Network Induction and Resolution Diagrams of the Brieskorn Singularities

By Akio YAMADA

Department of Mathematics, The College of Arts and Sciences,
The University of Tokyo, Komaba, Meguro-ku, Tokyo 153

and Yukio MATSUMOTO

Department of Mathematics, Faculty of Science, The University of Tokyo,
Hongo, Bunkyo-ku, Tokyo 113

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§ 1. Introduction

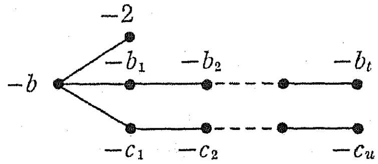
In our previous paper [5], we characterized those weighted graphs of type II that appear as resolution diagrams of the Brieskorn singularities of type $(2, q, r)$, and gave an outline of the proof. The purpose of the present paper is to fully formulate the "network induction" to which we referred there and to give detailed proofs of the theorems in [5] as an application of the network induction.

Roughly speaking, the network induction is a generalization of the usual mathematical induction, but it proceeds along a network, i. e., a directed graph, in which any two vertices may have infinitely many edges connecting them. We shall formulate it more precisely in § 2.

In this section, we will recall the main results of [5] for reader's convenience.

Let $V(2, q, r)$ be a complex hypersurface in $\boldsymbol{C}^3$ defined by the equation $z_1^2+z_2^q+z_3^r=0$, where 2, q, r are pairwise coprime integers with $2<q<r$. The hypersurface $V(2, q, r)$ has an isolated singular point at the origin called the *Brieskorn singularity* of type $(2, q, r)$.

As is well known [1], the (minimal) resolution diagram of $V(2, q, r)$ is a weighted graph of the following shape:

(1.1)  $(b_k, c_l \geqq 2)$

The positive integers (*weights*) b, b_k, c_l are computed as follows:

Let b, y, z be positive integers defined by

$$(1.2) \qquad \begin{cases} 2yr \equiv -1 \pmod{q} , & 2zq \equiv -1 \pmod{r} \\ 0<y<q , & 0<z<r \\ 2bqr = qr+2yr+2zq+1 . \end{cases}$$

Then $q/y=[b_1, b_2, \cdots, b_t]$ and $r/z=[c_1, c_2, \cdots, c_u]$, where $[n_1, n_2, \cdots, n_v]$ denotes the continued fraction

$$n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_v}}} \quad (n_i \geqq 2) .$$

The number b defined above is equal to 1 or 2.

We are interested in weighted graphs of type II, that is, graphs whose weights are even integers.

In weighted graphs of type II, the number b must be equal to 2. Thus the weighted (planar) graphs of type II such as (1.1) are in one to one correspondence to the arrays of positive even integers $X=\begin{bmatrix} b_1, b_2, \cdots, b_t \\ c_1, c_2, \cdots, c_u \end{bmatrix}$. We denote the weighted graph corresponding to X by $D(X)$.

Let S be the totality of arrays of integers. S has a structure of semi-group. Product in S is defined by juxtaposition:

$$\begin{bmatrix} m_1, & \cdots, & m_\mu \\ n_1, & \cdots, & n_\nu \end{bmatrix} \begin{bmatrix} m'_1, & \cdots, & m'_\xi \\ n'_1, & \cdots, & n'_\eta \end{bmatrix} = \begin{bmatrix} m_1, & \cdots, & m_\mu, & m'_1, & \cdots, & m'_\xi \\ n_1, & \cdots, & n_\nu, & n'_1, & \cdots, & n'_\eta \end{bmatrix} .$$

The identity element in S is the empty array $\begin{bmatrix} \phi \\ \phi \end{bmatrix}$.

To state our results we need to define three special types of elements in S called "joints," "molecules" and "head and tail," respectively.

A) *Joints.* There are four elements $\bar{Z}, \underline{Z}, \bar{T}, \underline{T} \in S$ called joints:

$$\bar{Z}=\begin{bmatrix} 2, 2, 6 \\ 6, 2, 2 \end{bmatrix} , \qquad \underline{Z}=\iota(\bar{Z}) , \qquad \bar{T}=\begin{bmatrix} 2, 2, 4, 2, 2 \\ 8 \end{bmatrix} , \qquad \underline{T}=\iota(\bar{T}) ,$$

where ι: $S \to S$ is the involution defined by

$$\iota(X)=\begin{bmatrix} n_1, & \cdots, & n_\nu \\ m_1, & \cdots, & m_\mu \end{bmatrix} \qquad \text{for} \quad X=\begin{bmatrix} m_1, & \cdots, & m_\mu \\ n_1, & \cdots, & n_\nu \end{bmatrix} .$$

We call $\iota(X)$ the *inverse* of X.

B) *Molecules.* Let $\bar{e}, \underline{e}, \bar{p}_n, \underline{p}_n \in S$ be defined as follows:

$$\bar{e}=\begin{bmatrix} 2 \\ \phi \end{bmatrix}, \qquad \underline{e}=\iota(\bar{e}), \qquad \bar{p}_n=\begin{bmatrix} (8n-1)*2 \\ 2n+2 \end{bmatrix}, \qquad \underline{p}_n=\iota(\bar{p}_n),$$

where $n\geqq 1$, and the notation $m*2$ stands for the sequence $2, 2, \cdots, 2$ consisting of m 2's. A *molecule* M is a product (in S) of these auxiliary elements ("particles") $\bar{e}$, $\underline{e}$, $\bar{p}_n$, $\underline{p}_n$ of the following form:

$$M=\begin{cases} \bar{e}\bar{p}_{n(1)}\underline{p}_{n(2)}\bar{p}_{n(3)}\cdots\underline{p}_{n(\mu)}\underline{e} \text{ or its inverse } (\mu\text{: even}\geqq 0), \\ \bar{e}\bar{p}_{n(1)}\underline{p}_{n(2)}\bar{p}_{n(3)}\cdots\bar{p}_{n(\mu)}\bar{e} \text{ or its inverse } (\mu\text{: odd}\geqq 1). \end{cases}$$

For a more precise construction rule and examples, see [5].

C) *Head and tail.* The *head* $H\in S$ is defined by $H=\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, the same element as the simplest molecule $\bar{e}\underline{e}=\underline{e}\bar{e}$. The *tail* $L\in S$ is defined by $L=\begin{bmatrix} \phi \\ 2, 2 \end{bmatrix}$.

Our main results in [5] are as follows:

THEOREM 1.1. *Let $X\in S$. Then $D(X)$ is a weighted graph of type II which appears as the resolution diagram of a Brieskorn singularity $(2, q, r)$ $(q<r)$ if and only if X is written as the following product in S:*

(1.3) $$X=HM_1J_1M_2J_2\cdots M_{\nu-1}J_{\nu-1}M_\nu L \quad (\nu\geqq 1),$$

where H is the head, L is the tail, each M_i is a molecule and each J_i is a joint.

Remark. The above decomposition of X is unique.

THEOREM 1.2. *Suppose that $D(HM_1J_1\cdots J_{\nu-1}M_\nu L)$ and $D(H\cdot\iota(M_1J_1\cdots J_{\nu-1}M_\nu)L)$ are resolution diagrams of the singularities $(2, q, r)$ and $(2, q', r')$, respectively. Then (q, r) and (q', r') are related by*

$$q'=-4q+3r$$
$$r'=-5q+4r.$$

The "if" part of Theorem 1.1 and Theorem 1.2 are proved by the network induction arguments (§§ 3, 5). We remark that also in the proof of the "only if" part of Theorem 1.1 (§ 4) the network induction applies in principle, but the proof slightly deviates from the formalism as given in § 2 in its most rigorous sense.

§ 2. Principle of Network Induction

Let Λ be a set and $E=(E(\alpha, \beta))_{(\alpha,\beta)\in\Lambda\times\Lambda}$ a family of sets indexed by $(\alpha, \beta)\in\Lambda\times\Lambda$ such that

$$E(\alpha, \beta)\cap E(\alpha', \beta')=\phi \quad \text{for} \quad (\alpha, \beta)\neq(\alpha', \beta').$$

Then we say that (Λ, E) is a *network*. An element of Λ is called a *vertex* and an element of $E(\alpha, \beta)$ is called an *edge* having α as its *starting point* and β

as its *terminal point*. Since $E(\alpha, \beta)$ and $E(\alpha', \beta')$ are disjoint for $(\alpha, \beta) \neq (\alpha', \beta')$, the starting point and the terminal one of an edge f are uniquely determined and denoted by $\sigma(f)$ and $\tau(f)$, respectively. A *path* ω starting at a vertex α and terminating at a vertex β is defined as a finite sequence $(f_n, f_{n-1}, \cdots, f_1)$ of edges f_i $(1 \leqq i \leqq n)$ such that

$$\sigma(f_1) = \alpha \ , \qquad \tau(f_i) = \sigma(f_{i+1}) \quad (1 \leqq i \leqq n-1) \ , \qquad \tau(f_n) = \beta \ .$$

The integer n is called the *length* of ω and denoted by $|\omega|$. We denote the totality of paths starting at α and terminating at β by $\Omega(\alpha, \beta)$. In the case $\alpha = \beta$, we include the *empty path*, i.e., empty sequence $\phi = (\ \ )$ of edges, in $\Omega(\alpha, \alpha)$ and define its length $|\phi|$ to be zero. Let $\omega = (f_n, \cdots, f_1)$ be a path. Then $\omega' = (f_j, f_{j-1}, \cdots, f_{i+1}, f_i)$ $(1 \leqq i, j \leqq n)$ is called a *subpath* of ω. The empty path ϕ is a subpath of any path. Let

$$\omega = (f_n, \cdots, f_1) \in \Omega(\alpha, \beta) \quad \text{and} \quad \omega' = (g_m, \cdots, g_1) \in \Omega(\beta, \gamma) \ .$$

Then we define the *composition* $\omega' \circ \omega$ of ω and ω' as

$$\omega' \circ \omega := (g_m, \cdots, g_1, f_n, \cdots, f_1) \in \Omega(\alpha, \gamma) \ .$$

Clearly we have

$$|\omega' \circ \omega| = |\omega'| + |\omega| \ , \qquad \omega \circ \phi = \omega \ .$$

PRINCIPLE OF NETWORK INDUCTION. *Let (A, E) be a network and fix a vertex $\alpha_0 \in A$. Suppose that for each vertex $\beta \in A$, a propositional function P_β is defined on $\Omega(\alpha_0, \beta)$, that is to say, to each path $\omega \in \Omega(\alpha_0, \beta)$, a proposition $P_\beta(\omega)$ is assigned. Assume that the following two assertions hold.*

1° *$P_{\alpha_0}(\phi)$ is true;*

2° *for $\beta, \gamma \in A$, $\omega \in \Omega(\alpha_0, \beta)$ and $f \in E(\beta, \gamma)$,*
$P_\beta(\omega)$ implies $P_\gamma((f) \circ \omega)$.

Then, for any $\beta \in A$ and any $\omega \in \Omega(\alpha_0, \beta)$, $P_\beta(\omega)$ is true.

This is shown by mathematical induction on the length of ω.

Remark. When A is a singleton $\{\alpha_0\}$ and $E(\alpha_0, \alpha_0)$ also is a singleton $\{f_0\}$, we have

$$\Omega(\alpha_0, \alpha_0) = \{\phi, (f_0), (f_0, f_0), (f_0, f_0, f_0), \cdots\} \ ,$$

which is in one to one correspondence to the set of all non-negative integers by assigning to $\omega \in \Omega(\alpha_0, \alpha_0)$ its length $|\omega|$. The network induction on this "Peano network" (A, E) is nothing other than the ordinary mathematical induction (Fig. 1).

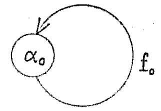


Fig. 1. Peano network

§ 3. Brieskorn Network of Type $(2, *, *)$

We proceed to construct a network, called *Brieskorn network of type* $(2, *, *)$, in order to prove the Theorems in § 1. Let $A := \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ and define

$$E(\mathrm{I}, \mathrm{II}) := \{\underline{\varepsilon}\}\ , \qquad E(\mathrm{I}, \mathrm{III}) := \{\bar{\varepsilon}\}$$
$$E(\mathrm{II}, \mathrm{III}) := \{\underline{p}_n \mid n \geqq 1\}\ , \qquad E(\mathrm{II}, \mathrm{IV}) := \{\bar{\delta}\}$$
$$E(\mathrm{III}, \mathrm{II}) := \{\bar{p}_n \mid n \geqq 1\}\ , \qquad E(\mathrm{III}, \mathrm{IV}) := \{\underline{\delta}\}$$
$$E(\mathrm{IV}, \mathrm{I}) := \{\bar{Z}, \underline{Z}, \bar{T}, \underline{T}\}\ .$$

For all other pairs $(\alpha, \beta) \in A \times A$, we define $E(\alpha, \beta)$ as the empty set. Then (A, E) is a network, which we call the "Brieskorn network of type $(2, *, *)$ (Fig. 2).

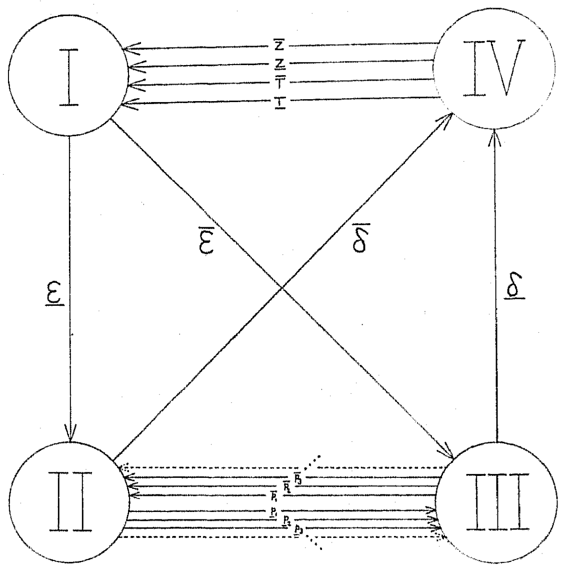


Fig. 2. Brieskorn network of type $(2, *, *)$

Define a map R from the set of all edges in (A, E) to the semi-group S of arrays of integers defined in § 1 as follows:

$$\begin{cases} R(\underline{\varepsilon}) := \underline{e}\ , \quad R(\bar{\varepsilon}) := \bar{e}\ , \quad R(\underline{\delta}) := \underline{e}\ , \quad R(\bar{\delta}) := \bar{e}\ , \\ R(\underline{p}_n) := \underline{p}_n\ , \quad R(\bar{p}_n) := \bar{p}_n \quad (n \geqq 1) \\ R(J) := J \quad \text{for } J = \bar{Z}, \underline{Z}, \bar{T} \text{ or } \underline{T}\ . \end{cases}$$

Then R is a map: $\bigcup_{(\alpha, \beta) \in A \times A} E(\alpha, \beta) \longrightarrow S$.

Let $\alpha, \beta \in A$ and $\omega = (f_n, \cdots, f_1) \in \Omega(\alpha, \beta)$. Define

$$\rho(\omega) := R(f_n) R(f_{n-1}) \cdots R(f_1) \in S\ .$$

Then ρ is a map: $\bigcup_{(\alpha,\beta)\in\Lambda\times\Lambda} \Omega(\alpha,\beta) \longrightarrow S$.

An array $M\in S$ is a molecule if and only if there exists a path $\omega=(f_n,\cdots,f_1)$ $\in\Omega(\mathrm{I},\mathrm{IV})$ such that $M=\rho(\omega)$ and $f_i\notin E(\mathrm{IV},\mathrm{I})$ for all $i\in\{1,\cdots,n\}$. Thus, an array $X\in S$ is written as (1.3) if and only if there exists a path $\omega\in\Omega(\mathrm{I},\mathrm{IV})$ such that $X=H\rho(\omega)L$.

Let $X=\begin{bmatrix} b_1,\cdots,b_t \\ c_1,\cdots,c_u \end{bmatrix}\in S$ be written as (1.3). Recall from [5] that a subarray $X'=\begin{bmatrix} b_1,\cdots,b_k \\ c_1,\cdots,c_l \end{bmatrix}$ of X $(1\leqq k\leqq t,\ 1\leqq l\leqq u)$ is said to end with M if there exists an integer λ such that $1\leqq\lambda\leqq\nu$ and $X'=HM_1J_1\cdots M_{\lambda-1}J_{\lambda-1}M_\lambda$. Suppose that $X=H\rho(\omega)L$ $(\omega\in\Omega(\mathrm{I},\mathrm{IV}))$. Then, a subarray X' of X ends with M if and only if there exists a subpath ω' of ω such that $X'=H\rho(\omega')$ and $\omega'\in\Omega(\mathrm{I},\mathrm{IV})$. Similarly, a subarray X' of X ends with $J(p,\bar{p}$, resp.) if and only if there exists a subpath ω' of ω such that $X'=H\rho(\omega')$ and $\omega'\in\Omega(\mathrm{IV},\mathrm{IV})$ $(\Omega(\mathrm{II},\mathrm{IV}),\Omega(\mathrm{III},\mathrm{IV})$, resp.).

Define four polynomials in $\boldsymbol{Z}[\eta,\eta',\zeta,\zeta']$ as follows:

$$\begin{aligned} F_{\mathrm{I}}(\eta,\eta',\zeta,\zeta') &:=1+\eta\zeta-4(\eta-\eta')(\zeta-\zeta') , \\ F_{\mathrm{II}}(\eta,\eta',\zeta,\zeta') &:=1+\eta\zeta'-4(\eta-\eta')(\zeta-\zeta') , \\ F_{\mathrm{III}}(\eta,\eta',\zeta,\zeta') &:=1+\eta'\zeta-4(\eta-\eta')(\zeta-\zeta') , \\ F_{\mathrm{IV}}(\eta,\eta',\zeta,\zeta') &:=1+\eta'\zeta'-4(\eta-\eta')(\zeta-\zeta') . \end{aligned}$$

Let β be a vertex in the Brieskorn network of type $(2,*,*)$, i.e., $\beta=\mathrm{I},\mathrm{II},\mathrm{III}$ or IV and let $Y=\begin{bmatrix} y_0,y_1,\cdots,y_{t+1} \\ z_0,z_1,\cdots,z_{u+1} \end{bmatrix}\in S$. For each pair (k,l) of integers with $0\leqq k\leqq t$ and $0\leqq l\leqq u$, we define

$$(Y|k,l)_\beta:=F_\beta(y_k,y_{k+1},z_l,z_{l+1}) .$$

This notation will be used later in § 4.

For each $\beta\in\Lambda$, define a propositional function P_β on $\Omega(\mathrm{I},\beta)$ as follows: Let $\omega\in\Omega(\mathrm{I},\beta)$ and define $X_\omega:=\rho(\omega)L(\in S)$. Write X_ω explicitly as an array of integers: $X_\omega=\begin{bmatrix} b_t,b_{t-1},\cdots,b_1 \\ c_u,c_{u-1},\cdots,c_1 \end{bmatrix}$. (Here, the indexing order of $\{b_k\}$ and $\{c_l\}$ is different from that written so far. This ordering is temporarily adopted for convenience of the proof of lemma 3.1 below). Define an array $Y_\omega=\begin{bmatrix} y_{t+1},y_t,\cdots,y_0 \\ z_{u+1},z_u,\cdots,z_0 \end{bmatrix}\in S$ by

$$(3.1)\qquad \begin{cases} y_0=0 , & y_1=1 , \quad y_{k+1}=b_ky_k-y_{k-1} \quad (1\leqq k\leqq t) , \\ z_0=0 , & z_1=1 , \quad z_{l+1}=c_lz_l-z_{l-1} \quad (1\leqq l\leqq u) . \end{cases}$$

Then, a proposition $P_\beta(\omega)$ is defined as follows:

$$P_\beta(\omega):\quad F_\beta(y_{t+1},y_t,z_{u+1},z_u)=0 .$$

LEMMA 3.1. *For any* $\beta\in\Lambda$ *and any* $\omega\in\Omega(\mathrm{I},\beta)$, $P_\beta(\omega)$ *holds.*

Remark. This lemma is essentially the same as the Fundamental Lemma in [5], and its proof contains main idea in proving Theorems 1.1 and 1.2.

Proof. We shall prove this lemma by network induction.

1°. When $\beta=\mathrm{I}$ and $\omega=\phi$, we have

$$X_\omega=L\,,\quad t=0\,,\quad u=2\,,\quad y_0=0\,,\quad y_1=1\,,\quad z_2=2\,,\quad z_3=3\,.$$

Hence, $F_{\mathrm{I}}(y_1, y_0, z_3, z_2)=0$, i. e., $P_{\mathrm{I}}(\phi)$ is true.

2°. Let $\beta, \gamma\in\Lambda$, $\omega\in\Omega(\mathrm{I}, \beta)$ and $f\in E(\beta, \gamma)$. Suppose $P_\beta(\omega)$ is true. Putting $\omega'=(f)\circ\omega\in\Omega(\mathrm{I}, \gamma)$, we have

$$X_{\omega'}=\rho(\omega')L=R(f)\rho(\omega)L=R(f)X_\omega\ .$$

When we write

$$X_\omega=\begin{bmatrix} b_t, b_{t-1}, \cdots, b_1 \\ c_u, c_{u-1}, \cdots, c_1 \end{bmatrix}, \qquad R(f)=\begin{bmatrix} b_{t'}, b_{t'-1}, \cdots, b_{t+1} \\ c_{u'}, c_{u'-1}, \cdots, c_{u+1} \end{bmatrix}$$

with $t'\geqq t$ and $u'\geqq u$, $Y_{\omega'}$ is written as

$$Y_{\omega'}=\begin{bmatrix} y_{t'+1}, y_{t'}, \cdots, y_{t+2}, y_{t+1}, \cdots, y_0 \\ z_{u'+1}, z_{u'}, \cdots, z_{u+2}, z_{u+1}, \cdots, z_0 \end{bmatrix},$$

where $\{y_k\}_{0\leqq k\leqq t'+1}$ and $\{z_l\}_{0\leqq l\leqq u'+1}$ are determined by (3.1) with t and u replaced by t' and u', respectively. Hence,

$$\begin{pmatrix} y_{k+1} \\ y_k \end{pmatrix}=\begin{pmatrix} b_k & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} \qquad \text{for } 1\leqq k\leqq t'\,,$$

$$\begin{pmatrix} z_{l+1} \\ z_l \end{pmatrix}=\begin{pmatrix} c_l & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_l \\ z_{l-1} \end{pmatrix} \qquad \text{for } 1\leqq l\leqq u'\,.$$

For any array $Q=\begin{bmatrix} b_{t'}, b_{t'-1}, \cdots, b_{t+1} \\ c_{u'}, c_{u'-1}, \cdots, c_{u+1} \end{bmatrix}\in S$, we define 2×2 matrices

$$U(Q):=\begin{pmatrix} b_{t'} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_{t'-1} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} b_{t+1} & -1 \\ 1 & 0 \end{pmatrix},$$

$$V(Q):=\begin{pmatrix} c_{u'} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_{u'-1} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_{u+1} & -1 \\ 1 & 0 \end{pmatrix}.$$

Under this notation, we have

$$\begin{pmatrix} y_{t'+1} \\ y_{t'} \end{pmatrix}=U(R(f))\begin{pmatrix} y_{t+1} \\ y_t \end{pmatrix}, \qquad \begin{pmatrix} z_{u'+1} \\ z_{u'} \end{pmatrix}=V(R(f))\begin{pmatrix} z_{u+1} \\ z_u \end{pmatrix}.$$

There are several cases to be treated separately, according to the pair (β, γ) such that $f\in E(\beta, \gamma)$.

The case in which $(\beta, \gamma)=(\mathrm{I}, \mathrm{II})$.

Since $f=\varepsilon$, we have

$$U(R(f))=U(\underline{e})=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad V(R(f))=V(\underline{e})=\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}.$$

Hence,

$$\begin{aligned}F_{\mathrm{II}}(y_{t'+1}, y_{t'}, z_{u'+1}, z_{u'})&=1+y_{t'+1}z_{u'}-4(y_{t'+1}-y_{t'})(z_{u'+1}-z_{u'})\\ &=1+y_{t+1}z_{u+1}-4(y_{t+1}-y_t)(z_{u+1}-z_u)\\ &=F_{\mathrm{I}}(y_{t+1}, y_t, z_{u+1}, z_u).\end{aligned}$$

Since $F_{\mathrm{I}}(y_{t+1}, y_t, z_{u+1}, z_u)=0$ by network induction hypothesis, we have $F_{\mathrm{II}}(y_{t'+1}, y_{t'}, z_{u'+1}, z_{u'})=0$, i.e., $P_{\mathrm{II}}(\omega')$ holds. The cases in which $(\beta,\gamma)=(\mathrm{I},\mathrm{III})$, $(\mathrm{II},\mathrm{IV})$ or $(\mathrm{III},\mathrm{IV})$ are proved similarly.

The case in which $(\beta,\gamma)=(\mathrm{II},\mathrm{III})$.

Since $f=p_n$ for some integer $n\geqq 1$, we have, denoting 2×2 matrix $\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}$ by G,

$$\begin{aligned}U(R(f))&=U(p_n)=\begin{pmatrix}2n+2 & -1\\ 1 & 0\end{pmatrix},\\ V(R(f))&=V(p_n)=G^{8n-1}=\begin{pmatrix}8n & -8n+1\\ 8n-1 & -8n+2\end{pmatrix}.\end{aligned}$$

Hence,

$$\begin{aligned}&F_{\mathrm{III}}(y_{t'+1}, y_{t'}, z_{u'+1}, z_{u'})\\ &=1+y_{t'}z_{u'+1}-4(y_{t'+1}-y_{t'})(z_{u'+1}-z_{u'})\\ &=1+y_{t+1}\{8nz_{u+1}+(-8n+1)z_u\}-4\{(2n+1)y_{t+1}-y_t\}(z_{u+1}-z_u)\\ &=1+y_{t+1}z_u-4(y_{t+1}-y_t)(z_{u+1}-z_u)\\ &=F_{\mathrm{II}}(y_{t+1}, y_t, z_{u+1}, z_u)\\ &=0 \quad \text{(by network induction hypothesis),}\end{aligned}$$

which implies $P_{\mathrm{III}}(\omega')$.

The case in which $(\beta,\gamma)=(\mathrm{III},\mathrm{II})$ is shown similarly.

The case in which $(\beta,\gamma)=(\mathrm{IV},\mathrm{I})$.

If $f=\bar{Z}$, we have

$$\begin{aligned}U(R(f))&=U(\bar{Z})=G^2\begin{pmatrix}6 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}16 & -3\\ 11 & -2\end{pmatrix},\\ V(R(f))&=V(\bar{Z})=\begin{pmatrix}6 & -1\\ 1 & 0\end{pmatrix}G^2=\begin{pmatrix}16 & -11\\ 3 & -2\end{pmatrix}.\end{aligned}$$

Hence,

$$\begin{aligned}
&F_{\mathrm{I}}(y_{t'+1}, y_{t'}, z_{u'+1}, z_{u'}) \\
&\quad=1+y_{t'+1}z_{u'+1}-4(y_{t'+1}-y_{t'})(z_{u'+1}-z_{u'}) \\
&\quad=1+(16y_{t+1}-3y_t)(16z_{u+1}-11z_u)-4(5y_{t+1}-y_t)(13z_{u+1}-9z_u) \\
&\quad=1-4y_{t+1}z_{u+1}+4y_{t+1}z_u+4y_tz_{u+1}-3y_tz_u \\
&\quad=F_{\mathrm{IV}}(y_{t+1}, y_t, z_{u+1}, z_u) \\
&\quad=0\ .
\end{aligned}$$

If $f=\bar{T}$, we have

$$U(R(f))=U(\bar{T})=\begin{pmatrix}24 & -17\\ 17 & -12\end{pmatrix}, \qquad V(R(f))=V(\bar{T})=\begin{pmatrix}8 & -1\\ 1 & 0\end{pmatrix}.$$

Hence,

$$\begin{aligned}
&F_{\mathrm{I}}(y_{t'+1}, y_{t'}, z_{u'+1}, z_{u'}) \\
&\quad=1+(24y_{t+1}-17y_t)(8z_{u+1}-z_u)-4(7y_{t+1}-5y_t)(7z_{u+1}-z_u) \\
&\quad=F_{\mathrm{IV}}(y_{t+1}, y_t, z_{u+1}, z_u) \\
&\quad=0\ .
\end{aligned}$$

Since the polynomials $F_{\mathrm{I}}(\eta, \eta', \zeta, \zeta')$ and $F_{\mathrm{IV}}(\eta, \eta', \zeta, \zeta')$ are invariant by the transformation $(\eta, \eta', \zeta, \zeta')\longmapsto(\zeta, \zeta', \eta, \eta')$, and $\underline{Z}=\iota(\bar{Z})$, $\underline{T}=\iota(\bar{T})$, the cases in which $f=\underline{Z}$ and $f=\underline{T}$ are reduced to those in which $f=\bar{Z}$ and $f=\bar{T}$, respectively. Thus, whatever edge f may be in $E(\mathrm{IV}, \mathrm{I})$, we have $F_{\mathrm{I}}(y_{t'+1}, y_{t'}, z_{u'+1}, z_{u'})=0$. Therefore, $P_{\mathrm{I}}(\omega')$ holds.

This completes the network induction. Q. E. D.

§ 4. Proof of a Weakened Version of Theorem 1.1

It is assumed in Theorem 1.1 that Brieskorn singularities $(2, q, r)$ always satisfy the condition $q<r$. In § 4, we prove a slightly weakened version of Theorem 1.1, not assuming $q<r$:

THEOREM 1.1′. *$D(X)$ is a weighted graph of type II which appears as the resolution diagram of a Brieskorn singularity $(2, q, r)$ (not necessarily satisfying $q<r$) if and only if X is written as the following product in S:*

$$X=HM_1J_1M_2J_2\cdots M_{\nu-1}J_{\nu-1}M_\nu L \quad (\nu\geqq 1) \tag{1.3}$$

or

$$X=HM_1J_1M_2J_2\cdots M_{\nu-1}J_{\nu-1}M_\nu\cdot\iota(L) \quad (\nu\geqq 1)\ , \tag{1.3$'$}$$

where H is the head, L is the tail, each M_i is a molecule and each J_i is a joint.

Remark. The decomposition of X ends with L or $\iota(L)$, according as $q<r$ or $q>r$. This will be proved later in § 5 with aid of Theorem 1.2.

Proof. The "if" part is already shown in [5], using Fundamental Lemma. In order to prove the "only if" part, suppose that $X=\begin{bmatrix} b_1, \cdots, b_t \\ c_1, \cdots, c_u \end{bmatrix} \in S$ is such that $D(X)$ is a weighted graph of type II which appears as the resolution diagram of a Brieskorn singularity $(2, q, r)$, where q and r are coprime odd integers greater than 2. We shall show that X can be decomposed as (1.3) or (1.3)$'$.

There exist integers y, z and b satisfying (1.2). Since b must be equal to 2 in our case, we get

$$4qr = qr + 2yr + 2qz + 1 .$$

Define two sequences $\{y_k\}_{0 \leqq k \leqq t+1}$ and $\{z_l\}_{0 \leqq l \leqq u+1}$ of integers inductively:

$$y_0 := q , \quad y_1 := y , \quad y_{k+1} := b_k y_k - y_{k-1} \quad \text{for } 1 \leqq k \leqq t ,$$
$$z_0 := r , \quad z_1 := z , \quad z_{l+1} := c_l z_l - z_{l-1} \quad \text{for } 1 \leqq l \leqq u .$$

Then, we have the relation

$$4y_0 z_0 = 1 + y_0 z_0 + 2y_1 z_0 + 2y_0 z_1 \tag{4.0}$$

and

$$0 \leqq y_{k+1} < y_k \quad (1 \leqq k \leqq t) , \quad y_t = 1 , \quad y_{t+1} = 0 ,$$
$$0 \leqq z_{l+1} < z_l \quad (1 \leqq l \leqq u) , \quad z_u = 1 , \quad z_{u+1} = 0 .$$

In order to prove that X is written as (1.3) or (1.3)$'$ we make use of the network induction again, but in reverse direction this time. To be more precise, it suffices to show the following five propositions (Cf. Fig. 2):

1° $\begin{bmatrix} b_1 \\ c_1 \end{bmatrix} = H$ *and* $(Y|1, 1)_{\mathrm{IV}} = 0$;

2° *if* $(Y|k, l)_{\mathrm{IV}} = 0$, *then one of the following two statements holds*:

(i) $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_{l+1} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \underline{e}$ *and* $(Y|k, l+1)_{\mathrm{III}} = 0$,

(ii) $\begin{bmatrix} b_1, \cdots, b_{k+1} \\ c_1, \cdots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \bar{e}$ *and* $(Y|k+1, l)_{\mathrm{II}} = 0$;

3° *if* $(Y|k, l)_{\mathrm{III}} = 0$, *then one of the following two statements holds*:

(i) $\begin{bmatrix} b_1, \cdots, b_{k+1} \\ c_1, \cdots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \bar{e}$ *and* $(Y|k+1, l)_{\mathrm{I}} = 0$,

(ii) $\begin{bmatrix} b_1, \cdots, b_{k+1} \\ c_1, \cdots, c_{l+8n-1} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} p_n$ *for some* $n \geqq 1$ *and* $(Y|k+1, l+8n-1)_{\mathrm{II}} = 0$;

4° *if* $(Y|k, l)_{\mathrm{II}} = 0$, *then one of the following two statements holds*:

(i) $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_{l+1} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \underline{e}$ *and* $(Y|k, l+1)_{\mathrm{I}}=0$,

(ii) $\begin{bmatrix} b_1, \cdots, b_{k+8n-1} \\ c_1, \cdots, c_{l+1} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \bar{p}_n$ *for some* $n \geqq 1$

and $(Y|k+8n-1, l+1)_{\mathrm{III}}=0$;

5° *if* $(Y|k, l)_{\mathrm{I}}=0$, *then one of the following two statements holds*:

(i) $\begin{bmatrix} b_1, \cdots, b_t \\ c_1, \cdots, c_u \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} L$ *or* $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \iota(L)$,

(ii) $\begin{bmatrix} b_1, \cdots, b_{k'} \\ c_1, \cdots, c_{l'} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} J$ *with* $k<k' \leqq t$, $l<l' \leqq u$ *and some joint* J,

and $(Y|k', l')_{\mathrm{IV}}=0$,

where Y *denotes the array* $\begin{bmatrix} y_0, \cdots, y_{t+1} \\ z_0, \cdots, z_{u+1} \end{bmatrix} \in S$.

(Proof of 1°) If y_0 were greater than or equal to $2y_1$, it would follow that $2y_1 \leqq y_0-1$ because y_0 is an odd integer. So we would have

$$\begin{aligned} 3y_0z_0-1 &= 2y_0z_1+2y_1z_0 \quad \text{(by (4.0))} \\ &\leqq 2y_0(z_0-1)+(y_0-1)z_0 \\ &= 3y_0z_0-2y_0-z_0 \ . \end{aligned}$$

Hence,

$$2y_0+z_0 \leqq 1 \ ,$$

which is a contradiction. Thus, $y_0<2y_1$.

Since $y_0=b_1y_1-y_2>(b_1-1)y_1$, we have

$$(b_1-1)y_1<2y_1 \ .$$

Hence, $b_1=2$. Similarly, $c_1=2$. Thus $\begin{bmatrix} b_1 \\ c_1 \end{bmatrix}=H$.

Substitution of $y_0=2y_1-y_2$ and $z_0=2z_1-z_2$ into (4.0) proves $(Y|1, 1)_{\mathrm{IV}}=0$.

(Proof of 2°) Suppose $(Y|k, l)_{\mathrm{IV}}=0$. Then, $k<t$ holds because $k=t$ implies $y_{k+1}=0$ which contradicts $(Y|k, l)_{\mathrm{IV}}=0$. Since $k+1 \leqq t$, b_{k+1} is defined. Similarly c_{l+1} is defined.

If $b_{k+1} \geqq 3$ and $c_{l+1} \geqq 3$, we would have

$$y_k=b_{k+1}y_{k+1}-y_{k+2} \geqq (b_{k+1}-1)y_{k+1}+1 \geqq 2y_{k+1}+1 \ .$$

Hence,

$$y_k-y_{k+1} \geqq y_{k+1}+1 \ .$$

Similarly

$$z_l - z_{l+1} \geqq z_{l+1} + 1 \ .$$

We would, therefore, obtain

$$\begin{aligned}(Y|k, l)_{\mathrm{IV}} &= 1 + y_{k+1}z_{l+1} - 4(y_k - y_{k+1})(z_l - z_{l+1}) \\ &\leqq 1 + y_{k+1}z_{l+1} - 4(y_{k+1}+1)(z_{l+1}+1) \\ &< 0 \ ,\end{aligned}$$

which is a contradiction. Therefore

$$b_{k+1} = 2 \quad \text{or} \quad c_{l+1} = 2 \ .$$

(i) When $c_{l+1}=2$, we have $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_{l+1} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \underline{e}$ and

$$\begin{aligned}(Y|k, l+1)_{\mathrm{III}} &= 1 + y_{k+1}z_{l+1} - 4(y_k - y_{k+1})(z_{l+1} - z_{l+2}) \\ &= 1 + y_{k+1}z_{l+1} - 4(y_k - y_{k+1})(z_l - z_{l+1}) \\ &= (Y|k, l)_{\mathrm{IV}} \\ &= 0 \ .\end{aligned}$$

(ii) When $b_{k+1}=2$, similarly we have

$$\begin{bmatrix} b_1, \cdots, b_{k+1} \\ c_1, \cdots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \bar{e} \quad \text{and} \quad (Y|k+1, l)_{\mathrm{II}} = (Y|k, l)_{\mathrm{IV}} = 0 \ .$$

(Proof of 3°) Suppose $(Y|k, l)_{\mathrm{III}}=0$. Then, $k<t$ and b_{k+1} is defined, similarly as in the proof of 2°.
(i) When $b_{k+1}=2$, one has

$$(Y|k+1, l)_{\mathrm{I}} = (Y|k, l)_{\mathrm{III}} = 0 \ .$$

(ii) When $b_{k+1} \geqq 3$, one considers the propositions

$$(P_i) \quad \text{for} \quad 0 \leqq i \leqq i_k + 1$$

and

$$(Q_i) \quad \text{for} \quad 0 \leqq i \leqq i_k \ ,$$

where $i_k := 4b_{k+1} - 10$, as follows:

$$(P_i) \quad \begin{cases} l+i \leqq u \quad \text{and} \\ 1 + y_{k+1}z_{l+i} - \{(i_k - i + 2)y_{k+1} + 4(y_{k+1} - y_{k+2})\}(z_{l+i} - z_{l+i+1}) = 0 \ , \end{cases}$$

$$(Q_i) \quad l+i < u \quad \text{and} \quad c_{l+i+1} = 2 \ .$$

We shall show by induction on i that (P_i) $(0 \leqq i \leqq i_k + 1)$ and (Q_i) $(0 \leqq i \leqq i_k)$ hold, as follows:

[First step] (P_0) holds because $(Y|k,l)_{\rm III}=0$.

[Second step] For each $i\leqq i_k$, (P_i) implies (Q_i).

(Proof) Suppose (P_i) holds.

If $l+i=u$, then $z_{l+i}=1$ and $z_{l+i+1}=0$, which would imply the relation

$$0=1-\{(i_k-i+1)y_{k+1}+4(y_{k+1}-y_{k+2})\}\leqq 1-(y_{k+1}+4)\ ,$$

contradicting $y_k>0$. Thus, $l+i<u$.

If $c_{l+i+1}\geqq 3$, then one would have

$$\begin{aligned} z_{l+i}&=c_{l+i+1}z_{l+i+1}-z_{l+i+2}\\ &\geqq(c_{l+i+1}-1)z_{l+i+1}+1\\ &\geqq 2z_{l+i+1}+1\ . \end{aligned}$$

Hence,

$$z_{l+i}-z_{l+i+1}\geqq\frac{z_{l+i}+1}{2}\ .$$

On the other hand,

$$(i_k-i+2)y_{k+1}+4(y_{k+1}-y_{k+2})\geqq 2y_{k+1}+4\ .$$

Combining these two inequalities, one would obtain

$$\begin{aligned} &\{(i_k-i+2)y_{k+1}+4(y_{k+1}-y_{k+2})\}(z_{l+i}-z_{l+i+1})\\ &\quad\geqq(y_{k+1}+2)(z_{l+i}+1)\\ &\quad>1+y_{k+1}z_{l+i}\ , \end{aligned}$$

which contradicts (P_i). Therefore, $c_{l+i+1}=2$.

[Third step] For each $i\leqq i_k$,

$$(P_i)\text{ and }(Q_i)\text{ imply }(P_{i+1})\ .$$

(Proof) Suppose (P_i) and (Q_i). Then, clearly $l+i+1\leqq u$. Substituting $z_{l+i}=2z_{l+i+1}-z_{l+i+2}$ into the equation in (P_i), we have

$$\begin{aligned} 0&=1+y_{k+1}(2z_{l+i+1}-z_{l+i+2})-\{(i_k-i+2)y_{k+1}+4(y_{k+1}-y_{k+2})\}(z_{l+i+1}-z_{l+i+2})\\ &=1+y_{k+1}z_{l+i+1}-\{(i_k-i+1)y_{k+1}+4(y_{k+1}-y_{k+2})\}(z_{l+i+1}-z_{l+i+2})\ , \end{aligned}$$

which proves (P_{i+1}).

Thus, (P_i) $(0\leqq i\leqq i_k+1)$ and (Q_i) $(0\leqq i\leqq i_k)$ hold. In particular, (P_{i_k+1}) reads as follows:

$$0=1+y_{k+1}z_{l+i_k+1}-\{y_{k+1}+4(y_{k+1}-y_{k+2})\}(z_{l+i_k+1}-z_{l+i_k+2})$$
$$=1+y_{k+1}z_{l+i_k+2}-4(y_{k+1}-y_{k+2})(z_{l+i_k+1}-z_{l+i_k+2})$$
$$=(Y|k+1, l+i_k+1)_{\mathrm{II}} .$$

For $0 \leqq i \leqq i_k$, (Q_i) means $c_{l+i+1}=2$. Since $D(X)$ is of type II, b_{k+1} can be written as $2n+2$ for some integer $n \geqq 1$. Then,

$$i_k+1=4b_{k+1}-9=8n-1 .$$

We obtain, therefore,

$$\begin{bmatrix} b_{k+1} \\ c_{l+1}, c_{l+2}, \cdots, c_{l+8n-1} \end{bmatrix} = \begin{bmatrix} 2n+2 \\ (8n-1)*2 \end{bmatrix} = p_n$$

and

$$(Y|k+1, l+8n-1)_{\mathrm{II}}=0$$

as desired.

(Proof of 4°) Similar to that of 3°.
(Proof of 5°) Suppose $(Y|k, l)_{\mathrm{I}}=0$.
(i) The case in which $k=t$ or $l=u$.

If $k=t$, then $y_k=1$ and $y_{k+1}=0$. Substituting these values into $(Y|k, l)_{\mathrm{I}}=0$, we have

$$1-3z_l+4z_{l+1}=0 . \tag{4.1}$$

The equation (4.1) does not hold for $l=u$. Thus, $l<u$ and c_{l+1} is defined. Then,

$$2z_{l+1}=\frac{3z_l-1}{2} \geqq z_l=c_{l+1}z_{l+1}-z_{l+2}>(c_{l+1}-1)z_{l+1} ,$$

which implies $c_{l+1}=2$.

Substituting $z_l=2z_{l+1}-z_{l+2}$ into (4.1), we obtain

$$1-2z_{l+1}+3z_{l+2}=0 ,$$

which shows $l+1<u$ and

$$2z_{l+2}=\frac{2}{3}(2z_{l+1}-1) \geqq z_{l+1}$$

because $z_{l+1} \geqq z_{l+2}+1 \geqq 2$. Hence, $c_{l+2}=2$ and

$$1-z_{l+2}+2z_{l+3}=0 .$$

If $l+2<u$, one would have

$$z_{l+2}=2z_{l+3}+1 \leqq 3z_{l+3}$$

and so $c_{l+3}=3$, which contradicts the assumption that $D(X)$ is of type II. Therefore $l+2=u$ and

$$\begin{bmatrix} b_1, \cdots, b_t \\ c_1, \cdots, c_u \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} L \ .$$

If $l=u$, a similar argument shows

$$\begin{bmatrix} b_1, \cdots, b_t \\ c_1, \cdots, c_u \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}_{\iota}(L) \ .$$

(ii) The case in which $k<t$ and $l<u$.

In this case our argument flows along the chart in Fig. 3. First we show that $b_{k+1}=2$ or $c_{l+1}=2$. If $b_{k+1}\geqq 3$ and $c_{l+1}\geqq 3$ then

$$4(y_k-y_{k+1})(z_l-z_{l+1})\geqq 4\cdot\frac{y_k+1}{2}\cdot\frac{z_l+1}{2}>1+y_k z_l \ ,$$

which contradicts $(Y|k, l)_{\mathrm{I}}=0$.

(A) The case in which $b_{k+1}=2$.

Substituting $y_k=2y_{k+1}-y_{k+2}$ into $(Y|k, l)_{\mathrm{I}}=0$, we obtain

(4.2) $$1+y_{k+1}z_l-(y_{k+1}-y_{k+2})(3z_l-4z_{l+1})=0 \ .$$

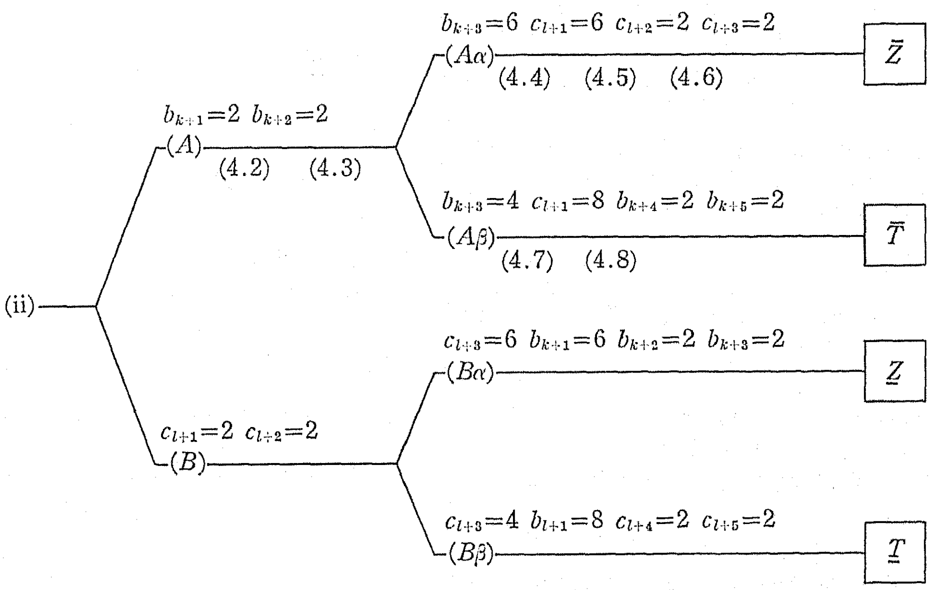


Fig. 3.

If $k+1=t$, then $y_{k+1}=1$ and $y_{k+2}=0$ and so

$$1-2z_l+4z_{l+1}=0\ ,$$

which is a contradiction. Thus, $k+1<t$.

If $z_l\geqq 3z_l-4z_{l+1}$, then one would have

$$(y_{k+1}-y_{k+2})(3z_l-4z_{l+1})<y_{k+1}z\ ,$$

which contradicts (4.2). Therefore, $z_l<3z_l-4z_{l+1}$, and so

$$z_l>2z_{l+1}\ .$$

Hence,

$$c_{l+1}\geqq 3\ .$$

Since $D(X)$ is of type II, we have $c_{l+1}\geqq 4$.

Hence,

$$z_{l+1}\leqq\frac{z_l-1}{3}$$

and

$$3z_l-4z_{l+1}\geqq\frac{5z_l+4}{3}\ .$$

If $b_{k+2}\geqq 4$, then one would have

$$(y_{k+1}-y_{k+2})(3z_l-4z_{l+1})\geqq\frac{2y_{k+1}+1}{3}\cdot\frac{5z_l+4}{3}>1+y_{k+1}z_l\ ,$$

which contradicts (4.2). Thus, $b_{k+2}\leqq 3$.

Since $D(X)$ is of type II, we have $b_{k+2}=2$. Substituting $y_{k+1}=2y_{k+2}-y_{k+3}$ into (4.2), we obtain

(4.3) $$1+y_{k+2}z_l-(y_{k+2}-y_{k+3})(2z_l-4z_{l+1})=0\ .$$

If $k+2=t$, then $y_{k+2}=1$, $y_{k+3}=0$ and

$$1-z_l+4z_{l+1}=0\ .$$

Hence, one would have $c_{l+1}=5$, which contradicts the assumption that $D(X)$ is of type II. Thus, $k+2<t$.

If $z_l\geqq 2z_l-4z_{l+1}$, it would follow that

$$(y_{k+2}-y_{k+3})(2z_l-4z_{l+1})<y_{k+2}z_l\ ,$$

which contradicts (4.3). Therefore, $z_l<2z_l-4z_{l+1}$.

Hence, $z_l>4z_{l+1}$, which implies $c_{l+1}\geqq 5$. Since $D(X)$ is of type II, we have

$c_{l+1} \geqq 6$. Hence,

$$z_{l+1} \leqq \frac{z_l - 1}{5} ,$$

and so

$$z_l - 2z_{l+1} \geqq \frac{3z_l + 2}{5} .$$

If $y_{k+2} \geqq 2(y_{k+2} - y_{k+3})$, then one would have

$$(y_{k+2} - y_{k+3})2(z_l - 2z_{l+1}) < y_{k+2} z_l ,$$

which contradicts (4.3). Therefore $y_{k+2} < 2(y_{k+2} - y_{k+3})$, i. e., $y_{k+2} > 2y_{k+3}$.

It follows that $b_{k+3} \geqq 4$ under the assumption that $D(X)$ is of type II. Hence,

$$y_{k+2} - y_{k+3} \geqq \frac{2y_{k+2} + 1}{3} .$$

If $c_{l+1} \geqq 9$, then the inequality

$$z_{l+1} \leqq \frac{z_l - 1}{8}$$

would hold. Hence

$$(y_{k+2} - y_{k+3})2(z_l - 2z_{l+1}) \geqq \frac{2y_{k+2} + 1}{3} \cdot 2 \cdot \frac{3z_l + 1}{4} > y_{k+2} z_l + 1$$

(by $z_l \geqq 2$). This inequality is contradictory to (4.3). We obtain, therefore,

$$c_{l+1} = 6 \text{ or } 8 .$$

If $b_{k+3} \geqq 7$, then it would follow that

$$(y_{k+2} - y_{k+3})2(z_l - 2z_{l+1}) \geqq \frac{5y_{k+2} + 1}{6} \cdot 2 \cdot \frac{3z_l + 2}{5} > y_{k+2} z_l + 1$$

(by $y_{k+2} \geqq 2$). This contradicts (4.3). We obtain, therefore,

$$b_{k+3} = 6 \text{ or } 4 .$$

($\mathrm{A}\alpha$) The case in which $b_{k+3} = 6$.

Substituting $y_{k+2} = 6y_{k+3} - y_{k+4}$ into (4.3), we have

(4.4) $$1 + y_{k+3} z_l - (5y_{k+3} - y_{k+4})(z_l - 4z_{l+1}) = 0 .$$

If $c_{l+1} \geqq 7$, it would follow that

$$(5y_{k+3}-y_{k+4})(z_l-4z_{l+1})\geqq(4y_{k+3}+1)\frac{z_l+2}{3}>1+y_{k+3}z_l\ ,$$

contradictory to (4.4). We obtain, therefore, $c_{l+1}=6$.

Substituting $z_l=6z_{l+1}-z_{l+2}$ into (4.4), we have

(4.5) $$1+4y_{k+3}z_{l+1}-(4y_{k+3}-y_{k+4})(2z_{l+1}-z_{l+2})=0\ .$$

If $l+1=u$, then $z_{l+1}=1$, $z_{l+2}=0$ and

$$1+4y_{k+3}-2(4y_{k+3}-y_{k+4})=0\ ,$$

which is a contradiction. Thus, $l+1<u$.

If $c_{l+2}\geqq 3$, then one would have

$$(4y_{k+3}-y_{k+4})(2z_{l+1}-z_{l+2})\geqq(3y_{k+3}+1)\frac{3z_{l+1}+1}{2}>1+4y_{k+3}z_{l+1}\ ,$$

which contradicts (4.5). Therefore, $c_{l+2}=2$.

Substituting $z_{l+1}=2z_{l+2}-z_{l+3}$ into (4.5), we have

(4.6) $$1+2y_{k+3}z_{l+2}-(2y_{k+3}-y_{k+4})(3z_{l+2}-2z_{l+3})=0\ .$$

If $l+2=u$, then $z_{l+2}=1$, $z_{l+3}=0$ and

$$0=1-4y_{k+3}+3y_{k+4}<1-y_{k+3}\ ,$$

which is a contradiction. Thus, $l+2<u$.

If $c_{l+3}\geqq 3$, then

$$(2y_{k+3}-y_{k+4})(3z_{l+2}-2z_{l+3})\geqq(y_{k+3}+1)(2z_{l+2}+1)>1+2y_{k+3}z_{l+2}\ ,$$

which is contradictory to (4.6). We obtain, therefore,

$$c_{l+3}=2\ .$$

To sum up, it is shown that

$$\begin{bmatrix} b_{k+1}, b_{k+2}, b_{k+3} \\ c_{l+1},\ c_{l+2},\ c_{l+3}\end{bmatrix}=\begin{bmatrix}2, 2, 6\\ 6, 2, 2\end{bmatrix}=\bar{Z}\ ,$$

i.e., $\begin{bmatrix} b_1, \cdots, b_{k'} \\ c_1, \cdots, c_{l'}\end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l\end{bmatrix}\bar{Z}$ with $k'=k+3$ and $l'=l+3$.

Substituting $z_{l+2}=2z_{l+3}-z_{l+4}$ into (4.6), we obtain finally

$$(Y|k', l')_{\mathrm{IV}}=(Y|k+3, l+3)_{\mathrm{IV}}=0\ .$$

(Aβ) The case in which $b_{k+3}=4$.

Substituting $y_{k+2}=4y_{k+3}-y_{k+4}$ into (4.3), we have

(4.7) $$1+y_{k+3}z_l-(3y_{k+3}-y_{k+4})(z_l-4z_{l+1})=0\ .$$

If $c_{l+1} \leqq 6$, then

$$(3y_{k+3}-y_{k+4})(z_l-4z_{l+1}) \leqq 3y_{k+3} \cdot \frac{z_l}{3} < 1+y_{k+3}z_l ,$$

which contradicts (4.7). We have, therefore, $c_{l+1}=8$.
Substituting $z_l=8z_{l+1}-z_{l+2}$ into (4.7), we obtain

$$1+4y_{k+3}z_{l+1}-(2y_{k+3}-y_{k+4})(4z_{l+1}-z_{l+2})=0 \ . \tag{4.8}$$

When one changes $(y_{k+3}, y_{k+4}, z_{l+1}, z_{l+2})$ into $(z_{l+1}, z_{l+2}, y_{k+3}, y_{k+4})$, (4.8) is converted into (4.5). The remaining part of the case (Aβ) is, therefore, reduced to the part following (4.5) in the case (Aα): we obtain

$$k+4<t, b_{k+4}=b_{k+5}=2 \quad \text{and} \quad (Y|k+5, l+1)_{\text{IV}}=0 \ .$$

Consequently,

$$\begin{bmatrix} b_{k+1}, b_{k+2}, b_{k+3}, b_{k+4}, b_{k+5} \\ c_{l+1} \end{bmatrix} = \begin{bmatrix} 2, 2, 4, 2, 2 \\ 8 \end{bmatrix} = \bar{T} \ .$$

Hence,

$$\begin{bmatrix} b_1, \cdots, b_{k'} \\ c_1, \cdots, c_{l'} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} \bar{T} \quad \text{with } k'=k+5 \text{ and } l'=l+1$$

and

$$(Y|k', l')_{\text{IV}}=0 \ .$$

(B) The case in which $c_{l+1}=2$.
Similarly to the case (A), one can show

$$c_{l+2}=2$$
$$b_{k+1}=6 \text{ or } 8$$
$$c_{l+3}=6 \text{ or } 4 \ .$$

(Bα) The case in which $c_{l+3}=6$.
It is shown as in the case (Aα) that

$$b_{k+1}=6 \quad \text{and} \quad b_{k+2}=b_{k+3}=2 \ .$$

Hence,

$$\begin{bmatrix} b_{k+1}, b_{k+2}, b_{k+3} \\ c_{l+1}, c_{l+2}, c_{l+3} \end{bmatrix} = \underline{Z} \ .$$

Thus,

$$\begin{bmatrix} b_1, \cdots, b_{k'} \\ c_1, \cdots, c_{l'} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} Z \quad \text{with } k'=k+3 \text{ and } l'=l+3 ,$$

and

$$(Y|k', l')_{\mathrm{IV}}=0 .$$

(Bβ) The case in which $c_{l+3}=4$.

As in (Aβ) one can show that

$$\begin{bmatrix} b_1, \cdots, b_{k'} \\ c_1, \cdots, c_{l'} \end{bmatrix} = \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix} T \quad \text{with } k'=k+1 \text{ and } l'=l+5 ,$$

and

$$(Y|k', l')_{\mathrm{IV}}=0 .$$

The proof of 5° being completed, the "only if" part of Theorem 1.1′ is proved. Q. E. D.

Remark 1. The uniqueness of decomposition of X is shown by the following two propositions:

(i) $MX=M'X'$ implies $M=M'$ and $X=X'$,
(ii) $JX=J'X'$ implies $J=J'$ and $X=X'$,
where M and M' are molecules, J and J' are joints, and X and X' are arrays. Both (i) and (ii) can be proved easily by the definitions of molecule and joint.

Remark 2. By a network induction scheme, one can observe that the number of vertices of a weighted graph of type II which appears as the resolution diagram of a Brieskorn singularity $(2, q, r)$ is a multiple of 8—which is a well-known fact.

§ 5. Proof of Theorem 1.2

In order to prove Theorem 1.2, we prepare some notations and a lemma. Define eight 2×2 matrices as follows:

$$B_{\mathrm{I}} := \begin{pmatrix} -8 & 8 \\ -6 & 8 \end{pmatrix}, \quad B_{\mathrm{II}} := \begin{pmatrix} -10 & 8 \\ -8 & 8 \end{pmatrix}, \quad B_{\mathrm{III}} := \begin{pmatrix} -8 & 8 \\ -8 & 10 \end{pmatrix}, \quad B_{\mathrm{IV}} := \begin{pmatrix} -8 & 6 \\ -8 & 8 \end{pmatrix},$$

$$C_{\mathrm{I}} := \begin{pmatrix} 8 & -8 \\ 6 & -8 \end{pmatrix}, \quad C_{\mathrm{II}} := \begin{pmatrix} 8 & -8 \\ 8 & -10 \end{pmatrix}, \quad C_{\mathrm{III}} := \begin{pmatrix} 10 & -8 \\ 8 & -8 \end{pmatrix}, \quad C_{\mathrm{IV}} := \begin{pmatrix} 8 & -6 \\ 8 & -8 \end{pmatrix}$$

Then, 4×4 matrices A_{I}, A_{II}, A_{III} and A_{IV} are defined by

$$A_* = \begin{pmatrix} B_* & 3E \\ -5E & C_* \end{pmatrix} \qquad *=\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV},$$

where E denotes the 2×2 unit matrix.

Let $Y=\begin{bmatrix} y_0, \cdots, y_{t+1} \\ z_0, \cdots, z_{u+1} \end{bmatrix} \in S$ and $Y'=\begin{bmatrix} y'_0, \cdots, y'_{u-1} \\ z'_0, \cdots, z'_{t+3} \end{bmatrix} \in S$.

For each pair (k, l) of integers with $1 \leqq k \leqq t$ and $1 \leqq l \leqq u-2$, we define four vectors denoted by $\langle Y, Y'|k, l\rangle_{\mathrm{I}}$, $\langle Y, Y'|k, l\rangle_{\mathrm{II}}$, $\langle Y, Y'|k, l\rangle_{\mathrm{III}}$ and $\langle Y, Y'|k, l\rangle_{\mathrm{IV}}$:

$$\langle Y, Y'|k, l\rangle_* := \begin{pmatrix} y'_l \\ y'_{l+1} \\ z'_k \\ z'_{k+1} \end{pmatrix} - A_* \begin{pmatrix} y_k \\ y_{k+1} \\ z_l \\ z_{l+1} \end{pmatrix} \qquad *=\mathrm{I, II, III, IV}.$$

LEMMA 5.1. *Let* $X=\begin{bmatrix} b_1, \cdots, b_t \\ c_1, \cdots, c_u \end{bmatrix} \in S$ *be written as* (1.3)

and

$$X'=\begin{bmatrix} b'_1, \cdots, b'_{u-2} \\ c'_1, \cdots, c'_{t+2} \end{bmatrix} \in S \quad as \quad X'=H \cdot {}_t(M_1 J_1 \cdots M_{\nu-1} J_{\nu-1} M_\nu) L \ .$$

Define

$$Y=\begin{bmatrix} y_0, \cdots, y_{t+1} \\ z_0, \cdots, z_{u+1} \end{bmatrix} \in S \quad by$$

$$\begin{cases} y_{t+1}=0\,, & y_t=1\,, & y_{k-1}=b_k y_k - y_{k+1} & (1 \leqq k \leqq t)\ , \\ z_{u+1}=0\,, & z_u=1\,, & z_{l-1}=c_l z_l - z_{l+1} & (1 \leqq l \leqq u)\ , \end{cases}$$

and

$$Y'=\begin{bmatrix} y'_0, \cdots, y'_{u-1} \\ z'_0, \cdots, z'_{t+3} \end{bmatrix} \in S \quad by$$

$$\begin{cases} y'_{u-1}=0\,, & y'_{u-2}=1\,, & y'_{l-1}=b'_l y'_l - y'_{l+1} & (1 \leqq l \leqq u-2)\ , \\ z'_{t+3}=0\,, & z'_{t+2}=1\,, & z'_{k-1}=c'_k z'_k - z'_{k+1} & (1 \leqq k \leqq t+2)\ . \end{cases}$$

Then, the following four propositions hold:

I. *if a subarray* $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}$ *of* X *ends with* M, *then* $\langle Y, Y'|k, l\rangle_{\mathrm{I}}=0$ *(zero vector)*;

II. *if a subarray* $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}$ *of* X *ends with* p, *then* $\langle Y, Y'|k, l\rangle_{\mathrm{II}}=0$;

III. *if a subarray* $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}$ *of* X *ends with* $\bar{p}$, *then* $\langle Y, Y'|k, l\rangle_{\mathrm{III}}=0$;

IV. *if a subarray* $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}$ *of* X *ends with* J, *then* $\langle Y, Y'|k, l\rangle_{\mathrm{IV}}=0$.

Proof. We prove this lemma by resorting to a network induction; it suffices to show the following eight assertions:

1° $\langle Y, Y'|t, u-2\rangle_{\mathrm{I}}=0$;

2° if $\langle Y, Y'|k, l\rangle_{\mathrm{I}}=0$ and $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_{l-1} \end{bmatrix}\underline{e}$, then $\langle Y, Y'|k, l-1\rangle_{\mathrm{II}}=0$;

3° if $\langle Y, Y'|k, l\rangle_{\mathrm{I}}=0$ and $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_{k-1} \\ c_1, \cdots, c_l \end{bmatrix}\bar{e}$, then $\langle Y, Y'|k-1, l\rangle_{\mathrm{III}}=0$;

4° if $\langle Y, Y'|k, l\rangle_{\mathrm{II}}=0$ and $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_{k-1} \\ c_1, \cdots, c_l \end{bmatrix}\bar{e}$, then $\langle Y, Y'|k-1, l\rangle_{\mathrm{IV}}=0$;

5° if $\langle Y, Y'|k, l\rangle_{\mathrm{II}}=0$ and $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_{k-1} \\ c_1, \cdots, c_{l-8n+1} \end{bmatrix}p_n$, for some $n\geqq 1$, then $\langle Y, Y'|k-1, l-8n+1\rangle_{\mathrm{III}}=0$;

6° if $\langle Y, Y'|k, l\rangle_{\mathrm{III}}=0$ and $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_{k-8n+1} \\ c_1, \cdots, c_{l-1} \end{bmatrix}\bar{p}_n$, for some $n\geqq 1$, then $\langle Y, Y'|k-8n+1, l-1\rangle_{\mathrm{II}}=0$;

7° if $\langle Y, Y'|k, l\rangle_{\mathrm{III}}=0$ and $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_{l-1} \end{bmatrix}\underline{e}$, then $\langle Y, Y'|k, l-1\rangle_{\mathrm{IV}}=0$;

8° if $\langle Y, Y'|k, l\rangle_{\mathrm{IV}}=0$ and $\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_{k'} \\ c_1, \cdots, c_{l'} \end{bmatrix}J$, where J is a joint, then $\langle Y, Y'|k', l'\rangle_{\mathrm{I}}=0$.

(Proof of 1°) Since

$$\begin{pmatrix} y_t \\ y_{t+1} \\ z_{u-2} \\ z_{u-1} \end{pmatrix}=\begin{pmatrix} y'_{u-2} \\ y'_{u-1} \\ z'_t \\ z'_{t+1} \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix},$$

a straightforward calculation shows

$$\langle Y, Y'|t, u-2\rangle_{\mathrm{I}}=0 .$$

(Proof of 2° through 8°) The hypotheses of 2° through 8° are of the form:

$$\langle Y, Y'|k, l\rangle_{*}=0 \quad \text{and} \quad \begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}=\begin{bmatrix} b_1, \cdots, b_i \\ c_1, \cdots, c_j \end{bmatrix}Q ,$$

where Q is a subarray of X and $*=\mathrm{I}, \mathrm{II}, \mathrm{III}$ or IV. For an array $Q=\begin{bmatrix} b_{i+1}, \cdots, b_k \\ c_{j+1}, \cdots, c_l \end{bmatrix}\in S$ we defined in §3

$$\begin{cases} U(Q):=\begin{pmatrix} b_{i+1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_{i+2} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} b_k & -1 \\ 1 & 0 \end{pmatrix}, \\ V(Q):=\begin{pmatrix} c_{j+1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_{j+2} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_l & -1 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

Then,

$$\begin{pmatrix} y_i \\ y_{i+1} \end{pmatrix}=U(Q)\begin{pmatrix} y_k \\ y_{k+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_j \\ z_{j+1} \end{pmatrix}=V(Q)\begin{pmatrix} z_l \\ z_{l+1} \end{pmatrix} .$$

On the other hand, since

$$\begin{bmatrix} b_1', \cdots, b_l' \\ c_1', \cdots, c_k' \end{bmatrix} = \iota\left(\begin{bmatrix} b_1, \cdots, b_k \\ c_1, \cdots, c_l \end{bmatrix}\right) \quad \text{for } 1 \leqq k \leqq t \text{ and } 1 \leqq l \leqq u-2 \ ,$$

we have

$$\begin{pmatrix} y_j' \\ y_{j+1}' \end{pmatrix} = V(Q)\begin{pmatrix} y_l' \\ y_{l+1}' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_i' \\ z_{i+1}' \end{pmatrix} = U(Q)\begin{pmatrix} z_k' \\ z_{k+1}' \end{pmatrix} .$$

Hence, if $\langle Y, Y'|k, l\rangle_* = 0$, then

$$\begin{pmatrix} y_j' \\ y_{j+1}' \\ z_i' \\ z_{i+1}' \end{pmatrix} = \begin{pmatrix} V(Q) & 0 \\ 0 & U(Q) \end{pmatrix} \begin{pmatrix} y_l' \\ y_{l+1}' \\ z_k' \\ z_{k+1}' \end{pmatrix} = \begin{pmatrix} V(Q) & 0 \\ 0 & U(Q) \end{pmatrix} A_* \begin{pmatrix} y_k \\ y_{k+1} \\ z_l \\ z_{l+1} \end{pmatrix}$$

$$= \begin{pmatrix} V(Q) & 0 \\ 0 & U(Q) \end{pmatrix} A_* \begin{pmatrix} U(Q)^{-1} & 0 \\ 0 & V(Q)^{-1} \end{pmatrix} \begin{pmatrix} y_i \\ y_{i+1} \\ z_j \\ z_{j+1} \end{pmatrix} .$$

The conclusions of 2° through 8° are of the form:

$$\langle Y, Y'|i, j\rangle_\# = 0 \quad (\# = \text{I, II, III or IV}) .$$

Thus, it suffices to show that

$$\begin{pmatrix} V(Q) & 0 \\ 0 & U(Q) \end{pmatrix} A_* \begin{pmatrix} U(Q)^{-1} & 0 \\ 0 & V(Q)^{-1} \end{pmatrix} = A_\# .$$

Furthermore, since we have

$$\begin{pmatrix} V(Q) & 0 \\ 0 & U(Q) \end{pmatrix} A_* \begin{pmatrix} U(Q)^{-1} & 0 \\ 0 & V(Q)^{-1} \end{pmatrix} = \begin{pmatrix} V(Q)B_*U(Q)^{-1} & 3E \\ -5E & U(Q)C_*V(Q)^{-1} \end{pmatrix}$$

and

$$B_*C_* = C_*B_* = -16E \quad (* = \text{I, II, III, IV}) ,$$

we have only to prove

$$V(Q)B_*U(Q)^{-1} = B_\# .$$

Here is a table of Q, $*$, $\#$ and the equality to be proved for each of 2° through 8°:

	$*$	Q	$\#$	the equality to be proved	
2°	I	$\underset{\sim}{e}$	II	$V(\underset{\sim}{e})B_{\text{I}}U(\underset{\sim}{e})^{-1} = B_{\text{II}}$	(5.1)
3°	I	$\bar{e}$	III	$V(\bar{e})B_{\text{I}}U(\bar{e})^{-1} = B_{\text{III}}$	(5.2)
4°	II	$\bar{e}$	IV	$V(\bar{e})B_{\text{II}}U(\bar{e})^{-1} = B_{\text{IV}}$	(5.3)
5°	II	p_n	III	$V(p_n)B_{\text{II}}U(p_n)^{-1} = B_{\text{III}}$	(5.4)
6°	III	$\bar{p}_n$	II	$V(\bar{p}_n)B_{\text{III}}U(\bar{p}_n)^{-1} = B_{\text{II}}$	(5.5)
7°	III	$\underset{\sim}{e}$	IV	$V(\underset{\sim}{e})B_{\text{III}}U(\underset{\sim}{e})^{-1} = B_{\text{IV}}$	(5.6)
8°	IV	J	I	$V(J)B_{\text{IV}}U(J)^{-1} = B_{\text{I}}$	(5.7)

[Proof of (5.1)] A straightforword calculation using $U(\underline{e})=E$ and $V(\underline{e})=G$ (where G denotes $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ as in the proof of Theorem 1.1′) shows (5.1).
[Proof of (5.2)] (5.1) implies

$$U(\underline{e})C_{\mathrm{I}}V(\underline{e})^{-1}=C_{\mathrm{II}}$$

as stated earlier. Since $C_{\mathrm{I}}=-B_{\mathrm{I}}$, $C_{\mathrm{II}}=-B_{\mathrm{III}}$ and $U(\iota(Q))=V(Q)$ for any $Q\in S$, we have

$$\begin{aligned} V(\bar{e})B_{\mathrm{I}}U(\bar{e})^{-1} &= U(\iota(\bar{e}))(-C_{\mathrm{I}})V(\iota(\bar{e}))^{-1} \\ &= -U(\underline{e})C_{\mathrm{I}}V(\underline{e})^{-1} \\ &= -C_{\mathrm{II}} \\ &= B_{\mathrm{III}}\ , \end{aligned}$$

which is no other than (5.2).
[Proof of (5.3)] A straightforward calculation shows (5.3).
[Proof of (5.6)] Since $C_{\mathrm{IV}}=-B_{\mathrm{IV}}$ and $C_{\mathrm{II}}=-B_{\mathrm{III}}$, (5.6) is to (5.3) what (5.2) is to (5.1). Hence, (5.6) is proved similarly to (5.2).
[Proof of (5.4)] Since $U(\underline{p}_n)=\begin{pmatrix} 2n+2 & -1 \\ 1 & 0 \end{pmatrix}$ and $V(\underline{p}_n)=G^{8n-1}=\begin{pmatrix} 8n & -8n+1 \\ 8n-1 & -8n+2 \end{pmatrix}$, (5.4) is verified directly.
[Proof of (5.5)] It is reduced to that of (5.4) similarly to the cases (5.2) and (5.6) because $C_{\mathrm{II}}=-B_{\mathrm{III}}$ and $C_{\mathrm{III}}=-B_{\mathrm{II}}$.
[Proof of (5.7)] We have only to show the equality in the cases in which $J=\bar{Z}$ or $\bar{T}$, the remaining cases in which $J=\underline{Z}$ or $\underline{T}$ being reduced to the former cases (respectively). Since we have

$$\begin{aligned} U(\bar{Z}) &= G^2\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 16 & -3 \\ 11 & -2 \end{pmatrix}, \\ V(\bar{Z}) &= \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}G^2=\begin{pmatrix} 16 & -11 \\ 3 & -2 \end{pmatrix}, \\ U(\bar{T}) &= G^2\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}G^2=\begin{pmatrix} 24 & -17 \\ 17 & -12 \end{pmatrix}, \\ V(\bar{T}) &= \begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

(5.7) is verified straightforwardly. Q. E. D.

Proof of Theorem 1.2. Suppose that X, X', Y and Y' are elements of S as in the assumption of Lemma 5.1. Since the subarray $\begin{bmatrix} b_1 \\ c_1 \end{bmatrix}$ of X ends with J, we obtain

$$\langle Y, Y'|1, 1\rangle_{\mathrm{IV}}=0$$

by Lemma 5.1. A similar argument as in the proof of Lemma 5.1 shows

(5.8) $$\langle Y, Y'|0,0\rangle_0=0\ ,$$

where

$$\langle Y, Y'|0,0\rangle_0 := \begin{pmatrix} y_0' \\ y_1' \\ z_0' \\ z_1' \end{pmatrix} - A_0 \begin{pmatrix} y_0 \\ y_1 \\ z_0 \\ z_1 \end{pmatrix},$$

$$A_0 := \begin{pmatrix} B_0 & 3E \\ -5E & C_0 \end{pmatrix},\quad B_0 := \begin{pmatrix} -4 & 0 \\ -6 & 4 \end{pmatrix},\quad C_0 := \begin{pmatrix} 4 & 0 \\ 6 & -4 \end{pmatrix},$$

because of the equality

$$V(H)B_{\mathrm{IV}}U(H)^{-1}=B_0\ .$$

Since $y_0=q$, $z_0=r$, $y_0'=q'$ and $z_0'=r'$, (5.8) reads as

$$\begin{pmatrix} q' \\ y_1' \\ r' \\ z_1' \end{pmatrix} = \begin{pmatrix} -4 & 0 & 3 & 0 \\ -6 & 4 & 0 & 3 \\ -5 & 0 & 4 & 0 \\ 0 & -5 & 6 & -4 \end{pmatrix} \begin{pmatrix} q \\ y_1 \\ r \\ z_1 \end{pmatrix}.$$

Thus, we obtain the desired result. Q. E. D.

Proof of Theorem 1.1. Since Theorem 1.1′ has already been proved, nothing remains but to show that, if $X \in S$ is written as (1.3), then $q<r$. Let $X \in S$ be written as (1.3). Define X' by

$$X' := H\cdot\iota(M_1J_1\cdots M_{\nu-1}J_{\nu-1}M_\nu)L\ .$$

Then, by Theorem 1.1′, $D(X')$ is a weighted graph of type II which appears as the resolution diagram of a Brieskorn singularity $(2, q', r')$ where q' and r' are coprime odd integers greater than 2. Theorem 1.2 says that

$$q'=-4q+3r\ .$$

Since $q'>0$, we have

$$q<\frac{3}{4}r<r\ ,$$

which completes the proof. Q. E. D.

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