Cohomology Groups in the Cartesian Product of Countably Infinite Number of Complex Planes

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Introduction.

The properties of holomorphic functions on the Cartesian product of countably infinite number of complex planes, which will be denoted by $\Pi\boldsymbol{C}$, have been discussed by many authors. Among others, Rickart proved in [9] that for any polynomially convex compact subset Ω of $\Pi\boldsymbol{C}$ $H^k(\Omega, \mathcal{O})=0$ for $k\geqq 1$ by using a theorem on sheaves represented as the inductive limit of some sheaves. On the other hand, Dineen proved in [3] that for any open set U in $\Pi\boldsymbol{C}, H^1(U, \mathcal{O})\neq 0$ by showing that the first Cousin problem is not solvable on U.

In this paper, we will introduce a fine subsheaf $\mathcal{E}_b$ of the sheaf of infinitely differentiable functions on $\Pi\boldsymbol{C}$. Then, we will show that we have a fine resolution of the sheaf $\mathcal{O}$ on $\Pi\boldsymbol{C}$:

$$0\longrightarrow\mathcal{O}\longrightarrow\mathcal{E}_b^{0,0}\longrightarrow\mathcal{E}_b^{0,1}\longrightarrow\mathcal{E}_b^{0,2}\longrightarrow\cdots.$$

By using this resolution, we will show that for some convex subset Q_n of $\Pi\boldsymbol{C}$, $H^1(Q_n, \mathcal{O})\neq 0$ and $H^k(Q_n, \mathcal{O})=0$ for $k\geqq 2$.

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§1. Topological properties of $\Pi\boldsymbol{C}$.

We recall some topological properties of $\Pi\boldsymbol{C}$. We note a fundamental system of neighborhoods and compact sets in $\Pi\boldsymbol{C}$. Let p_n be the projection from $\Pi\boldsymbol{C}$ onto $\boldsymbol{C}^n$ and p_n^{n+1} the projection from $\boldsymbol{C}^{n+1}$ onto $\boldsymbol{C}^n$.

(1.1) A fundamental system of neighborhoods of a point in $\Pi\boldsymbol{C}$ consists of the open sets of the form

$$\left\{\prod_{i=1}^{n}\left\{z_i\in\boldsymbol{C}_i\,;\,|z_i|<\frac{1}{n}\right\}\times\prod_{i=n+1}^{\infty}\boldsymbol{C}_i\right\},\ n=1,2,3,\cdots,$$

where $(z_i)\in\Pi\boldsymbol{C}$.

(1.2) The family of the sets $\left\{\prod_{i=1}^{\infty} K_i\right\}$ forms a basis of compact sets in $\Pi\boldsymbol{C}$, where K_i runs over compact sets in $\boldsymbol{C}$.

(1.3) Let $P=\{(z_i)\in\Pi\boldsymbol{C}\,;\,|z_i|\leqq r_i\}$. Then, P has a fundamental system of neighborhoods of the following form:

$$U_n=\prod_{i=1}^{n}\left\{z_i\in\boldsymbol{C}_i\,;\,|z_i|<r_i+\frac{1}{n}\right\}\times\prod_{i=n+1}^{\infty}\boldsymbol{C}_i\,.$$

Let U be an arbitrary open set in $\Pi\boldsymbol{C}$. We denote by $\mathcal{O}(U)$ the space of all holomorphic functions on U. We can easily see that the presheaf $\{\mathcal{O}(U)\}$ constitutes a sheaf on $\Pi\boldsymbol{C}$, so that we denote it by $\mathcal{O}$.

In the sequel, we will use the abbreviation: $U^n=p_n(U)$. We will denote by $\mathcal{O}_n$ the sheaf of germs of holomorphic functions on $\boldsymbol{C}^n$. The mappings u_{n+1}^n: $\mathcal{O}_n(U^n)\to\mathcal{O}_{n+1}(U^{n+1})$ and $u_n:\mathcal{O}_n(U)\to\mathcal{O}(U)$ are defined by $u_{n+1}^n(f)=f\circ p_n^{n+1}$ and $u_n(f)=f\circ p_n$, respectively. These mappings are obviously injective.

PROPOSITION 1.1. ([4]) *Let U be an open set with a finite number of connected components in $\Pi\boldsymbol{C}$. Then, we have the following algebraic isomorphism:*

$$\mathcal{O}(U)\xleftarrow{\sim}\varinjlim_{n}\{\mathcal{O}_n(U^n)\,;\,u_{n+1}^n\}\,.$$

This implies in particular that an element of $\mathcal{O}(U)$ depends on only a finite number of variables.

Errata in [4]: Proposition 2.5 in [4] should be replaced by our Proposition 1.1. There we forgot the assumption of the finiteness of the connected components. Also Prof. Dineen kindly pointed out that the isomorphism holds only in the algebraic sense. The proof in [4] is valid for this corrected form.

Put $\tilde{\mathcal{O}}(U)=\varinjlim_{n}\mathcal{O}_n(U^n)$. Then, we remark that the presheaf $\{\tilde{\mathcal{O}}(U)\}$ does not define a sheaf on $\Pi\boldsymbol{C}$. In fact, let U be an open set with an infinite number of connected components, i.e., $U=\bigcup_{i=1}^{\infty}U_i$, $U_i\cap U_j=\phi$ $(i\neq j)$. Then, we take the holomorphic function $f_i(z)=z_i$ on U_i for every $i>0$ and define the function $f(z)$ on U such that $f(z)=f_i(z)$ if $z\in U_i$. The function f does not belong to $\tilde{\mathcal{O}}(U)$. The sheaf $\mathcal{O}$ is associated with this presheaf.

The maximum principle and the principle of analytic continuation hold on $\Pi\boldsymbol{C}$ (see for example [1]). Now, we can show the following.

PROPOSITION 1.2. *Let U be a connected open set in $\Pi\boldsymbol{C}$ and K a compact subset of U. Then, for any function $f\in\mathcal{O}(U\backslash K)$ there exists a unique function $g\in\mathcal{O}(U)$ such that* $f=g$ on $U\backslash K$.

[Proof] Since $U\backslash K$ is a connected open set, by Proposition 1.1 there exists an integer k such that $f\in\mathcal{O}_k(p_k(U\backslash K))$. As $p_k(K)$ is compact and $p_k(U\backslash K)$ is open in $\boldsymbol{C}^k$, by Hartogs' theorem there exists a unique function $g_k\in\mathcal{O}_k(p_k(U))$ such that $f=g_k$ on $p_k(U)\backslash p_k(K)$. For any $m\geqq k$ we can find the function g_m as in the above discussion. But in view of the principle of analytic continuation $g_m=g_{m'}$ holds on $p_{m'}(U)$ for any m and m', $m\geqq m'\geqq k$. Thus, there exists a unique function $g\in\mathcal{O}(U)$ such that $f=g$ on $U\backslash K$. [Q.E.D.]

§ 2. Resolution of the sheaf $\mathcal{O}$.

In this section we will construct a fine resolution of the sheaf $\mathcal{O}$. Let $\mathcal{E}_n{}^{0,q}$ (resp. $\mathcal{E}_n$) denote the sheaf of germs of infinitely differentiable $(0, q)$ forms (resp. functions) on $\boldsymbol{C}^n$. Let $u_{n+1}^n : \mathcal{E}_n{}^{0,q}(U^n)\to\mathcal{E}_{n+1}^{0,q}(U^{n+1})$ be defined by

$$u_{n+1}^n\Big(\sum_{i_1<\cdots<i_q} f_{i_1,\dots,i_q} d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q}\Big) = \sum_{i_1<\cdots<i_q} f_{i_1,\dots,i_q}\circ p_n^{n+1} d\bar{z}_{i_1}\wedge\cdots\wedge d\bar{z}_{i_q}.$$

Put

$$\mathcal{E}_a{}^{0,q}(U)=\varinjlim_n \mathcal{E}_n{}^{0,q}(U^n).$$

The presheaf $\{\mathcal{E}_a{}^{0,q}(U)\}$ does not constitute a sheaf. Thus, we denote by $\mathcal{E}_b{}^{0,q}$ the sheaf associated with this presheaf on $\Pi\boldsymbol{C}$. The sheaf $\mathcal{E}_b$ is the sheaf associated with the presheaf $\left\{\varinjlim_n \mathcal{E}_n(U^n)\right\}$. We remark on the space $\mathcal{E}(U)$ of all infinitely differentiable functions on U which are, by definition, infinitely differentiable on every finite dimensional subspace and continuous on U.

While the isomorphism in Proposition 1.1 holds for the case of holomorphic functions, we have

$$\mathcal{E}(U) \not\leftarrow \mathcal{E}_b(U).$$

In fact, for example

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{2^k}e^{-x_k^2}$$

belongs to $\mathcal{E}(\Pi\boldsymbol{R})$, but not to $\mathcal{E}_b(\Pi\boldsymbol{R})$, because it depends on the whole variables on any small neighborhood of the origin.

Now, we return to the sheaf $\mathcal{E}_b{}^{0,q}$.

PROPOSITION 2.1. *The sheaf $\mathcal{E}_b{}^{0,q}$ is a fine sheaf on* $\Pi\boldsymbol{C}$.

We need the following lemma to prove this proposition.

LEMMA 2.2. *Let F be an arbitrary closed set in $\Pi\boldsymbol{R}$. Then, for any open neighborhood X of F there exists a function $f\in\mathcal{E}_b(\Pi\boldsymbol{R})$ such that $f=1$ on F and $f=0$ on $\Pi\boldsymbol{R}\backslash X$.*

[Proof] *Part* 1. First, we construct a locally finite covering of F. We assume without loss of generality that F contains the origin. Let $\mathfrak{U}$ be an open covering of F whose element is of the form:

$$U=\Delta^n(w,r)=D(w_1,r_1)\times D(w_2,r_2)\times\cdots\times D(w_n,r_n)\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i$$

where $D(w_i,r_i)=\{z\in\boldsymbol{R}\,;\,|z-w_i|<r_i\}$. We use the notation:

$$D^n((w_1,\cdots,w_n),(r_1,\cdots,r_n))=D(w_1,r_1)\times\cdots\times D(w_n,r_n).$$

Put

$$\mathfrak{U}_1=\{U\in\mathfrak{U}\,;\,U\cap\boldsymbol{R}\neq\phi\}.$$

Then, $\{D(w_{\alpha,1},r_{\alpha,1})\,;\,U_\alpha=\Delta^n(w_\alpha,r_\alpha)\in\mathfrak{U}_1\}$ covers $F\cap\boldsymbol{R}$. Now, we choose a refinement $\mathfrak{U}'_1=\{D(w_{\beta,1},r_{\beta,1})\}$ of the covering $\{D(w_{\alpha,1},r_{\alpha,1})\,;\,U_\alpha\in\mathfrak{U}_1\}$ which satisfies the following conditions:

(i) There exists $c>0$ such that $r_{\beta,1}<c$ for every β.

(ii) $\mathfrak{U}'_1$ is a locally finite covering of $F\cap\boldsymbol{R}$.

(iii) There exists $r'_{\beta,1}$, $0<r'_{\beta,1}<r_{\beta,1}$ such that

  (1) $\mathfrak{U}''_1=\{D(w_{\beta,1},r'_{\beta,1})\}$ also covers $F\cap\boldsymbol{R}$,

  (2) there exists $d>0$ such that for each pair $D(w_{\beta,1},r_{\beta,1})$, $D(w_{\beta',1},r_{\beta',1})(\beta\neq\beta')$, $D(w_{\beta,1},r_{\beta,1})\cap D(w_{\beta',1},r_{\beta',1})\neq\phi$, we have $D(w_{\beta,1},r'_{\beta,1}-d)\cap D(w_{\beta',1},r_{\beta',1})=\phi$.

For each $D(w_{\beta,1},r_{\beta,1})\in\mathfrak{U}'_1$, let $\Delta^n(w_\alpha,r_\alpha)\in\mathfrak{U}_1$ be such that n is the minimum integer among

$$\{m>0\,;\,\Delta^m(w_\alpha,r_\alpha)\supset D(w_{\beta,1},r_{\beta,1})\,;\,\Delta^m(w_\alpha,r_\alpha)\in\mathfrak{U}_1\}.$$

Set

$$\tilde{U}_\beta^{(1)}=D^n((w_{\beta,1},0,\cdots,0),(r_{\beta,1},\cdots,r_{\beta,1}))\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i\cap\Delta^n(w_\alpha,r_\alpha).$$

Then, we can find $\tilde{r}_{\beta,i}$, $0<\tilde{r}_{\beta,i}\leqq r_{\beta,i}$, $i=1,\cdots,n$ such that

$$\tilde{U}_\beta^{(1)}=D^n((w_{\beta,1},0,\cdots,0),(\tilde{r}_{\beta,1},\tilde{r}_{\beta,2},\cdots,\tilde{r}_{\beta,n}))\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i.$$

Choosing $\tilde{r}'_{\beta,i}$ such that $\tilde{r}_{\beta,1}-(d/2^{i+1})<\tilde{r}'_{\beta,i}<\tilde{r}_{\beta,i}$ for $2\leqq i\leqq n$, we put

$$\tilde{U}'^{(1)}_\beta=D^n((w_{\beta,1},0,\cdots,0),(\tilde{r}'_{\beta,1},\tilde{r}'_{\beta,2},\cdots,\tilde{r}'_{\beta,n}))\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i,$$

where $\tilde{r}'_{\beta,1}=r'_{\beta,1}$. Now, we set

$$\tilde{\mathfrak{U}}_1=\{\tilde{U}_\beta^{(1)}\} \quad \text{and} \quad \tilde{\mathfrak{U}}_1'=\{\tilde{U}_\beta'^{(1)}\}.$$

Next, we put

$$\mathfrak{U}_2=\{U\in\mathfrak{U}\,;\,U\cap \boldsymbol{R}^2\neq\phi\}.$$

Then, we choose a refinement $\mathfrak{U}_2'=\{D^2((w_{\beta,1},w_{\beta,2}),(r_{\beta,1},r_{\beta,1}))\}$ of the covering $\{D^2((w_{\alpha,1},w_{\alpha,2}),(r_{\alpha,1},r_{\alpha,2}))\,;\,U_\alpha\in\mathfrak{U}_2\}$ of $(F\backslash\bigcup_\beta\tilde{U}_\beta'^{(1)})\cap\boldsymbol{R}^2$ which satisfies the following conditions:

(i) $r_{\beta,1}<c/2$ for every β.

(ii) $\tilde{\mathfrak{U}}_1\cup\mathfrak{U}_2'$ is a locally finite covering of $F\cap\boldsymbol{R}^2$.

(iii) There exists $r_{\beta,1}'$, $0<r_{\beta,1}'<r_{\beta,1}$ such that

(1) $\tilde{\mathfrak{U}}_1'\cup\{D^2((w_{\beta,1},w_{\beta,2}),(r_{\beta,1}',r_{\beta,2}'))\}$ also covers $F\cap\boldsymbol{R}^2$,

(2) for each pair of different elements

$$V=D^2((w_1,w_2),(r_1,r_2))\in\tilde{\mathfrak{U}}_1|_{\boldsymbol{R}^2}\cup\mathfrak{U}_2' \text{ and } V'=D^2((w_{\beta,1},w_{\beta,2}),(r_{\beta,1},r_{\beta,1}))\in\mathfrak{U}_2'$$

such that $V\cap V'=\phi$, there exists ε, $0<\varepsilon<\min\{r_1',r_2',d/2\}$ such that $D((w_1,w_2),(r_1'-\varepsilon,r_2'-\varepsilon))\cap V'=\phi$, where if $V\in\mathfrak{U}_2'$ then $r_1=r_2$ and $r_1'=r_2'$.

For each $D^2((w_{\beta,1},w_{\beta,2}),(r_{\beta,1},r_{\beta,1}))\in\mathfrak{U}_2'$ let $\Delta^n(w_\alpha,r_\alpha)\in\mathfrak{U}_2$ be such that n is the minimum integer among

$$\{m>0\,;\,\Delta^m((w_\alpha,r_\alpha))\supset D^2((w_{\beta,1},w_{\beta,2})(r_{\beta,1},r_{\beta,1}))\,;\,\Delta^m(w_\alpha,r_\alpha)\in\mathfrak{U}_2\}\,.$$

Set

$$\tilde{U}_\beta^{(2)}=D^n((w_{\beta,1},w_{\beta,2},0,\cdots,0),(r_{\beta,1},\cdots,r_{\beta,1}))\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i\cap\Delta^n(w_\alpha,r_\alpha)\,.$$

Then, we can find $\tilde{r}_{\beta,i}$, $0<\tilde{r}_{\beta,i}\leqq r_{\beta,1}$, $i=1,2,\cdots,n$ such that

$$\tilde{U}_\beta^{(2)}=D^n((w_{\beta,1},w_{\beta,2},0,\cdots,0),(\tilde{r}_{\beta,1},\tilde{r}_{\beta,2},\cdots,\tilde{r}_{\beta,n}))\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i\,.$$

Choosing $\tilde{r}_{\beta,i}'$ such that $\tilde{r}_{\beta,i}-(d/2^{i+2})<\tilde{r}_{\beta,i}'<\tilde{r}_{\beta,i}$ for $3\leqq i\leqq n$, we put

$$\tilde{U}_\beta'^{(2)}=D^n((w_{\beta,1},w_{\beta,2},0,\cdots,0),(\tilde{r}_{\beta,1}',\tilde{r}_{\beta,2}',\cdots,\tilde{r}_{\beta,n}'))\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i\,,$$

where $\tilde{r}_{\beta,i}'=r_{\beta,1}'$ $(i=1,2)$. Now, we set

$$\tilde{\mathfrak{U}}_2=\{\tilde{U}_\beta^{(2)}\} \quad \text{and} \quad \tilde{\mathfrak{U}}_2'=\{\tilde{U}_\beta'^{(2)}\}\,.$$

Suppose that we have $\tilde{\mathfrak{U}}_i=\{\tilde{U}_\beta^{(i)}\}$ and $\tilde{\mathfrak{U}}_i'=\{\tilde{U}_\beta'^{(i)}\}$ for $i\leqq k-1$. Now, we put

$$\mathfrak{U}_k=\{U\in\mathfrak{U}\,;\,U\cap\boldsymbol{R}^k\neq\phi\}\,.$$

Then, we choose a refinement $\mathfrak{U}_k'=\{D^k((w_{\beta,1},\cdots,w_{\beta,k}),(r_{\beta,1},\cdots,r_{\beta,1}))\}$ of the covering $\{D^k((w_{\alpha,1},\cdots,w_{\alpha,k}),(r_{\alpha,1},\cdots,r_{\alpha,k})\,;\,U_\alpha\in\mathfrak{U}_k\}$ of $(F\backslash\bigcup_{1\leqq i\leqq k-1,\beta}\tilde{U}_\beta'^{(i)})\cap\boldsymbol{R}^k$ which satisfies the following conditions:

(i) $0<r_{\beta,1}<c/k$ for every β.

(ii) $(\tilde{\mathfrak{U}}_1 \cup \cdots \cup \tilde{\mathfrak{U}}_{k-1}) \cup \mathfrak{U}'_k$ is a locally finite covering of $F \cap \boldsymbol{R}^k$.

(iii) There exists $r'_{\beta,1}, 0 < r'_{\beta,1} < r_{\beta,1}$ such that

(1) $(\tilde{\mathfrak{U}}'_1 \cup \cdots \cup \tilde{\mathfrak{U}}'_{k-1}) \cup \{D^k((w_{\beta,1}, \cdots, w_{\beta,k}), (r'_{\beta,1}, \cdots, r'_{\beta,1}))\}$ also covers $F \cap \boldsymbol{R}^k$,

(2) for each pair of different elements

$$V = D^k((w_1, \cdots, w_k), (r_1, \cdots, r_k)) \in (\tilde{\mathfrak{U}}_1 \cup \cdots \cup \tilde{\mathfrak{U}}_{k-1})|_{\boldsymbol{R}^k} \cup \mathfrak{U}'_k$$

and

$$V' = D^k((w_{\beta,1}, \cdots, w_{\beta,k}), (r_{\beta,1}, \cdots, r_{\beta,1})) \in \mathfrak{U}'_k$$

such that $V \cap V' \neq \phi$, there exists $\varepsilon, 0 < \varepsilon < \min\{r'_1, r'_2, \cdots, r'_k, (d/2^{k-1})\}$ such that $D^k((w_1, \cdots, w_k), (r'_1 - \varepsilon, \cdots, r'_k - \varepsilon)) \cap V' = \phi$, where if $V \in \mathfrak{U}'_k$ then $r_1 = \cdots = r_k$ and $r'_1 = \cdots = r'_k$.

For each $D^k((w_{\beta,1}, \cdots, w_{\beta,1}), (r_{\beta,1}, \cdots, r_{\beta,1})) \in \mathfrak{U}'_k$ let $\Delta^n(w_\alpha, r_\alpha) \in \mathfrak{U}_k$ be such that n is the minimum integer among

$$\{m > 0 ; \Delta^m(w_\alpha, r_\alpha) \supset D^k((w_{\beta,1}, \cdots, w_{\beta,k}), (r_{\beta,1}, \cdots, r_{\beta,1})) ; \Delta^m(w_\alpha, r_\alpha) \in \mathfrak{U}_k\}.$$

Set

$$\tilde{U}^{(k)}_\beta = D^n((w_{\beta,1}, \cdots, w_{\beta,k}, 0, \cdots, 0), (r_{\beta,1}, \cdots, r_{\beta,1})) \times \prod_{i=n+1}^{\infty} \boldsymbol{R}_i \cap \Delta^n(w_\alpha, r_\alpha),$$

Then, we can find $\tilde{r}_{\beta,i}, 0 < \tilde{r}_{\beta,i} \leqq r_{\beta,1}, i = 1, \cdots, n$ such that

$$\tilde{U}^{(k)}_\beta = D^n((w_{\beta,1}, \cdots, w_{\beta,k}, 0, \cdots, 0), (\tilde{r}_{\beta,1}, \cdots, \tilde{r}_{\beta,n})) \times \prod_{i=n+1}^{\infty} \boldsymbol{R}_i .$$

Choosing $\tilde{r}'_{\beta,i}$ such that $\tilde{r}_{\beta,i} - (d/2^{i+k}) < \tilde{r}'_{\beta,i} < \tilde{r}_{\beta,i}$ for $k < i \leqq n$, we put

$$\tilde{U}'^{(k)}_\beta = D^n((w_{\beta,1}, \cdots, w_{\beta,k}, 0, \cdots, 0), (\tilde{r}'_{\beta,1}, \cdots, \tilde{r}'_{\beta,n})) \times \prod_{i=n+1}^{\infty} \boldsymbol{R}_i ,$$

where $\tilde{r}'_{\beta,i} = r'_{\beta,1}$ $(i = 1, \cdots, k)$. We set

$$\tilde{\mathfrak{U}}_k = \{U^{(k)}_\beta\} \quad \text{and} \quad \tilde{\mathfrak{U}}'_k = \{U'^{(k)}_\beta\} .$$

Thus, we put

$$\tilde{\mathfrak{U}} = \bigcup_{k=1}^{\infty} \tilde{\mathfrak{U}}_k .$$

Then, $\tilde{\mathfrak{U}}$ is a covering of F. In fact, let x be an arbitrary point in F. Then, there exists $U = \Delta^n(w, r) \in \mathfrak{U}$ such that $x \in \Delta^n(w, r)$. Suppose that there exists no $\tilde{U} \in \tilde{\mathfrak{U}}$ such that $x \in \tilde{U}$. Then, in view of the condition (i), there exists a sufficiently large integer m such that $p_m(x) \in \tilde{U}^{(m)}$ for some $\tilde{U}^{(m)} \in \tilde{\mathfrak{U}}_m$ and $p_m(\tilde{U}^{(m)}) \subset p_m(U)$. Thus, the construction of $\tilde{U}^{(m)}$ implies that $\tilde{U}^{(m)} = \Delta^s(w', r')$ for some $s \leqq m, w', r'$, i.e., $x \in \tilde{U}^{(m)}$, which contradicts the assumption. Therefore, $\tilde{\mathfrak{U}}$ is a covering of F. Now, we will show that $\tilde{\mathfrak{U}}$ is locally finite. For any point $x \in F$, there exists $\tilde{U}'^{(m)} = \Delta^n(w, r') \in \tilde{\mathfrak{U}}'_m$ such that $x \in \tilde{U}'^{(m)}$. Taking a neighborhood V' of z in $\boldsymbol{R}^n$

such that $\bar{V}'\subset p_n(\Delta^n(w, r'))$, we put $V=V'\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i\subset\Delta^n(w, r')$. Then, V is a neighborhood of x which intersects with only finitely many elements of $\tilde{\mathfrak{U}}$, in view of the condition (iii).

Part 2. Let $\tilde{U}_{\beta}^{(m)}=D^n((w_{\beta,1}, \cdots, w_{\beta,n}), (\tilde{r}_{\beta,1}, \cdots, \tilde{r}_{\beta,n}))\times\prod_{i=n+1}^{\infty}\boldsymbol{R}_i$ be an arbitrary element of $\tilde{\mathfrak{U}}$. Put

$$W_{m,\beta}=D^n((w_{\beta,1}, \cdots, w_{\beta,n}), (\tilde{r}_{\beta,1}, \cdots, \tilde{r}_{\beta,n}))$$

and

$$W'_{m,\beta}=D^n((w_{\beta,1}, \cdots, w_{\beta,n}), (\tilde{r}'_{\beta,1}, \cdots, \tilde{r}'_{\beta,n}))\,.$$

Then, we choose a function $h_{m,\beta}\in C^{\infty}(\boldsymbol{R}^n)$ such that

(i) $h_{m,\beta}|_{\bar{W}'_{m,\beta}}=1$,

(ii) $h_{m,\beta}|_{\boldsymbol{R}^n\setminus W_{n,\beta}}=0$,

(iii) $0\leqq h_{m,\beta}\leqq 1$.

Put $g_{m,\beta}=h_{m,\beta}\circ p_n$. Then, the function $g_{m,\beta}\in\mathcal{E}_b(\Pi\boldsymbol{R})$ satisfies the conditions:

(i) $g_{m,\beta}|_{\bar{\tilde{U}}_{\beta}^{\prime(m)}}=1$,

(ii) $g_{m,\beta}|_{\Pi\boldsymbol{R}\setminus\tilde{U}_{\beta}^{(m)}}=0$,

(iii) $0\leqq g_{m,\beta}\leqq 1$.

Put

$$g=\prod_{m,\beta}(1-g_{m,\beta})\,.$$

Then, $g\in\mathcal{E}_b(\Pi\boldsymbol{R})$ holds, because $\tilde{\mathfrak{U}}$ is a locally finite covering. The function $f=1-g$ is the required function. [Q. E. D.]

Remark. Lemma 2.2 is valid for $\Pi\boldsymbol{C}\cong\Pi\boldsymbol{R}^2$.

[Proof of Proposition 2.1.]

In view of Theorem 2, §4, Chapter A in [6], it follows from Lemma 2.1 that the sheaf $\mathcal{E}_b$ is soft. Hence, by Theorem 9.12, Chapter 2 in [2], the sheaf $\mathcal{E}_b^{0,q}$ is a fine sheaf on $\Pi\boldsymbol{C}$. [Q.E.D.]

Then, we have the following fine resolution of the sheaf $\mathcal{O}$.

THEOREM 2.3. *The following is a fine resolution of* $\mathcal{O}$ *on* $\Pi\boldsymbol{C}$:

$$0\longrightarrow\mathcal{O}\longrightarrow\mathcal{E}_b^{0,0}\longrightarrow\mathcal{E}_b^{0,1}\longrightarrow\cdots.$$

[Proof] For an arbitrary point z in $\Pi\boldsymbol{C}$ there exists a fundamental system of neighborhoods $\mathfrak{B}_z$ of z whose element is of the form as in (1.1) in §1.

For any $U \in \mathfrak{B}_z$, by Corollary 4.2.6 in [7] the following sequence is exact:

$$0 \longrightarrow \mathcal{O}_n(U^n) \longrightarrow \mathcal{E}_n^{0,0}(U^n) \longrightarrow \mathcal{E}_n^{0,1}(U^n) \longrightarrow \cdots \longrightarrow \mathcal{E}_n^{0,n}(U^n) \longrightarrow 0.$$

Taking the inductive limit with respect to n, the following sequence is exact:

$$0 \longrightarrow \varinjlim_n \mathcal{O}_n(U^n) \longrightarrow \varinjlim_n \mathcal{E}_n^{0,0}(U^n) \longrightarrow \varinjlim_n \mathcal{E}_n^{0,1}(U^n) \longrightarrow \cdots.$$

i.e.

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{E}_a^{0,0}(U) \longrightarrow \mathcal{E}_a^{0,1}(U) \longrightarrow \cdots.$$

Taking the inductive limit with respect to U, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_z \longrightarrow \mathcal{E}_{b,z}^{0,0} \longrightarrow \mathcal{E}_{b,z}^{0,1} \longrightarrow \cdots.$$

Since the sheaf $\mathcal{E}_b^{0,q}$ is a fine sheaf by Proposition 2.1, the proof is completed. [Q.E.D.]

Remark. Kajiwara [8] employed a fine resolution of the same type as in the above theorem on the space X of the product of infinite number of Riemann spheres. Our result is independent of that in [8], because $\Pi \boldsymbol{C}$ is not locally closed in X.

§ 3. Cohomology groups in $\Pi \boldsymbol{C}$.

We will calculate the cohomology groups by using the fine resolution of the sheaf $\mathcal{O}$ given in the preceeding section. Let $Q_n = \boldsymbol{C}^n \times \prod_{i=n+1}^{\infty} \{z_i \in \boldsymbol{C}_i ; |z_i| \leqq r_i\}$ $(r_i > 0)$. We will show below that $H^1(Q_n, \mathcal{O}) \neq 0$ and $H^k(Q_n, \mathcal{O}) = 0$ for $k \geqq 2$.

First, we will discuss the cohomology groups of a compact set.

LEMMA 3.1. *Let* $P = \{(z_i) \in \Pi \boldsymbol{C} ; |z_i| \leqq r_i\}$ $(0 \leqq r_i < \infty)$. *For any* $g \in \mathcal{E}_b^{0,q}(P)$ *such that* $\bar{\partial} g = 0$, *there exists* $f \in \mathcal{E}_b^{0,q-1}(P)$ *such that* $\bar{\partial} f = g$ *for* $q \geqq 1$.

[Proof] $\mathcal{E}_b^{0,q}(P) = \varinjlim_U \mathcal{E}_b^{0,q}(U)$ holds by Theorem 4.11.1 in [5], where U runs over open neighborhoods of P. Thus, $g \in \mathcal{E}_b^{0,q}(U)$ for some open neighborhood of P. For any $z \in P$, there exists $\Delta^n(z, r)$ such that $g|_{\Delta^n(z,r)} \in \mathcal{E}_n^{0,q}(p_n(\Delta^n(z, r)))$. Since P is compact in $\Pi \boldsymbol{C}$, there exist finitely many points $z^1, \cdots, z^s$ such that $\{\Delta^{n_i}(z^i, r^i)\}_{i=1,\cdots,s}$ covers P. Then, owing to the property (1.3), there exists m such that $\bigcup_{i=1}^{s} \Delta^{n_i}(z^i, r^i) \supset U_m$. Since $p_m(U_m)$ is convex in $\boldsymbol{C}^m$, there exists $f' \in \mathcal{E}_m^{0,q-1}(p_m(U_m))$ such that $\bar{\partial} f' = g|_{p_m(U_m)}$ by Corollary 4.2.6 in [7]. Putting $f = f' \circ p_m$, we have

$$f \in \mathcal{E}_b^{0,q-1}(U_m)$$

and

$$\bar{\partial} f = g|_{U_m}. \qquad \text{[Q.E.D]}$$

THEOREM 3.2. *Let $P = \{(z_i) \in \Pi C; |z_i| \leqq r_i\}$ $(r_i > 0)$. Then, we have*

$$H^k(P, \mathcal{O}) = 0 \quad (k \geqq 1).$$

[Proof] By Theorem 2.3 the following is a fine resolution of $\mathcal{O}$:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}_b^{0,0} \longrightarrow \mathcal{E}_b^{0,1} \longrightarrow \cdots.$$

It is sufficient to show that the following sequence is exact:

$$0 \longrightarrow \Gamma(P, \mathcal{O}) \longrightarrow \Gamma(P, \mathcal{E}_b^{0,0}) \longrightarrow \Gamma(P, \mathcal{E}_b^{0,1}) \longrightarrow \cdots.$$

By Lemma 3.1, we can show that the above sequence is exact. [Q.E.D.]

Now, we treat the cohomology groups of Q_n below. First, we will show that $H^k(Q_n, \mathcal{O}) = 0$ for $k \geqq 2$.

LEMMA 3.3. *Let $Q_n = C^n \times \prod_{i=n+1}^{\infty} \{z_i \in C_i; |z_i| \leqq r_i\}$ $(0 \leqq r_i < \infty)$. For any $g \in \mathcal{E}_b^{0,q}(Q_n)$ such that $\bar{\partial} g = 0$, there exists $f \in \mathcal{E}_b^{0,q-1}(Q_n)$ such that $\bar{\partial} f = g$* for $q \geqq 2$.

[Proof] $\mathcal{E}_b^{0,q}(Q_n) = \varinjlim_U \mathcal{E}_b^{0,q}(U)$ holds by Theorem 4.11.1 in [5], where U runs over open neighborhoods of Q_n. Then, there exists an open neighborhood U of Q_n such that $g \in \mathcal{E}_b^{0,q}(U)$. Put $K_k = \prod_{i=1}^{n} \{z_i \in C_i; |z_i| \leqq k\}$. Since $C^n = \bigcup_{k=1}^{\infty} K_k$,

$$\left\{\tilde{K}_k = K_k \times \prod_{i=k+1}^{\infty} \{z_i \in C_i; |z_i| \leqq r_i\}\right\}$$

covers Q_n. By the same argument as in Lemma 3.1, we can find $U_{m_k}^{(k)}$ subordinate to each $\tilde{K}_k$ for which there exists $\tilde{f}'_k \in \mathcal{E}_{m_k}^{0,q-1}(p_{m_k}(U_{m_k}^{(k)}))$ such that

$$\bar{\partial} \tilde{f}'_k = g|_{p_{m_k}(U_{m_k}^{(k)})}.$$

Then, putting

$$\tilde{f}_k = \tilde{f}'_k \circ p_{m_k},$$

we have $\tilde{f}_k \in \mathcal{E}_b^{0,q-1}(U_{m_k}^{(k)})$ such that

$$\bar{\partial} \tilde{f}_k = g|_{U_{m_k}^{(k)}}.$$

Now setting

$$W^{(k)}=U^{(k)}_{m_k}\cap U^{(k+1)}_{m_{k+1}},$$

we have

$$\bar\partial(\tilde f_k-\tilde f_{k+1})|_{W^{(k)}}=0 .$$

Then, there exists $h_k\in\mathcal{E}^{0,q}_{m_{k+1}}(p_{m_{k+1}}(W^{(k)}))$

such that

$$\bar\partial h_k=\tilde f'_k\circ p^{m_{k+1}}_{m_k}-\tilde f'_{k+1}$$

holds on $p_{m_{k+1}}(W^{(k)})$ by Corollary 4.2.6 in [7]. Since

$$\overline{W'^{(k)}}=\overline{U^{(k)}_{m_{k+1}+1}}\subset W^{(k)},$$

there exists $\theta\in\mathcal{E}_{m_{k+1}+1}(p_{m_{k+1}+1}(W^{(k)})), 0\leqq\theta\leqq 1$ such that

$$\theta=\begin{cases}1 \text{ on } p_{m_{k+1}+1}(W'^{(k)})\\ 0 \text{ on } C^{m_{k+1}+1}\setminus p_{m_{k+1}+1}(W^{(k)}) .\end{cases}$$

Put

$$f_{k+1}=\tilde f_{k+1}+\bar\partial((\theta\circ p_{m_{k+1}+1})(h_k\circ p_{m_{k+1}})) .$$

Then, $f_{k+1}|_{W'^{(k)}}=\tilde f_k$. Thus, we have $\{f_k\}$ which satisfies the following conditions:

(i) $f_k\in\mathcal{E}_b^{0,q-1}(W^{(k)})$,

(ii) $\bar\partial f_k=g|_{W^{(k)}}$,

(iii) $f_{k+1}|_{W'^{(k)}}=f_k$.

Therefore, define $f=f_k$ on $W'^{(k)}$. Then, $f\in\mathcal{E}_b^{0,q-1}(\bigcup_k W'^{(k)})$ such that

$$\bar\partial f=g \text{ on } \bigcup_k W'^{(k)}.$$

By Theorem 2.3 we have the desired result. [Q.E.D.]

THEOREM 3.4. *Let* $Q_n=C^n\times\prod_{i=n+1}^{\infty}\{z_i\in C_i\,;\,|z_i|\leqq r_i\}$ $(r_i>0)$. *Then, we have*

$$H^k(Q_n,\mathcal{O})=0 \quad\text{for}\quad k\geqq 2 .$$

[Proof] The result follows from Theorem 2.3 and Lemma 3.3. [Q.E.D.]

Next, we will show that $H^1(Q_n,\mathcal{O})\neq 0$ below. The way of proving this result is inspired by Dineen [3]. Let $\mathcal{M}$ be the sheaf of germs of meromorphic functions on ΠC (see [3]).

DEFINITION 3.1. Additive Cousin data for Q_n is a collection $(U_i,f_i)_{i\in I}$, where $(U_i)_{i\in I}$, is an open covering of $Q_n, f_i\in\Gamma(U_i,\mathcal{M})$ for all $i,j\in I$ and f_i-

$f_j \in \Gamma(U_i \cap U_j, \mathcal{O})$ for all $i, j \in I$.

DEFINITION 3.2. The first Cousin problem is solvable on Q_n if for any additive Cousin data on Q_n, $(U_i, f_i)_{i \in I}$, there exists an open neighborhood $W \subset \bigcup_{i \in I} U_i$ for which there exists a meromorphic function f on W, i.e., $f \in \Gamma(W, \mathcal{M})$ such that $f - f_i \in \Gamma(U_i \cap W, \mathcal{O})$ for all $i \in I$.

Now, we will show the following

THEOREM 3.5. *Let* $Q_n = \boldsymbol{C}^n \times \prod_{i=n+1}^{\infty} \{z_i \in \boldsymbol{C}_i \,;\, |z_i| \leqq r_i\}$. *Then we have*

$$H^1(Q_n, \mathcal{O}) \neq 0.$$

[Proof] We divide the proof into two parts.
Part 1. We will show that if $H^1(Q_n, \mathcal{O}) = 0$, then the first Cousin problem is solvable on Q_n.

ASSERTION 1. *Let* $\{U_j\}$ *be an open covering of* Q_n. *Suppose that* $H^1(Q_n, \mathcal{O}) = 0$. *If* $g_{j,k} \in \mathcal{O}(U_j \cap U_k)$, $j, k = 1, 2, \cdots$ *and*

$$g_{j,k} = -g_{k,j}\,;\, g_{i,j} + g_{j,k} + g_{k,i} = 0 \text{ on } U_i \cap U_j \cap U_k$$

for all i, j, k, *then there exists an open neighborhood* W *of* Q_n *for which one can find functions* $g_j \in \mathcal{O}(U_j \cap W)$ *such that*

$$g_{j,k} = g_k - g_j \text{ on } U_j \cap U_k \cap W$$

for all j *and* k.

[Proof of Assertion 1] Since $\mathcal{E}_b$ is fine, we can choose functions φ_ν and positive integers i_ν, $\nu = 1, 2, \cdots$, such that

(i) $\varphi_\nu \in \mathcal{E}_b(U_{i_\nu})$ such that $\operatorname{supp} \varphi_\nu \subset U_{i_\nu}$,

(ii) $\{\operatorname{supp} \varphi_\nu\}$ is a locally finite covering of $\bigcup_j U_j$,

(iii) $\sum \varphi_\nu = 1$ on $\bigcup_j U_j$.

Put

$$h_k = \sum_\nu \varphi_\nu g_{i_\nu, k}\,.$$

Then, $h_k \in \mathcal{E}_b(U_k)$ and

$$h_k - h_j = g_{j,k} \text{ on } U_j \cap U_k\,.$$

This implies that

$$\bar{\partial} h_k = \bar{\partial} h_j \text{ on } U_j \cap U_k\,,$$

so that there exists a form $\psi\in\mathcal{E}_b^{0,1}(\bigcup_j U_j)$ such that

$$\psi=\bar{\partial}h_k \text{ on } U_k$$

for every k. By the hypothesis, there exists an open neighborhood W of Q_n for which we can find $u\in\mathcal{E}_b(W)$ such that $\psi=\bar{\partial}u$. The function $g_k=h_k-u$ is the required one. [Q.E.D. of Assertion 1]

Put

$$g_{j,k}=f_j-f_k\,.$$

Since the hypothesis of Assertion 1 is fulfilled, there exists an open neighborhood W of Q_n for which we can find $g_j\in\mathcal{O}(U_j\cap W)$ such that

$$f_j-f_k=g_{j,k}=g_k-g_j \text{ on } U_j\cap U_k\cap W$$

for each j, k. This implies that

$$f_j+g_j=f_k+g_k \text{ on } U_j\cap U_k\cap W\,.$$

Hence, there exists a meromorphic function f on W such that

$$f=f_j+g_j \text{ on } U_j\cap W$$

for every j. Since

$$f-f_j=g_j\in\mathcal{O}(U_j\cap W)\,,$$

this proves that the first Cousin problem is solvable on Q_n.

Part 2. Now, we will show the following

ASSERTION 2. *The first Cousin problem on Q_n is not solvable.*

[Proof of Assertion 2] Let U be an arbitrary connected open neighborhood of Q_n. Put

$$U_1=\left\{(z_i)\in\Pi\boldsymbol{C}\,;\operatorname{Im} z_1<\frac{7}{4}\right\}\cap U\,,$$
$$U_n=\left\{(z_i)\in\Pi\boldsymbol{C}\,;n-\frac{3}{4}<\operatorname{Im} z_1<n+\frac{3}{4}\right\}\cap U \quad\text{for}\quad n>1.$$

Set

$$f_n(z)=\frac{z_{n+1}}{z_1-\sqrt{-1}n}\,.$$

Then, $(f_i, U_i)_{i=1,2,\dots}$ is an additive Cousin data for Q_n. Now, suppose that there exists $f\in\Gamma(Q_n,\mathcal{M})$ such that

$$f-f_n\in\Gamma(U_n,\mathcal{O})$$

for all n. Then, the function f depends on all variables z_i. This contradicts Lemma 1 in [3] (a meromorphic variant of Proposition 1.1). Thus, the first Cousin problem on Q_n is not solvable. [Q.E.D. of Assertion 2]

Assertion 2, together with the result in Part 1 leads to the conclusion that $H^1(Q_n,\mathcal{O})\neq 0$. [Q.E.D.]

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