A Note on the $K(\pi, 1)$ Property of the Orbit Space of the Unitary Reflection Group $G(m, l, n)$

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1. Introduction

A *unitary reflection* g in C^n with its reflecting hyperplane V is an element g of $U(n)$ whose fixed point set is a hyperplane V in C^n. A *unitary reflection group* G in $U(n)$ is a finite subgroup of $U(n)$ generated by unitary reflections in C^n. Given a unitary reflection group G, the collection $\mathcal{V}(G)=\{V_i|i\in I\}$ of *walls* of G is the set of all hyperplanes V_i each of which is a reflecting hyperplane of some unitary reflection belonging to G. Here we introduce the notation $|\mathcal{V}(G)|$ to denote the union $\cup\{V_i|i\in I\}$ of walls of G. Then G is known to operate freely on the complement $C^n\backslash|\mathcal{V}(G)|$ of the set $|\mathcal{V}(G)|$ in C^n.

As well-known, the irreducible unitary reflection groups were classified by Shephard and Todd [5]. According to their results, there exists a series of unitary reflection groups $G(m, l, n)$ with index being the triple of natural numbers (m, l, n) satisfying the conditions $2\leqq n$ and $l|m$ and all other groups are "exceptional". Now we recall the definition of $G(m, l, n)$: $G(m, l, n)$ is a subgroup of $U(n)$ consisting of all linear transformations of the form

$$z_i'=\zeta_m^{a_i}z_{\sigma(i)} \qquad (\zeta_m=\exp(2\pi\sqrt{-1}/m))$$

where σ is an arbitrary permutation in the symmetric group of degree n and $(a_1, \cdots, a_n)$ is an arbitrary sequence of n integers satisfying the relation

$$a_1+\cdots+a_n\equiv 0 \qquad \mod l,$$

For this group $G=G(m, l, n)$, we shall use the abbreviated notation $\mathcal{V}(G)=\mathcal{V}(m, l, n)$ in the sequel.

Now, the objective of this note is to give an elementary proof to the following theorem.

THEOREM. *For every triple of natural numbers (m, l, n) satisfying the conditions $2\leqq n$ and $l|m$, the complement $C^n\backslash|\mathcal{V}(m, l, n)|$ of $|\mathcal{V}(m, l, n)|$ in C^n is $K(\pi, 1)$.*

This theorom has been first proved by Brieskorn [1] and Deligne [2] for the case of $G(m,l,n)$'s being the Weyl group and announced by Etsuko Bannai for general $G(m,l,n)$ in 1976.

Before proving the theorem, we list some known results needed in the subsequent argument.

LEMMA [4], [5]. *For each triple of natural numbers (m,l,n) satisfying the conditions $2\leqq n$ and $l|m$, the collection of walls $\mathcal{V}(m,l,n)$ of the group $G(m,l,n)$ has the form as stated in what follows.*

If $m>l$, we have

$$\mathcal{V}(m,l,n)=\{V(z_h),\ V(z_i-\zeta_m^a z_j)|1\leqq h\leqq n, 1\leqq i<j\leqq n, 0\leqq a\leqq m-1\}$$

If $m=l$, we have

$$\mathcal{V}(m,m,n)=\{V(z_i-\zeta_m^a z_j)|1\leqq i<j\leqq n, 0\leqq a\leqq m-1\}\,.$$

Here we used the notation $V(f)\subset \boldsymbol{C}^n$ to denote the variety represented as $f^{-1}(0)$ in terms of a polynomial $f(z)\in \boldsymbol{C}[z]$.

Eventually, we obtain

$$\mathcal{V}(m,l,n)=\mathcal{V}(m,1,n)$$

in the case $m>l\geqq 1$.

Above facts allow us to reduce the proof of the theorem to the one of the next proposition.

PROPOSITION. *For every triple of natural numbers (m,l,n) satisfying the conditions $2\leqq n$ and $l=1$ or $2\leqq n$ and $m=l$, the complement $\boldsymbol{C}^n\backslash|\mathcal{V}(m,l,n)|$ of $|\mathcal{V}(m,l,n)|$ in $\boldsymbol{C}^n$ is $K(\pi,1)$.*

Ending this section, we remark that the groups appearing in the proposition have generators as stated in the following.

$G(m,1,n)$ is a subgroup of $U(n)$ generated by the set of linear transformations g, s_i with $1\leqq i\leqq n-1$ written in the form

$$g: z_1'=\zeta_m z_1,\ z_j'=z_j \qquad 2\leqq j\leqq n$$
$$s_i: z_i'=z_{i+1},\ z_{i+1}'=z_i,\ z_j'=z_j \qquad 1\leqq j\leqq n, j\neq i, i+1$$

$G(m,m,n)$ is generated by the set of linear transformations h, s_i with $1\leqq i\leqq n-1$ written in the form

$$h: z_1'=\zeta_m^{-1}z_2,\ z_2'=\zeta_m z_1,\ z_j'=z_j \qquad 3\leqq j\leqq n$$
$$s_i: z_i'=z_{i+1},\ z_{i+1}'=z_i,\ z_j'=z_j \qquad 1\leqq j\leqq n, j\neq i, i+1\,.$$

2. Proof of the proposition

For later convenience, we introduce the notations to denote some collections of hyperplanes:

$$\mathcal{V}(m, n)' = \{V(z_h),\ V(z_i - \zeta_m^a z_j) | 1 \leqq h \leqq n, 1 \leqq i < j \leqq n, 0 \leqq a \leqq m-1\}$$
$$\mathcal{V}(m, n)'' = \{V(z_i - \zeta_m^a z_j) | 1 \leqq i < j \leqq n, 0 \leqq a \leqq m-1\}.$$

Further we designate the union of hyperplanes in each of the above collections by adding vertical lines on both sides, that is

$$|\mathcal{V}(m, n)'| = V\Big(\prod_{1 \leqq h \leqq n} z_h \cdot \prod_{1 \leqq i < j \leqq n} (z_i^m - z_j^m)\Big)$$
$$|\mathcal{V}(m, n)''| = V\Big(\prod_{1 \leqq i < j \leqq n} (z_i^m - z_j^m)\Big).$$

Under these conventions, we are going to prove the previous proposition. The proof proceeds by verifying successive three assertions step by step.

i) $\boldsymbol{C}^n \backslash |\mathcal{V}(1, n)'|$ *is* $K(\pi, 1)$.

The assertion i) is proved by induction on n.

For $n=2$, the fact in i) can be shown by considering the natural projection from $\boldsymbol{C}^2 \backslash \{0\}$ onto $P\boldsymbol{C}^2 \approx S^2$.

For $n \geqq 3$, we prove that the $K(\pi, 1)$ property of $\boldsymbol{C}^{n-1} \backslash |\mathcal{V}(1, n-1)'|$ implies the one of $\boldsymbol{C}^n \backslash |\mathcal{V}(1, n)'|$.

Now we consider the projection

$$\tilde{\omega} : \boldsymbol{C}^n \backslash V\Big(\prod_{0 \leqq h \leqq n-1} z_h \cdot \prod_{0 \leqq i < j \leqq n-1} (z_i - z_j)\Big) \to \boldsymbol{C}^{n-1} \backslash V\Big(\prod_{1 \leqq h \leqq n-1} w_h \cdot \prod_{1 \leqq i < j \leqq n-1} (w_i - w_j)\Big)$$

defined by

$$\tilde{\omega}(z_0, z_1, \cdots, z_{n-1}) = (w_1, \cdots, w_{n-1})$$

where $w_i = z_0 - z_i$ for $1 \leqq i \leqq n-1$.

We put

$$\boldsymbol{z}_1 = (z_0, z_1, \cdots, z_{n-1}) \in \boldsymbol{C}^n, \quad \boldsymbol{w} = (w_1, \cdots, w_{n-1}) \in \boldsymbol{C}^{n-1}$$

Then assuming $\tilde{\omega}(\boldsymbol{z}) = \boldsymbol{w}$, we can easily verify that each of the conditions

$$w_h = 0,\ w_i - w_j = 0 \quad 1 \leqq h \leqq n-1,\ 1 \leqq i < j \leqq n-1$$

holds if and only if each of the conditions

$$z_0 - z_h = 0,\ z_i - z_j = 0 \quad 1 \leqq h \leqq n-1,\ 1 \leqq i < j \leqq n-1$$

holds respectively. Thus $\tilde{\omega}$ is surjective.

Next we fix a point $\boldsymbol{w}$ in $\boldsymbol{C}^{n-1}$ satisfying the conditions

$$w_h \neq 0,\ w_i - w_j \neq 0$$

for all h, i, j with $1 \leqq h \leqq n-1, 1 \leqq i < j \leqq n-1$. Then the point $\boldsymbol{z}$ in $\boldsymbol{C}^n$ is in $\tilde{\omega}^{-1}(\boldsymbol{w})$

if and only if z satisfies the conditions

$$z_i = z_0 - w_i, \ z_i \neq 0$$

for all i with $1 \leqq i \leqq n-1$. By projecting $\tilde{\omega}^{-1}(\boldsymbol{w})$ into its its z_0-component, we can show that $\tilde{\omega}^{-1}(\boldsymbol{w})$ is diffeomorphic to the space consisting of points in $\boldsymbol{C}$ with the coordinate z_0 satisfying the conditions

$$z_0 \neq 0, w_i$$

for all i with $1 \leqq i \leqq n-1$.

Now by integrating an appropriately chosen vector field on $\boldsymbol{C}$ with the coordinate z_0, we can prove that $\tilde{\omega}$ defines a C^∞ fibre bundle.

Moreover, the above argument shows that the fibre $\tilde{\omega}^{-1}(\boldsymbol{w})$ is homotopy equivalent to the join $\overset{n}{\vee} S^1$ of n copies of S^1 and so it has the $K(\pi, 1)$ property.

Here applying the homotopy exact sequence of the fibre bundle $\tilde{\omega}$, we can show that the $K(\pi, 1)$ property of the base space implies the one of the total space. Further we have an exact sequence of the fundamental groups

$$1 \longrightarrow \overset{n}{*} \boldsymbol{Z} \longrightarrow \pi_1(\boldsymbol{C}^n \backslash |\mathcal{V}(1, n)'|) \longrightarrow \pi_1(\boldsymbol{C}^{n-1} \backslash |\mathcal{V}(1, n-1)'|) \longrightarrow 1$$

where $\overset{n}{*} \boldsymbol{Z}$ denotes a free group with n free generators.

ii) $\boldsymbol{C}^n \backslash |\mathcal{V}(m, n)''|$ *is* $K(\pi, 1)$.

First we consider the simplest case $m=1$.

For $n=2$, the proof is as in i).

Assume $n \geqq 3$. Paying regard to the assertion i), we find that it is enough to prove that the $K(\pi, 1)$ property of $\boldsymbol{C}^{n-1} \backslash |\mathcal{V}(1, n-1)'|$ implies the one of $\boldsymbol{C}^n \backslash \mathcal{V}(1, n)''|$.

We consider the map

$$\tilde{\omega}: \boldsymbol{C}^n \backslash V\Big(\prod_{0 \leqq i < j \leqq n-1} (z_i - z_j)\Big) \longrightarrow \boldsymbol{C}^{n-1} \backslash V\Big(\prod_{1 \leqq h \leqq n-1} w_h \cdot \prod_{1 \leqq i < j \leqq n-1} (w_i - w_j)\Big)$$

defined by

$$\tilde{\omega}(z_0, z_1, \cdots, z_{n-1}) = (w_1, \cdots, w_{n-1})$$

where $w_i = z_0 - z_i$ for $1 \leqq i \leqq n-1$.

Again the map $\tilde{\omega}$ is surjective.

Put

$$\boldsymbol{z} = (z_0, z_1, \cdots, z_{n-1}) \in \boldsymbol{C}^n, \ \boldsymbol{w} = (w_1, \cdots, w_{n-1}) \in \boldsymbol{C}^{n-1}.$$

Take a point $\boldsymbol{w}$ in $\boldsymbol{C}^{n-1} \backslash |\mathcal{V}(1, n-1)'|$. Then $\boldsymbol{z}$ is in $\tilde{\omega}^{-1}(\boldsymbol{w})$ if and only if the relations

$$z_i = z_0 - w_i$$

hold for all i with $1 \leqq i \leqq n-1$. Now the natural projection from $\tilde{\omega}^{-1}(\boldsymbol{w})$ into its z_0-component gives a diffeomorphism between $\tilde{\omega}^{-1}(\boldsymbol{w})$ and the whole complex line.

As in the proof of i), $\tilde{\omega}$ also defines a C^∞ fibre bundle

In this case each fibre is contractible, so $\tilde{\omega}$ induces a homotopy equivalence from the total space into the base space.

Next we treat the general case $m \geqq 1$.

For $n=2$, the proof is as before.

Assume $n \geqq 3$. From i), we only need to show that the $K(\pi, 1)$ property of $C^{n-1} \backslash |\mathcal{V}(1, n-1)'|$ assures the one of $C^n \backslash |\mathcal{V}(m, n)''|$.

Now we define the map

$$\tilde{\omega}: C^n \backslash V\Big(\prod_{0 \leqq i < j \leqq n-1} (z_i^m - z_j^m)\Big) \longrightarrow C^{n-1} \backslash V\Big(\prod_{1 \leqq h \leqq n-1} w_h \cdot \prod_{1 \leqq i < j \leqq n-1} (w_i - w_j)\Big)$$

to be the map given by

$$\tilde{\omega}(z_0, z_1, \cdots, z_{n-1}) = (w_1, \cdots, w_{n-1})$$

where $w_i = z_0^m - z_i^m$ for $1 \leqq i \leqq n-1$.

Put

$$z = (z_0, z_1, \cdots, z_{n-1}) \in C^n, \quad w = (w_1, \cdots, w_{n-1}) \in C^{n-1}.$$

When we assume $\tilde{\omega}(z) = w$, each of the conditions

$$w_h = 0, \; w_i - w_j = 0 \quad 1 \leqq h \leqq n-1, \; 1 \leqq i < j \leqq n-1$$

holds if and only if each of the conditions

$$z_0^m - z_h^m = 0, \; z_i^m - z_j^m = 0 \quad 1 \leqq h \leqq n-1, \; 1 \leqq i < j \leqq n-1$$

holds respectively. This shows the surjectivity of $\tilde{\omega}$.

Suppose given a point w in C^{n-1} satisfying the condition

$$w_h \neq 0, \quad w_i - w_j \neq 0$$

for all h, i, j with $1 \leqq h \leqq n-1$, $1 \leqq i < j \leqq n-1$. Then z is in $\tilde{\omega}^{-1}(w)$ if and only if z satisfies the conditions

$$z_i^m = z_0^m - w_i$$

for all i with $1 \leqq i \leqq n-1$.

Here we consider the natural projection $\pi_i : C^n \to C$ sending z to z_i for $0 \leqq i \leqq n-1$. And we define $\pi_0' : C^n \to C$ to be the map transforming z into z_0^m. Now we observe the composition $\pi_0' \circ \pi_i^{-1} : \pi_i \tilde{\omega}^{-1}(w) \to C$ for $0 \leqq i \leqq n-1$. Then the relation between all the components of the point z in $\tilde{\omega}^{-1}(z)$ shows that $\pi_0' \circ \pi_i^{-1}$ gives rise to an m-fold branched covering with m branching points $w_i \neq 0$ for $1 \leqq i \leqq n-1$ and with single branching point 0 for $i=0$. Further we can regard $\tilde{\omega}^{-1}(w)$ as the fibre product of the branched coverings $\pi_0' \circ \pi_i^{-1}$ with $0 \leqq i \leqq n-1$. These observations enable us to show that $\tilde{\omega}$ defines a C^∞ fibre bundle with the fibre $\tilde{\omega}^{-1}(w)$ being a nonsingular punctured irreducible algebraic curve.

By the way, we compute the first Betti number of the fibre $\tilde{\omega}^{-1}(\boldsymbol{w})$. We draw simple arcs $I_i:[0,1]\to \boldsymbol{C}$ on $\boldsymbol{C}$ so that $I_i(0)=0$, $I_i(1)=w_i$ with $1\leqq i\leqq n-1$ and $I_i\cap I_j=\{0\}$ if $i\neq j$. We put $X=\cup\{I_i|1\leqq i\leqq n-1\}$. Then X is a deformation retract of $\boldsymbol{C}$ and $\pi_0'|\tilde{\omega}^{-1}(\boldsymbol{w})$ is an unramified covering map outside X, so $\tilde{\omega}^{-1}(\boldsymbol{w})\cap(\pi_0')^{-1}(X)$ is a deformation retract of $\tilde{\omega}^{-1}(\boldsymbol{w})$. Now easy computation shows that the Euler characteristic of $\tilde{\omega}^{-1}(\boldsymbol{w})\cap(\pi_0')^{-1}(X)$ is $m^{n-1}(n-m(n-1))$ and hence the first Betti number of $\tilde{\omega}^{-1}(\boldsymbol{w})$ is equal to $1+m^{n-1}(m(n-1)-n)$.

Combining above results, $\tilde{\omega}^{-1}(\boldsymbol{w})$ is homotopy equivalent to $\overset{b}{\vee}S^1$ with $b=1+m^{n-1}(m(n-1)-n)$ and consequently endowed with the $K(\pi,1)$ property. The same argument as in i) can be applied to show that the $K(\pi,1)$ property of $\boldsymbol{C}^{n-1}\backslash|\mathcal{V}(1,n-1)'|$ implies the one of $\boldsymbol{C}^n\backslash|\mathcal{V}(m,n)''|$. Moreover we have an exact sequence of the fundamental groups

$$1\longrightarrow \overset{b}{*}\boldsymbol{Z}\longrightarrow\pi_1(\boldsymbol{C}^n\backslash|\mathcal{V}(m,n)''|)\longrightarrow\pi_1(\boldsymbol{C}^{n-1}\backslash|\mathcal{V}(1,n-1)'|)\longrightarrow 1$$

iii) $\boldsymbol{C}^n\backslash|\mathcal{V}(m,n)'|$ *is* $K(\pi,1)$.

We define the map

$$\tilde{\omega}:\boldsymbol{C}^n\backslash V\Big(\prod_{1\leqq h\leqq n}z_h\cdot\prod_{1\leqq i<j\leqq n}(z_i^m-z_j^m)\Big)\longrightarrow\boldsymbol{C}^n\backslash V\Big(\prod_{1\leqq h\leqq n}w_h\cdot\prod_{1\leqq i<j\leqq n}(w_i-w_j)\Big)$$

by the function

$$\tilde{\omega}(z_1,\cdots,z)=(w_1,\cdots,w_n)$$

with $w_i=z_i^m$ for $1\leqq i\leqq n$.

Clealy $\tilde{\omega}$ is an mn-fold unramified covering map, so the $K(\pi,1)$ property of the base space $\boldsymbol{C}^n\backslash|\mathcal{V}(1,n)'|$ implies the one of the covering space $\boldsymbol{C}\backslash|\mathcal{V}(m,n)'|$.

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