every $k\geqq 0$.

Proof. We consider the diagram:

$$\begin{tikzcd}
(1.4)_\infty & 0 \arrow[r] & \Gamma_{\Sigma\boldsymbol{R}\cap U}(U,\mathcal{F}^0_\infty) \arrow[r] \arrow[d] & \Gamma_{\Sigma\boldsymbol{R}\cap U}(U,\mathcal{F}^1_\infty) \arrow[r] \arrow[d] & \Gamma_{\Sigma\boldsymbol{R}\cap U}(U,\mathcal{F}^2_\infty) \arrow[r] \arrow[d] & \cdots \\
& & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
(1.4)_{n+1} & 0 \arrow[r] & \Gamma_{\boldsymbol{R}^{n+1}\cap U_{n+1}}(U_{n+1},\mathcal{F}^0_{n+1}) \arrow[r] \arrow[d] & \Gamma_{\boldsymbol{R}^{n+1}\cap U_{n+1}}(U_{n+1},\mathcal{F}^1_{n+1}) \arrow[r] \arrow[d] & \Gamma_{\boldsymbol{R}^{n+1}\cap U_{n+1}}(U,\mathcal{F}^2_{n+1}) \arrow[r] \arrow[d] & \cdots \\
(1.4)_{n} & 0 \arrow[r] & \Gamma_{\boldsymbol{R}^{n}\cap U_{n}}(U_{n},\mathcal{F}^0_{n}) \arrow[r] \arrow[d] & \Gamma_{\boldsymbol{R}^{n}\cap U_{n}}(U_{n},\mathcal{F}^1_{n}) \arrow[r] \arrow[d] & \Gamma_{\boldsymbol{R}^{n}\cap U_{n}}(U_{n},\mathcal{F}^2_{n}) \arrow[r] \arrow[d] & \cdots \\
& & \; & \; & \; &
\end{tikzcd}$$

In the above diagram every row $(1.4)_n$ is exact except the n-th in each row by Theorem 1.6. By using Lemma 1.2, the similar diagram chase as in proposition 1.5 leads to the conclusion that the sequence $(1.4)_\infty$ is exact.

Remark. In view of the long exact sequence of relative cohomology groups, we have also $H^k_S(\sum \boldsymbol{C},\mathcal{O})=0$ $(k\geqq 0)$ for any locally closed subset S of $\sum \boldsymbol{R}$.

§ 2. Analytic functionals with compact carrier and preudo-convex open sets

In this section we will investigate the relative cohomology groups with support in a compact set K.

Let W be an open set in C^n. We denote by $H_c^k(W, \mathcal{O}_n)$ the relative cohomology groups with compact support. As is well known that the following is a soft resolution of the sheaf $\mathcal{O}_n$;

$$0 \longrightarrow \mathcal{O}_n \longrightarrow \mathcal{D}_n'^{0,0} \xrightarrow{\bar{\partial}} \mathcal{D}_n'^{0,1} \xrightarrow{\bar{\partial}} \mathcal{D}_n'^{0,2} \xrightarrow{\bar{\partial}} \cdots,$$

where $\mathcal{D}_n'^{0,k}$ denotes the sheaf of differential $(0, k)$-forms with distribution coefficients on C^n and $\mathcal{D}_n'^{0,0} = \mathcal{D}_n'$. Then, we have

$$H_c^k(W, \mathcal{O}_n) = \frac{\operatorname{Ker}\{\Gamma_c(W, \mathcal{D}_n'^{0,k}) \xrightarrow{\bar{\partial}} \Gamma_c(W, \mathcal{D}_n'^{0,k+1})\}}{\operatorname{Im}\{\Gamma_c(W, \mathcal{D}_n'^{0,k-1}) \xrightarrow{\bar{\partial}} \Gamma_c(W, \mathcal{D}_n'^{0,k})\}}.$$

Let U be an open set in C^{n+1}. We denote by $\mathcal{E}_n$ the sheaf of germs of infinitely differentiable functions on C^n. Now we define the mapping

$$v_{n+1}^n : H_c^n(U \cap C^n, \mathcal{O}_n) \longrightarrow H_c^{n+1}(U, \mathcal{O}_{n+1})$$

by

$$\langle v_{n+1}^n([\phi d\bar{z}_1 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n]), f\rangle = \langle [\phi d\bar{z}_1 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n], f|_{C^n}\rangle$$

for $f \in \mathcal{E}_{n+1}(U)$, where [.] denotes the equivalence class and $\langle \, , \rangle$ denotes the bilinear form.

We refer to the following

THEOREM 2.1 [14]. *Let W be an open set in C^n such that*

$$\dim H^k(W, \mathcal{O}_n) < \infty \qquad \text{for } k \geqq 1.$$

Then $H^k(W, \mathcal{O}_n)$ and $H_c^{n-k}(W, \mathcal{O}_n)$ are Fréchet-Schwartz space and dual Fréchet-Schwartz space, respectively and they are strongly dual to each other.

COROLLARY 2.2. *Let W be a pseudo-convex open set in C^n. Then we have*

$$\mathcal{O}_n'(W) \cong H_c^n(W, \mathcal{O}_n).$$

We cite another theorem from the theory of analytic functionals.

THEOREM 2.3 [11]. *Let K be a compact set in C^n such that*

$$H^k(K, \mathcal{O}_n) = 0 \qquad \text{for } k \geqq 1. \tag{2.1}$$

Then for any open neighborhood W of K we have

$$(2.2) \qquad H_K^k(W, \mathcal{O}_n)=0 \qquad \textit{for } k\neq n\,,$$
$$(2.3) \qquad H_K^n(W, \mathcal{O}_n)\cong\mathcal{O}'(K)\,.$$

Now we well prove the following

THEOREM 2.4. *Let K be a compact set in $\sum \boldsymbol{C}$ such that $K\subset\boldsymbol{C}^m$ and*

$$(2.4) \qquad H^k(K, \mathcal{O}_m)=0 \qquad \textit{for } k\geqq 1\,.$$

Then, for any open neighborhood U of K in $\sum \boldsymbol{C}$ we have

$$(2.5) \qquad H_K^k(U, \mathcal{O})=0 \qquad \textit{for } k\geqq 0\,.$$

Proof. It is easy to see that $H^k(K, \mathcal{O}_n)=0$ for $n\geqq m$ and $k\geqq 1$. Thus, by Theorem 2.3, we have

$$(2.6) \qquad H_K^k(U_n, \mathcal{O}_n)=0 \qquad \text{for } k\neq n,\ n\geqq m\,.$$

Let us consider the following diagram:

$$\begin{array}{llll}
(2.7)^\infty & \Gamma_K(U, \mathcal{F}_\infty^{k-1}) & \longrightarrow \Gamma_K(U, \mathcal{F}_\infty^{k}) & \longrightarrow \Gamma_K(U, \mathcal{F}_\infty^{k+1}) \\
& \downarrow & \downarrow & \downarrow \\
& \vdots & \vdots & \vdots \\
& \downarrow & \downarrow & \downarrow \\
(2.7)_{n+1} & \Gamma_K(U_{n+1}, \mathcal{F}_{n-1}^{k-1}) & \longrightarrow \Gamma_K(U_{n+1}, \mathcal{F}_{n+1}^{k}) & \longrightarrow \Gamma_K(U_{n+1}, \mathcal{F}_{n+1}^{k+1}) \\
& \downarrow & \downarrow & \downarrow \\
(2.7)_n & \Gamma_K(U_n, \mathcal{F}_n^{k-1}) & \longrightarrow \Gamma_K(U_n, \mathcal{F}_n^{k}) & \longrightarrow \Gamma_K(U_n, \mathcal{F}_n^{k+1})
\end{array}$$

By (2.6), the sequence $(2.7)_n$ is exact if $n\geqq m$. In view of Lemma 1.2 the similar diagram chase as in Proposition 1.5 leads to the conclusion that the sequence $(2.7)_\infty$ is exact. This implies that $H_K^k(U, \mathcal{O})=0$. [Q. E. D.]

THEOREM 2.5. *Let K be a compact se in $\sum \boldsymbol{C}$ satisfying the following condition:*

(2.8) *K has a fundamental system $\mathfrak{B}$ of neighborhoods consisting of pseudoconvex open sets in $\sum \boldsymbol{C}$.*

Then we have

$$(2.9) \qquad \mathcal{O}'(K)\cong \varprojlim_{K\subset U}\ \varinjlim_{n} H_c^n(U_n, \mathcal{O}_n)\,,$$

where U runs over $\mathfrak{B}$.

Proof. Let U be a pseudo-convex open neighborhood of K. Then by Corollary 2.2 we have

$$\mathcal{O}'_n(U_n) \cong H^n_c(U_n, \mathcal{O}_n).$$

Owing to Proposition 2.11 in [3], we have

$$\begin{aligned}\mathcal{O}'(U) &= \varinjlim_n \mathcal{O}'_n(U_n) \\ &\cong \varinjlim_n H^n_c(U_n, \mathcal{O}_n).\end{aligned}$$

Taking into account that $\mathcal{O}(K) \cong \varinjlim_{K \subset U} \mathcal{O}(U)$, generally we have

$$\mathcal{O}'(K) = \varprojlim_{K \subset U} \mathcal{O}'(U).$$

Thus we obtain

$$\mathcal{O}'(K) \cong \varprojlim_{K \subset U} \varinjlim_n H^n_c(U_n, \mathcal{O}_n). \qquad \text{[Q. E. D.]}$$

Now as an application of Theorem 2.4 we discuss the relation between pseudo-convexity and the vanishing of the cohomology groups. It is well known that for an open set U in $\boldsymbol{C}^n$ the following conditions are equivalent (see for example [9]):

(2.10) $\quad U$ is pseudo-convex,

(2.11) $\quad H^k(U, \mathcal{O}_n) = 0 \quad$ for $k \geqq 1$.

On the other hand we have Theorem 0.1 on the space $\sum \boldsymbol{C}$. Then we will discuss whether the inverse holds true or not in the case of $\sum \boldsymbol{C}$. As a corollary of Theorem 2.4, we have the following answer.

COROLLARY 2.6. *There exists an open set U which is not pseudo-convex such that*

$$H^k(U, \mathcal{O}) = 0 \quad \textit{for } k \geqq 1.$$

Proof. Let W be a pseudo-convex open set in $\sum \boldsymbol{C}$ and let K be a compact set in $\boldsymbol{C}^m \cap W$ such that

$$H^k(K, \mathcal{O}_n) = 0 \quad \text{for } k \geqq 1,\ n \geqq m.$$

For example, it is sufficient to choose a compact set in $\boldsymbol{R}^m \cap W$ as K. By Theorem 2.3, we have

$$H^k_K(W_n, \mathcal{O}_n) = 0 \quad \text{for } k \neq n,$$

$$H_K^n(W_n, \mathcal{O}_n) \cong \mathcal{O}_n'(K).$$

In view of the long exact squence

$$\begin{aligned} 0 &\longrightarrow H_K^0(W_n, \mathcal{O}_n) \longrightarrow H^0(W_n, \mathcal{O}_n) \longrightarrow H^0(W_n - K, \mathcal{O}_n) \longrightarrow \\ &\longrightarrow H_K^1(W_n, \mathcal{O}_n) \longrightarrow H^1(W_n, \mathcal{O}_n) \longrightarrow H^1(W_n - K, \mathcal{O}_n) \longrightarrow \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\longrightarrow H_K^k(W_n, \mathcal{O}_n) \longrightarrow H^k(W_n, \mathcal{O}_n) \longrightarrow H^k(W_n - K, \mathcal{O}_n) \longrightarrow \cdots \end{aligned}$$

we have

$$\begin{aligned} &H^0(W_n, \mathcal{O}_n) = H^0(W_n - K, \mathcal{O}_n) \\ &H^k(W_n - K, \mathcal{O}_n) = 0 \qquad \text{for } n-2 \geqq k \geqq 1 \\ &H^{n-1}(W_n - K, \mathcal{O}_n) = \mathcal{O}'(K). \end{aligned}$$

Thus, $W_n - K$ is not pseudo-convex. Hence $W-K$ is not pseudo-convex (see for example Lemma 2.1.5. in [12]). However, considering the long exact sequence;

$$\begin{aligned} 0 &\longrightarrow H_K^0(W, \mathcal{O}) \longrightarrow H^0(W, \mathcal{O}) \longrightarrow H^0(W - K, \mathcal{O}) \longrightarrow \\ &\longrightarrow H_K^1(W, \mathcal{O}) \longrightarrow H^1(W, \mathcal{O}) \longrightarrow H^1(W - K, \mathcal{O}) \longrightarrow \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\longrightarrow H_K^k(W, \mathcal{O}) \longrightarrow H^k(W, \mathcal{O}) \longrightarrow H^k(W - K, \mathcal{O}) \longrightarrow \cdots, \end{aligned}$$

by Theorem 0.1 and Theorem 2.4 we have

$$H^k(W-K, \mathcal{O}) = 0 \qquad \text{for } k \geqq 1.$$

Thus $U = W-K$ is the required one. [Q. E. D.]

§ 3. Hyperfunctions with an infinite number of holomorphic parameters

We will show the following

THEOREM 3.1. *$\boldsymbol{R}^r \times \sum \boldsymbol{C}$ is purely r codimensional to the sheaf $\mathcal{O}$ of holomorphic functions on $\sum \boldsymbol{C}$.*

Before starting the proof we cite the following theorem from the theory of hyperfunctions.

THEOREM 3.2 ([12]). *$\boldsymbol{R}^r \times \boldsymbol{C}^n$ is purely r codimensional to the sheaf $\mathcal{O}_{n+r}$ of germs of holomorphic functions on $\boldsymbol{C}^{n+r}$.*

Proof of THEOREM 3.1. Let us consider the following diagram.

$$
\begin{array}{llll}
(3.1)_\infty & & & \\
0\to\Gamma_{\boldsymbol{R}^r\times\Sigma\boldsymbol{C}\cap U}(U,\mathcal{F}^0_\infty) & \to\Gamma_{\boldsymbol{R}^r\times\Sigma\boldsymbol{C}\cap U}(U,\mathcal{F}^1_\infty) & \to\Gamma_{\boldsymbol{R}^r\times\Sigma\boldsymbol{C}\cap U}(U,\mathcal{F}^2_\infty) & \to\cdots \\
\quad\downarrow & \quad\downarrow & \quad\downarrow & \\
\quad\vdots & \quad\vdots & \quad\vdots & \\
\quad\downarrow & \quad\downarrow & \quad\downarrow & \\
(3.1)_{n+1} & & & \\
0\to\Gamma_{\boldsymbol{R}^r\times\boldsymbol{C}^{n+1}\cap U_{n+1}}(U_{n+1},\mathcal{F}^0_{n+1}) & \to\Gamma_{\boldsymbol{R}^r\times\boldsymbol{C}^{n+1}\cap U_{n+1}}(U_{n+1},\mathcal{F}^1_{n+1}) & \to\Gamma_{\boldsymbol{R}^r\times\boldsymbol{C}^{n+1}\cap U_{n+1}}(U_{n+n},\mathcal{F}^2_{n+1}) & \to\cdots \\
\quad\downarrow & \quad\downarrow & \quad\downarrow & \\
(3.1)_n & & & \\
0\to\Gamma_{\boldsymbol{R}^r\times\boldsymbol{C}^n\cap U_n}(U_n,\mathcal{F}^0_n) & \to\Gamma_{\boldsymbol{R}^r\times\boldsymbol{C}^n\cap U_n}(U_n,\mathcal{F}^1_n) & \to\Gamma_{\boldsymbol{R}^r\times\boldsymbol{C}^n\cap U_n}(U_n,\mathcal{F}^2_n) & \to\cdots.
\end{array}
$$

In the above diagram every row $(3.1)_n$ is exact except the r-th by Theorem 3.2. Owing to Lemma 1.2, the similar diagram chase as in Proposition 1.5 leads to the conclusion that the sequence $(3.1)_\infty$ is exact except the r-th. [Q. E. D.]

Now let U be an open set in $\sum \boldsymbol{C}$. We put

$$
\begin{aligned}
&V_0=U,\\
&V_i=\{(z_k)\in\sum\boldsymbol{C};\ \operatorname{Im} z_i\neq 0\},\quad i\geqq 1,\\
&\mathfrak{M}_r=\{V_j;\ j=0,1,\cdots,r\},\\
&\mathfrak{M}'_r=\{V_j;\ j=1,2,\cdots,r\}.
\end{aligned}
$$

Then, it is easily see that following proposition holds.

PROPOSITION 3.3. *Let U be an pseudo-convex open set in $\sum \boldsymbol{C}$. Then we have*

$$H^r_{\boldsymbol{R}^r\times\Sigma\boldsymbol{C}\cap U}(U,\mathcal{O})\cong H^r(\mathfrak{M}_r,\mathfrak{M}'_r;\ \mathcal{O}).$$

For an arbitrary $f\in H^r(\mathfrak{M}_r,\mathfrak{M}'_r;\ \mathcal{O})$ we denote by F the defining function such that $f=[F]$, $F\in\mathcal{O}(V_0\cap V_1\cap\cdots\cap V_r)$. We define the mapping

$$\iota^r_{r+1}:\ H^r(\mathfrak{M}_r,\mathfrak{M}'_r;\ \mathcal{O})\longrightarrow H^{r+1}(\mathfrak{M}_{r+1},\mathfrak{M}'_{r+1};\ \mathcal{O})$$

by $\iota^r_{r+1}([F])=[F\cdot 1]$, where 1 denotes the function such that 1 on $V_0\cap V_1\cap\cdots\cap\{z\in\sum\boldsymbol{C};\ \operatorname{Im} z_{r+1}>0\}$ and 0 on $V_0\cap V_1\cap\cdots\cap\{z\in\sum\boldsymbol{C};\ \operatorname{Im} z_{r+1}<0\}$.

Now we will call the sheaf associated with the presheaf

$$\{\varinjlim_r H^r(\mathfrak{M}_r,\mathfrak{M}'_r;\ \mathcal{O})=\varinjlim_r H^r_{\boldsymbol{R}^n\times\Sigma\boldsymbol{C}\cap U}(U,\mathcal{O})\}$$

the sheaf of hyperfunctions with an infinite number of holomorphic parameters.

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