On Some Algorithm of the Number of the Branch Points and Their Degrees Starting from the Distance Matrix of a Given Tree

By Nobuko IWAHORI*

Department of Mathematics, Aoyama Gakuin University, Chitosedai, Setagaya, Tokyo 157

(Introduced by K. Yamazaki)

(Received September 12, 1983)

Introduction

It is well-known that a tree T is characterized by the so-called distance matrix $D=(d_{ij})$, $1\leqq i, j\leqq n$, defined by

$$d_{ij}=d(e_i, e_j)$$

where $\{e_1, e_2, \cdots, e_n\}$ is the totality of the extreme points of the tree T and $d(e_i, e_j)$ is the distance between e_i and e_j (cf. [1], [2]).

Thus in principle one should be able to obtain all the invariants associated to the given tree T in terms of the distance matrix $D=(d_{ij})$. In particular given two trees T_1, T_2 with distance matrices D_1 and D_2 respectively, one should be able to give a criterion for the topological equivalence of T_1 and T_2 (i.e. the existence of a homeomorphism f from T_1 onto T_2). Let us write $T_1\cong T_2$ if T_1 and T_2 are isomorphic as trees, i.e. if there exists a bijective mapping σ from the set V_1 of all vertices of T_1 onto the set V_2 of all vertices of T_2 such that $(\sigma(x), \sigma(y))$ is an edge of T_2 if and only if (x, y) is an edge of T_1.

Now let us write $T_1\simeq T_2$ if T_1 and T_2 are homeomorphic as topological spaces. It is obvious that if $f: T_1\to T_2$ is a homeomorphism, then f maps the extreme points of T_1 onto those of T_2 and the branch points of T_1 onto those of T_2. Moreover, if β is a branch point of T_1 of degree k (i.e. k is the number of edges issuing from the branch point β), $f(\beta)$ is a branch point of T_2 of degree k. This observation suggests the following notion of "the reduced tree $\tilde{T}$ of T is defined as follows. Let $E=E(T)$ be the totality of the extreme points of T and let $B=B(T)$ be the totality of the branch points of T. Then the disjoint union

* Supported by Grant Aid for Scientific Research.

$\tilde{V}=E\cup B$ is the set of vertices of $\tilde{T}$. Now let v_1, v_2 be points of $\tilde{V}$. We say that a pair (v_1, v_2) is an arc in $\tilde{T}$ if and only if $v_1\neq v_2$ and the path $P_T(v_1, v_2)$ in T connecting v_1 and v_2 satisfies $P_T(v_1, v_2)\cap\tilde{V}=\{v_1, v_2\}$. Then it is not difficult to verify that $\tilde{T}$ is a tree homeomorphic to T.

Also it is not difficult to verify that for trees T_1 and T_2, T_1 and T_2 are homeomorphic if and only if the reduced trees $\tilde{T}_1$ and $\tilde{T}_2$ are isomorphic. Thus, all the topological invariants of a tree T must be obtained from the distance matrix $D(\tilde{T})$ of the reduced tree $\tilde{T}$.

In this paper we give an algorithm to give $D(\tilde{T})$ starting from the distance matrix $D(T)$ of a given tree T. Moreover in order to get the number of branch points of T together with their degrees we introduce an operation called "cut-off" for the reduced tree $\tilde{T}$ (in §1).

The explicit results for these algorithms are given in Theorem 4.1 and Theorem 4.11.

§0 Notations

We denote by $T=T(V, A)$ a tree with the vertex-set V and the arc-set A. Thus A is a subset of $V\times V$. For a vertex $v\in V$, we denote by $A(v)$ the subset of V defined by

$$A(v)=\{u\in V\,|\,(v, u)\in A\}$$

$A(v)$ is called the neighborhood of v in T. Noting by $\# M$ or by $|M|$ the cardinality of a set M, $v\in V$ is an extreme point of T iff $\# A(v)=1$. Also $v\in V$ is a branch point of T iff $\# A(v)\geqq 3$.

In the following we denote by E the subset of V consisting of all extreme points of T and by B the subset of V consisting of all branch points of T. We put

$$E=\{e_1, \cdots, e_n\},\quad n=|E|,$$
$$B=\{\beta_1, \cdots, \beta_m\},\quad m=|B|.$$

A tree $T(V, A)$ has no branch point (i.e. $|B|=0$) iff $|E|=2$, and then this is the case where $T(V, A)$ is a segment-like tree. Therefore in the following we consider the case where $|E|\geqq 3$.

Given two vertices u, v $(u\neq v)$ of the tree T, there exists uniquely a sequence $v_0, v_1, \cdots, v_p$ of vertices such that

(i) $u=v_0,\quad v_p=v$

(ii) $|\{v_0, v_1, \cdots, v_p\}|=p+1$

(iii) $(v_i, v_{i+1})\in A$ (for $i=0, 1, \cdots, p-1$).

We denote then the subset $\{v_0, v_1, \cdots, v_p\}$ by $P_T(u, v)$ and we call it the path in T connecting u and v. Also we call p the distance between u, v in T and denote it by $p=d(u, v)$. Thus we have $P_T(u, v)=P_T(v, u)$ and $d(u, v)=d(v, u)$. We define

$P_T(u, u)$ to be the empty set ϕ and put $d(u, u)$ to be zero.

Given three distinct extreme points e_i, e_j, e_k, the intersection $P_T(e_i, e_j) \cap P_T(e_j, e_k) \cap P_T(e_k, e_i)$ consists of a single vertex β in B. We denote this point β by β_{ijk}. Thus β_{ijk} is independent of the permutations of i, j, k.

§1 Distance matrix of a tree T and of the reduced tree

Given a tree $T(V, A)$, we define an $n \times n$ symmetric matrix $D=(d_{ij})$ (where $n=|E|$) with non-negative integral entries d_{ij} as follows:

$$d_{ij}=d(e_i, e_j) \quad \text{for} \quad e_i, e_j \in E$$

The matrix $D=(d_{ij})$ is called the distance matrix of the given tree T.

It is well-known (cf. [1]) that the distance matrix D characterizes the original tree T up to isomorphism. Moreover given an $n \times n$ matrix $S=(s_{ij})$ with non-negative integral entries s_{ij}, a necessary and sufficient condition for S to be the distance matrix of some tree is also known. (cf. [1], [2])
This condition is given as follows:

(1.1) $$s_{ii}=0, \quad \text{for } i=1, \cdots, n$$

(1.2) $$s_{ij}=s_{ji}>0, \quad \text{for } i \neq j$$

(1.3) $s_{ij}+s_{jk}-s_{ik}$ is an even positive integer if $\#\{i, j, k\}=3$

(1.4) If $\#\{i, j, p, , q\}=4$ then exactly one of the following cases occurs:

(a) $$s_{ij}+s_{pq}<s_{ip}+s_{jq}=s_{iq}+s_{jp}$$
(b) $$s_{ip}+s_{jq}<s_{iq}+s_{jp}=s_{ij}+s_{pq}$$
(c) $$s_{iq}+s_{jp}<s_{ij}+s_{pq}=s_{ip}+s_{jq}$$
(d) $$s_{ij}+s_{pq}=s_{ip}+s_{jq}=s_{iq}+s_{jp}$$

Now given a tree $T=T(V, A)$, let us define a reduced tree $\tilde{T}=T(\tilde{V}, \tilde{A})$ of T as follows.

(i) The set $\tilde{V}$ of all vertices of $\tilde{T}$ is defined to be a subset of V as follows.

(1.5) $$\tilde{V}=\{v \in V \mid \#A(v)=1 \text{ or } \#A(v) \geqq 3\}$$

i.e.

(1.5)* $$\tilde{V}=\{v \in V \mid \#A(v) \neq 2\}$$

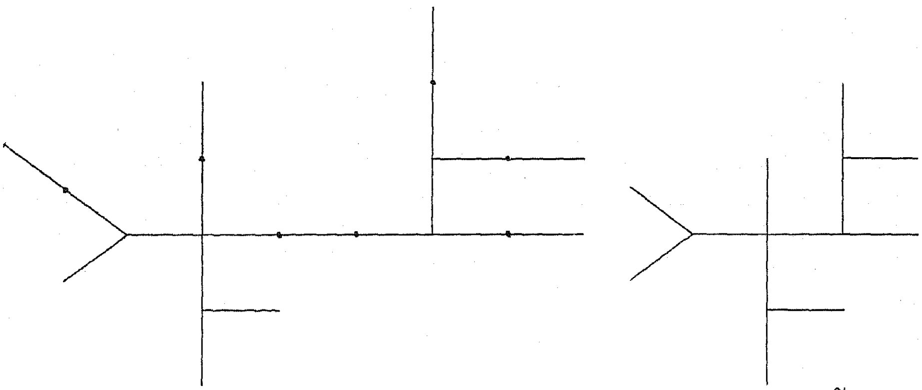
Thus $\tilde{V}$ is the set of all vertices of T which are extreme points or branch points.

(ii) The set $\tilde{A}$ of all arcs of $\tilde{T}$ is defined to be a subset of $\tilde{V} \times \tilde{V}$ as follows:

(1.6) $$\tilde{A}=\{(x, y) \in \tilde{V} \times \tilde{V} \mid P_T(x, y) \cap \tilde{V}=\{x, y\}\}$$

Then it is easy to verify that $T(\tilde{V}, \tilde{A})$ satisfies the definition of a tree; i.e. for every two distinct points x, y in $\tilde{V}$ there exists uniquely a subset $P_{\tilde{T}}(x, y)$ of $\tilde{V}$ (see Notation).

We denote by $\tilde{T}$ the tree $T(\tilde{V}, \tilde{A})$ thus obtained and we call $\tilde{T}$ the reduced tree of T. In other words, the reduced tree $\tilde{T}$ is obtained from T by ignoring the ordinary points (see Fig. 1.1).



(Fig. 1.1)

It is easy to see that if T_1 and T_2 are trees, then T_1 and T_2 are homeomorphic as topological spaces if and only if the corresponding reduced trees $\tilde{T}_1$ and $\tilde{T}_2$ are isomorphic as non-oriented graphs.

We also note that the set $\tilde{E}$ of all extreme points of $\tilde{T}$ coincides with the set E of all extreme points of T. So we denote $\tilde{E}$ also by E. Thus the set $\tilde{V}-E$ coincides with the set B of all branch points of T.

Since a tree $\tilde{T}$ is characterized by its distance matrix $\tilde{D}$ (up to the transformation $P\cdot\tilde{D}\cdot P^{-1}$ by a permutation matrix P), we may conclude that, given trees T_1 and T_2, then T_1 and T_2 are homeomorphic if and only if there exists a permutation matrix P such that $P\cdot\tilde{D}_1\cdot P^{-1}=\tilde{D}_2$, where $\tilde{D}_i$ is the distance matrix of the reduced tree $\tilde{T}_i$ $(i=1, 2)$.

In the following we fix our notation once for all as follows.

Let $T=T(V, A)$ be a given tree with the distance matrix $D=(d_{ij})$. Let $\tilde{T}=T(\tilde{V}, \tilde{A})$ be the corresponding reduced tree of T.

Denoting by $\tilde{d}(u, v)$ the distance between $u, v\in\tilde{V}$ in $\tilde{T}$, we note that the distance matrix $\tilde{D}=(\tilde{d}_{ij})$ of the reduced tree $\tilde{T}$ is given by the following.

PROPOSITION 1.1 *$\tilde{d}_{ij}=\tilde{d}(e_i, e_j)=\nu(i, j)+1$ where $\tilde{d}(e_i, e_j)$ is the distance between e_i, e_j in $\tilde{T}$ and the number $\nu(i, j)$ is defined as follows:*

$$\nu(i, j)=|P_T(e_i, e_j)\cap B|$$

Proof. We consider the map g

$$g: P_T(e_i, e_j) \cap B \to N \text{ (the set of positive integers)}$$

defined by $g(\beta)=d(e_i, \beta)$.

Clearly the map g is injective. Furthermore for every $\beta \in P_T(e_i, e_j) \cap B$ there exists an extreme point $e_p \in E-\{e_i, e_j\}$ such the $\beta=\beta_{ijp}$. Thus we have

$$\#\{d(e_i, \beta_{ijp}) | e_p \in E-\{e_i, e_j\}\}=|P_T(e_i, e_j) \cap B|=\nu(i, j), \quad \text{Q.E.D.}$$

PROPOSITION 1.2 *Suppose* $i \neq j, 1 \leqq i, j \leqq n$. *Then we have the following equality:*

$$\nu(i, j)=\#\{d_{ip}-d_{jp} | e_p \in E-\{e_i, e_j\}\}$$

Proof. We have easily the following equality for any distinct extreme points e_i, e_j, e_p

$$d_{ij}+d_{ip}-d_{jq}=2 \cdot d(e_i, \beta_{ijp})$$

where d_{ij}'s are the entries of the distance matrix D of the tree T. Hence we have

$$d(e_i, \beta_{ijp})=\frac{1}{2}\{d_{ij}+d_{ip}-d_{jp}\}$$

Then our assertion is an immediate consequence of the above equality, Q.E.D.

§2. An equivalence relation on E defined by mapping ψ

DEFINITION. We define a map $\psi: E \to B$ as follows:

$$\tilde{d}(e, \psi(e))=1 \text{ for } e \in E$$

i.e. $\psi(e)$ is the nearest branch point in $\tilde{T}$ from e.

The map ψ defines an equivalence relation $\sim$ on E, i.e. $e, e' \in E$ are equivalent (in notation $e \sim e'$) iff $\psi(e)=\psi(e')$. Thus, recalling the definition of $\tilde{D}=(\hat{d}_{ij})$, $e_i \sim e_j$ is equivalent to either (i) $e_i=e_j$ or (ii) $\nu(i, j)=1$.

PROPOSITION 2.1. *Let* $\psi(e_i)=\beta$, $\psi(e_j)=\beta'$. *Then* $P_{\tilde{T}}(e_i, e_j) \cap B=P_{\tilde{T}}(\beta, \beta') \cap B$.

Proof. Put $P_{\tilde{T}}(e_i, e_j) \cap B=\{\beta_1, \cdots, \beta_\nu\}$, $\nu=\nu(i, j)$ where we may assume that

$$\tilde{d}(e_j, \beta_1)<\cdots<\tilde{d}(e_i, \beta_\nu).$$

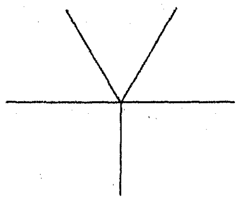
Then $\psi(e_i)=\beta, \psi(e_j)=\beta'$ imply that $\beta_1=\beta, \beta_\nu=\beta'$. Hence $P_{\tilde{T}}(\beta, \beta') \cap B=\{\beta_1, \cdots, \beta_\nu\}$, Q.E.D.

COLOLLARY 2.2. *Suppose* $e_i \sim e_{i'}$, $e_j \sim e_{j'}$. *Then* $\tilde{d}_{ij} = \hat{d}_{i'j'}$.

Proof. It is obvious from the definition of $\tilde{d}_{ij}$ and Prop. 2.1, Q.E.D.

We denote the ψ-equivalence class containing e by $[e]_0$. Thus $[e]_0 = \psi^{-1}(\psi(e))$. We fix a subset F_1 of E which forms a complete representative system of the ψ-equivalence classes. We call such a set F_1 the representative set of $E/\sim$.

If $|F_1|=1$, then $|B|=1$ and $\tilde{T}$ is the form of Fig. 2.1. So in the following, we consider the case where $|F_1| \geqq 2$.



(Fig. 2.1)

DEFINITION. Given $e_i \in E$ and $e_j \in E-[e_i]_0$. We define an integer $f_{e_j}(e_i)$ by

$$f_{e_j}(e_i) = \underset{e_p \in E-([e_i]_0 \cup \{e_j\})}{\text{Minimum}} \tilde{d}(e_i, \beta_{ijp})$$

PROPOSITION 2.3. *$f_{e_j}(e_i)$ is independent of $e_j \in E-[e_i]_0$ and is determined solely by the extreme point e_i. Thus we denote it by $f(e_i)$.*

Proof. It is enough to show that if e_i, e_j, e_k are distinct extreme points, then

$$f_{e_j}(e_i) = f_{e_k}(e_i).$$

We take e_p, e_q such that the minimums are attained:

$$(1) \qquad f_{e_j}(e_i) = \tilde{d}(e_i, \beta_{ijp}),$$

$$(2) \qquad f_{e_k}(e_i) = d(e_i, \beta_{ikq}).$$

Then β_{ijk} satisfies the following inequalities:

$$\tilde{d}(e_i, \beta_{ijk}) \geqq \tilde{d}(e_i, \beta_{ijp}),$$
$$\tilde{d}(e_i, \beta_{ijk}) \geqq \tilde{d}(e_i, \beta_{ikq}).$$

Hence we have

$$\beta_{ijp} \in P_{\tilde{T}}(e_i, \beta_{ijk}) \subset P_{\tilde{T}}(e_i, e_k).$$

This means $\hat{d}(e_i, \beta_{ikq}) \leqq \tilde{d}(e_i, \beta_{ijp})$. Also we have $\beta_{ikq} \in P_{\tilde{T}}(e_i, \beta_{ijk}) \subset P_{\tilde{T}}(e_i, e_j)$, which means $\tilde{d}(e_i, \beta_{ijp}) \leqq \tilde{d}(e_i, \beta_{ikp})$. Thus we have

$$f_{e_j}(e_i)=\tilde{d}(e_i, \beta_{ijq})=\tilde{d}(e_i, \beta_{ikq})=f_{e_k}(e_i), \text{ Q.E.D.}$$

PROPOSITION 2.4. *$f(e_i)$ is equal to 1 or 2. Also $f(e_i)$ is independent of the choice of e_i in $[e_i]_0$.*

Proof. Let $e_j \in E-[e_i]_0$. Then we have $\tilde{d}(e_i, \beta_{ijk}) \geqq \tilde{d}(e_i, \psi(e_i))=1$ for every $e_k \in E-([e_i]_0 \cup \{e_j\})$. On the other hand there exists a point $e_p \in E-([e_i]_0 \cup \{e_j\})$ satisfying $\tilde{d}(e_i, \beta_{ijp})=2$. Thus we get $f(e_i)=1$ or 2.

Let e_q be a point such that $e_i \sim e_q$. Then we have $\beta_{ijp}=\beta_{qjp}$ for any two points e_p, e_j in $E-[e_i]_0$. Therefore $f(e_i)$ is indpendent of the choice of e_i in $[e_i]_0$, Q.E.D.

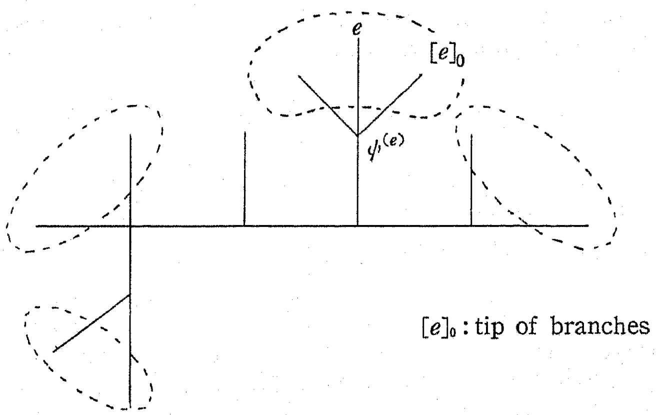
§ 3. A bud, a tip of branches, and some algorithm

DEFINITION. An extreme point $e \in F_1$ is called a bud of $\tilde{T}$ iff either

(i) $|F_1|=1, F_1=\{e\}$

or (ii) $|F_1| \geqq 2$ and $f(e)=2$.

DEFINITION. Suppose $e \in F_1$ is a bud of $\tilde{T}$. Then the equivalence class $[e]_0$ is called a tip of branches of $\tilde{T}$ (see Fig. 3.1).



(Fig. 3.1)

PROPOSITION 3.1. *If $|F_1|=2$, then F_1 consists of only buds of $\tilde{T}$.*

Proof. Since $|B|=2$ in this case, put $B=\{\beta_1, \beta_2\}$ and $F_1=\{e_1, e_2\}$. Let $\psi(e_1)=\beta_1, \psi(e_2)=\beta_2$. It is clear that $E-[e_1]_0=[e_2]_0$. Then we have

$$f(e_1)=\tilde{d}(e_1, \beta_2)=2.$$

This means e_1 is a bud or $\tilde{T}$. Similarly e_2 is a bud of $\tilde{T}$, Q.E.D.

In order to show the existence of buds of $\tilde{T}$ for $|F_1|\geqq 3$, let us prove the following first.

PROPOSITION 3.2. *A point e_i in F_1 is a bud of $\tilde{T}$ iff*
(i) $\#[e_i]_0\geqq 2$ *and*
(ii) *for some* $e_k\notin[e_i]_0$ *if* $\phi(e_i)=\beta_{ijk}$ *implies* $e_j\in[e_i]_0$.

Proof. ($\Rightarrow$) Let us assume $e_j\not\sim e_i$ for e_j satisfying $\phi(e_i)=\beta=_{ijk}$. Since $e_j, e_k\in E-[e_i]_0$, by difinition of $f(e_i)$, we have

$$f(e_i)\leqq\tilde{d}(e_i, \beta).$$

On the other hand, by the definition of ϕ, we get $f(e_i)=1$. This means e_i is not a bud of $\tilde{T}$, which is a contradiction.

($\Rightarrow$) Suppose now, $\phi(e_i)=\beta=\beta_{ijk}$ implies , $e_k\not\sim e_i$, $e_j\sim e_i$. Take e_p which attains $f(e_i)$, i.e.

$$f(e_i)=f_{e_k}(e_i)=\tilde{d}(e_i, \beta_{ikp}).$$

By definition of $f(e_i)$, $e_p\not\sim e_i$, then we have $\beta\neq\beta_{ikp}$ and $\tilde{d}(e_i, \beta)<\tilde{d}(e_i, \beta_{ikp})$. This means $f(e_i)=2$ and e_i is a bud of $\tilde{T}$, Q.E.D.

DEFINITION. We call the diameter of a tree T the maximum entry of the distance matrix $D=(d_{ij})$ of T, and denote it by $d(T)$.

THEOREM 3.3. *If $|F_1|\geqq 2$, then F_1 contains at least two buds.*

Proof. By Prop. 3.1, we may prove the theorem only for the case $|F_1|\geqq 3$. Let $d_0=d(\tilde{T})$ and put $d_0=\tilde{d}_{i_0j_0}=\nu(i_0, j_0)+1$. Denote $\nu_0=\underset{1\leqq i,j\leqq n}{\mathrm{Max}}\ \nu(i, j)=\nu(i_0, j_0)$. Let us show that e_{i_0}, e_{j_0} are both buds of $\tilde{T}$. Put

$$E(e_{i_0})=\{e_p\in E-\{e_{i_0}, e_{j_0}\}\,|\,\phi(e_{i_0})=\beta_{i_0j_0p}\}.$$

Clearly $E(e_{i_0})\neq\phi$. $e_p\in E(e_{i_0})$ imslies $e_{i_0}\sim e_p$. In fact, if we assume that $|P_{\tilde{T}}(e_p, \beta_{i_0j_0p})\cap B|\geqq 2$, then we get $\nu(p, j_0)>\nu(i_0, j_0)=\nu_0$ which is impossible. Thus $|P_{\tilde{T}}(e_p, \beta_{i_0j_0p})\cap B|=1$. Hence $P_{\tilde{T}}(e_p, e_{i_0})\cap B=\{\beta_{i_0j_0p}\}$. We have $e_{i_0}\sim e_p$. Thus e_{i_0} is a bud of $\tilde{T}$ by Prop. 3.2. Similarly e_{j_0} is a bud of $\tilde{T}$, Q.E.D.

Thus we have shown that $|F_1|\geqq 2$ implies that the subset S_1 of F_1 consisting of all buds of $\tilde{T}$ satisfies $|S_1|\geqq 2$. We then define a subset E_1 of E by

$$E_1=\Big(E-\bigcup_{e\in S_1}[e]_0\Big)\cup S_1.$$

Since $e\in S_1$ implies $\#[e]_0\geqq 2$, we have $E\supsetneqq E_1$. We then define a subtree $T_1=T(V_1, A_1)$ of $\tilde{T}$ as follows.

$$V_1=\tilde{V}-\bigcup_{e\in S_1}([e]_0-\{e\})$$

$$A_1=\tilde{A}-\bigcup_{e\in S_1}\cdot\bigcup_{e'\in[e]_0-\{e\}}(\phi(e), e')$$

Then it is easy to see that $T_1=T(V_1, A_1)$ forms a subtree of $T(\tilde{V}, \tilde{A})$. Furthermore the set of all extreme points of T_1 is E_1. Let us call this operation to get a new tree T_1 out of $\tilde{T}$ "cut-off tip of branches" of the tree $\tilde{T}$. Also let us call T_1 the cut-off tree of $\tilde{T}$.

Let us now describe the distance matrix D_1 of the tree T_1. It is easy to see that $D_1=(d_{ij}^{(1)})$ is a submatrix of $\tilde{D}=(\tilde{d}_{ij})$ corresponding to $e_i, e_j\in E_1$. Also the set B_1 of all branch points of T_1 is given by

$$B_1=B-\{\phi(e)|e\in S_1\}.$$

Thus we get $|B_1|=|B|-|S_1|$. Now the distance matrix $\tilde{D}_1=(\tilde{d}_{ij}^{(1)})$ of the reduced tree $\tilde{T}_1$ of T_1 i.e. $\tilde{d}_{ij}^{(1)}=\nu_1(i, j)+1$, where $\nu_1(i, j)=|P_{T_1}(e_i, e_j)\cap B_1|$ for $e_i, e_j\in E_1$, is given by the following:

THEOREM 3.4. *For $e_i, e_j\in E_1$, we have*

$$\nu_1(i, j)=\nu(i, j)-|\{e_i, e_j\}\cap S_1|.$$

Proof. If $|E_1|=n_1=2$, then T_1 is a segment and our assertion is trivial. So assume that $|E_1|=n_1\geqq 3$. Put $P_{\tilde{T}}(e_i, e_j)\cap B=P_T(e_i, e_j)\cap B=\{\beta_1, \beta_2, \cdots, \beta_\nu\}$, where $\nu=\nu(i, j)$. We assume

$$\tilde{d}(e_i, \beta_1)>\tilde{d}(e_i, \beta_2)<\cdots<\tilde{d}(e_i, \beta_\nu).$$

Then $\phi(e_i)=\beta_1$ and $\phi(e_j)=\beta_\nu$ by definition of the map ϕ.

Now we distinguish three cases according to

$$|\{e_i, e_j\}\cap S_1|=2, 1, 0.$$

Case I. $|\{e_i, e_j\}\cap S_1|=2$.

$\nu(i, j)\geqq 2$ since $e_i\nsim e_j$. Let us show $\nu(i, j)\geqq 3$. In fact assume that $\nu(i, j)=2$. Then putting $P_T(e_i, e_j)\cap B=\{\beta_1, \beta_2\}$ one has $\beta_{ijk}=\beta_1$ or $\beta_{ijk}=\beta_2$ for every $e_k\in E_1-\{e_i, e_j\}$, which is a contradiction to $|E_1|\geqq 3$. Thus we have $\nu(i, j)\geqq 3$.

Now by definition

$$\nu_1(i, j)=|P_{T_1}(e_i, e_j)\cap B_1|=|P_{T_1}(\beta_1, \beta_\nu)\cap B_1|.$$

But since $\beta_1\notin B_1, \beta_\nu\notin B_1$, we get

$$\nu_1(i, j)=\nu(i, j)-2=\nu(i, j)-|\{e_i, e_j\}\cap S_1|.$$

Case II. $|\{e_i, e_j\}\cap S_1|=1$.

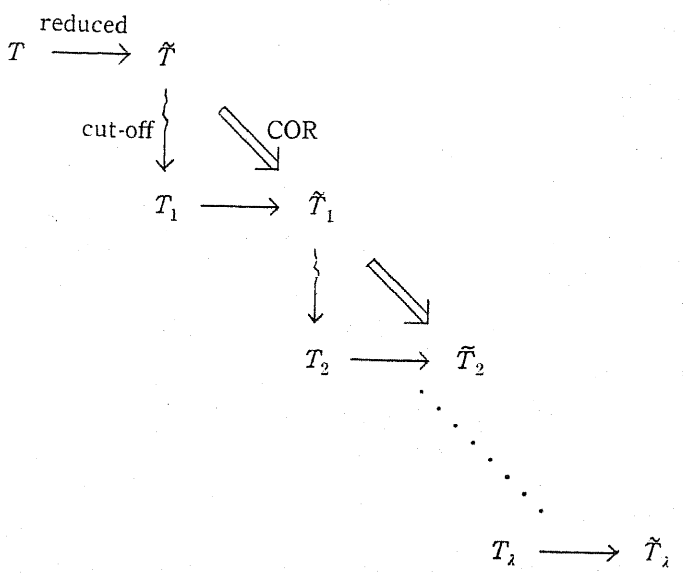
Let $e_i\in S_1$ and $e_j\notin S_1$. Then we see that this implies $\beta_1\notin B_1$ and $\beta_\nu\in B_1$. Thus we get $\nu_1(i, j)=|P_{T_1}(e_i, e_j)\cap B_1|=|\{\beta_2, \beta_3, \cdots, \beta_\nu\}|=\nu(i, j)-1=\nu(i, j)-|\{e_i, e_j\}\cap S_1|$.

Case III. $|\{e_i, e\}\cap S_1|=0$.

We have in this case $\beta_1\in B_1$ and $\beta_\nu\in B_1$. Thus $\nu_1(i, j)=|P_{T_1}(e_i, e_j)\cap B_1|=|P_{T_1}(e_i, e_j)\cap B|=\nu(i, j)$, Q.E.D.

Thus we have established an algorithm starting from a tree T (given in terms of its distance matrix D) to get firstly the reduced tree $\tilde{T}$ of T and then the cut-off tree T_1 of the tree $\tilde{T}$. Let $\tilde{T}_1$ be the reduced tree of the tree T_1 and let $\tilde{D}_1$ be the distance matrix of $\tilde{T}_1$. Our algorithm gives $\tilde{D}_1$ starting from D. Note that the size of $\tilde{D}_1$ is smaller then that of D. We denote $\tilde{T}_1$ by COR($\tilde{T}$).

Therefore by iteration of those operations "reduced" and "cut-off tip of branches" we get a series of trees $\tilde{T}, \tilde{T}_1, \tilde{T}_2, \cdots, \tilde{T}_\lambda$ where $\tilde{T}_1=$COR($\tilde{T}$), $\tilde{T}_{i+1}=$COR($\tilde{T}_i$) $(i=1, 2, \cdots, \lambda-1)$. (see Fig. 3.2)



(Fig. 3.2)

It is obvious by the operations that the set E_i of the extreme points of $\tilde{T}_i$ satisfies

$$E\supsetneqq E_1\supsetneqq E_2\supsetneqq\cdots\supsetneqq E_\lambda$$

with $|E_\lambda|=1$ or 2.

If $|E_\lambda|=1$, then $\tilde{T}_\lambda$ consists of a single vertex and if $|E_\lambda|=2$, then $\tilde{T}_\lambda$ is a segment of diameter 1.

Let S_α be the set of buds of the tree $\tilde{T}_{\alpha-1}$ and for a bud $e\in S_\alpha$ denoted by $[e]_{\alpha-1}$, the equivalence class (corresponding to $[e]_0$ in $\tilde{T}$) i.e. a tip of branches of $\tilde{T}_{\alpha-1}$ with a bud e.

THEOREM 3.5 *Given a tree* $T=T(V, A)$. *Let* $\tilde{T}$ *be its reduced tree and put* $\tilde{T}_1=\mathrm{COR}(\tilde{T})$. *Then* $d(\tilde{T})$ *and* $d(\tilde{T}_1)$ *satisfy*

$$d(\tilde{T}_1)=d(\tilde{T})-2$$

Proof. Put $d(\tilde{T})=\tilde{d}_{i_0j_0}=\nu(i_0, j_0)+1$ and $d(\tilde{T}_1)=\tilde{d}^{(1)}_{i_1j_1}=\nu(i_1, j_1)+1$. Let us put $\nu_0=\nu(i_0, j_0)$ and $\nu_0^{(1)}=\nu_1(i_1, j_1)$. Then our theorem claims that $\nu_0^{(1)}=\nu_0-2$. By the proof of Theorem 3.3, we may assume that e_{i_0}, e_{j_0} are in S_1. So one may assume that $e_{i_0}, e_{j_0}\in E_1$. Thus

$$\nu_1(i_0, j_0)=\nu_0-2$$

which implies

$$\nu_0^{(1)}\geqq\nu_0-2 \tag{1}$$

Hence it is enough to show $\nu_0^{(1)}\leqq\nu_0-2$.
We distinguish three cases according to the values

$$|\{e_{i_1}, e_{j_1}\}\cap S_1|=0, 1, 2.$$

Case I. $|\{e_{i_1}, e_{j_1}\}\cap S_1|=0$

Put $\beta=\psi(e_{i_1}), \beta'=\psi(e_{j_1})$. Since $e_{i_1}\notin S_1$ and $e_{j_1}\notin S_1$, there exists e_p, e_q in E such that

$$\beta=\beta_{i_1j_1p},\quad e_p\sim e_{i_1}$$

$$\beta'=\beta_{i_1j_1q},\quad e_q\sim e_{i_1}$$

by Prop. 3.2. Hence $|P_T(\beta, e_p)\cap B|\geqq 2, |P_T(\beta', e_q)\cap B|\geqq 2$. Therefore we have $\nu(p, q)\geqq\nu(i_1, j_1)+2$. By Theorem 3.4, we have $\nu_1(i_1, j_1)=\nu(i_1, j_1)$ and one gets $\nu(p, q)\geqq\nu_0^{(1)}+2$.On the other hand, because of the choice of ν_0 one has $\nu_0\geqq\nu(p, q)$. Thus we have $\nu_0-2\geqq\nu_0^{(1)}$.

Case II. $|\{e_{i_1}, e_{j_1}\}\cap S_1|=1.$

Let $e_{i_1}\in S_1, e_{j_1}\notin S_1$. Then there exists an extreme point e_q satisfying $\psi(e_{j_1})=\beta'=\beta_{i_1j_1q}$ and $e_{j_1}\sim e_q$ by Prop. 3.2. Thus we have

$$\begin{aligned}\nu_1(i_1, q)&=|P_T(e_{i_1}, e_q)\cap B_1|=|P_{T_1}(e_{i_1}, \beta')\cap B_1|+|P_{T_1}(\beta', e_q)\cap B_1|-|\{\beta'\}|\geqq\nu_1(i_1, j_1)+1\\&=\nu_0^{(1)}+1.\end{aligned}$$

On the other hand, $\nu_1(i_1, q)\leqq\nu(i_1, q)-1$ by Theorem 3.4. Combining these inequalities we get

$$\nu(i_1, q)-1\geqq\nu_1(i_1, q)\geqq\nu_0^{(1)}+1.$$

Hence

$$\nu_0\geqq\nu(i_1, q)\geqq\nu_0^{(1)}+2.$$

Case III. $|\{e_{i_1}, e_{j_1}\}\cap S_1|=2$.

We have $\nu_1(i_1, j_1)=\nu(i_1, j_1)-2$. So we get $\nu_0^{(1)}\leqq\nu_0-2$, Q.E.D.

THEOREM 3.6. *Let T be a tree. Let $\tilde{T}_1, \tilde{T}_2, \cdots, \tilde{T}_\lambda$ be a sequence of trees defined by $\tilde{T}_1=\mathrm{COR}(\tilde{T})$, $\tilde{T}_{i+1}=\mathrm{COR}(\tilde{T}_i)$ $(i=1, 2, \cdots, \lambda-1)$. Let E_i be the set of extreme points of $\tilde{T}_i$ and suppose $|E_\lambda|=1$ or 2. (Note that $E_1 \supsetneqq E_2 \supsetneqq \cdots \supsetneqq E_\lambda$). Then λ and the value $|E_\lambda|$ are determined from $d(\tilde{T})$ as following:*

(i) *if $d(\tilde{T})=2t$ then $\lambda=t$, $|E_\lambda|=2$,*

(ii) *if $d(\tilde{T})=2t-1$ then $\lambda=t$, $|E_\lambda|=1$.*

$\left(\textit{Thus } \lambda=\left[\dfrac{d(\tilde{T})+1}{2}\right]\right)$.

Proof. The diameters of the trees $\tilde{T}_1, \tilde{T}_2, \cdots, \tilde{T}_{\lambda-1}$ are given by $d(\tilde{T})-2$, $d(\tilde{T})-4, \cdots, d(\tilde{T})-2(\lambda-1)$ respectively by Theorem 3.5. Thus if $d(\tilde{T})=2t$, then we get $\lambda=t$ and $d(\tilde{T}_{\lambda-1})=d(\tilde{T})-2(\lambda-1)=2$, and if $d(\tilde{T})=2t-1$ then we get $\lambda=t$ and $d(\tilde{T}_{\lambda-1})=d(\tilde{T})-2(\lambda-1)=1$. Thus for the tree $\tilde{T}_\lambda$, the representative set F_λ of $E_{\lambda-1}/\sim$ must satisfiy $|F_\lambda|=2$ or 1. If $|F_\lambda|=2$, then $|E_\lambda|=2$ and $\tilde{T}_\lambda$ is a segment. If $|F_\lambda|=1$, then $|E_\lambda|=1$ and $\tilde{T}_\lambda$ is a single point, Q.E.D.

§4. A formula of the number of branch points and their degrees

THEOREM 4.1 *For a tree $T(V, A)$, the number $|B|$ of the branch point of T is given by*

$$|B|=\sum_{\alpha=1}^{\lambda}|S_\alpha|$$

where $\tilde{T}_1=\mathrm{COR}(\tilde{T})$, $\tilde{T}_{i+1}=\mathrm{COR}(\tilde{T}_i)$ $(i=1, 2, \cdots, \lambda-1)$, $\lambda=\left[\dfrac{d(\tilde{T})+1}{2}\right]$, and S_α is the set of buds of the tree $\tilde{T}_{\alpha-1}$ (we put $\tilde{T}_0=\tilde{T}$) for $\alpha=1, 2, \cdots, \lambda$. (Note that the final tree $\tilde{T}_\lambda$ has no buds).

Proof. Let B_α be the set of branch points of the tree $\tilde{T}_\alpha(\alpha=1, \cdots, \lambda)$. Then we have $|B_1|=|B|-|S_1|$, and similarly $|B_\alpha|=|B_{\alpha-1}|-|S_\alpha|$, for $\alpha=2, 3, \cdots, \lambda$. Since $|B_\lambda|=0$, we get $|B|=\sum_{\alpha=1}^{\lambda}|S_\alpha|$, Q.E.D.

We now give an algorithm giving the degree $\#A(\beta)$ of the branch point $\beta\in B$ using only the distance matrix $D=(d_{ij})$ of the tree T.

Put $\#A(\beta)=r$, where $A(\beta)=\{v_1, v_2, \cdots, v_r\}$ is the neibourhood of β in T. We associate to β an r-dimensional vector $M(\beta)$ by

$$M(\beta)=(\mu(v_1), \mu(v_2), \cdots, \mu(v_r))$$

where

$$\mu(v_i)=\underset{\substack{e\in E \\ \beta\notin P_T(v_i,e)}}{\text{Maximum}}\ |P_T(v_i, e)\cap B|.$$

Thus $\mu(v_i)$ is the largest number of the branch points on the path of T connecting v_i to the extreme points without passing through β. It is clear that every v_i satisfies

$$0\leqq\mu(v_i)\leqq\nu_0-1, \quad \text{where } \nu_0=\underset{1\leqq k,l\leqq n}{\text{Maximum}}\ \nu(k, l)$$

We index the v_i's in such a way that

$$\mu(v_1)\leqq\mu(v_2)\leqq\cdots\leqq\mu(v_r)$$

Let us show that $\mu(v_i)$ can be obtained also from the distance matrix D. Since β is a branch point, there exist $e_a, e_b, e_c\in E$ such that $\beta=\beta_{abc}$. Fixing two points in $\{e_a, e_b, e_c\}$, say e_a, e_b, put $P_T(e_a, e_b)\cap B=\{\beta_1, \cdots, \beta_\nu\}$ indexed as follows:

$$d(e_a, \beta_1)<d(e_a, \beta_2)<\cdots<d(e_a, \beta_\nu)$$

where $\nu=\nu(a, b)$. Then for each β_t $(1<t<\nu)$ put

$$E_{ab}(\beta_t)=\{e_p\in E\,|\,\beta_t=\beta_{abp}\}.$$

PROPOSITION 4.2. *Put* $E^*_{ab}(\beta_1)=E_{ab}(\beta_1)\cup\{e_a\}$ *and* $E^*_{ab}(\beta_\nu)=E_{ab}(\beta_\nu)\cup\{e_b\{$. *Then* E *is partitioned as follows:*

$$(**)\qquad E=E^*_{ab}(\beta_1)\coprod\cdots\coprod E_{ab}(\beta_t)\coprod\cdots\coprod E^*_{ab}(\beta_\nu).$$

Proof. Clearly every term on the right hand side of (**) is not empty. The mutual disjointness is also obvious by the fact that any three extreme points e_p, e_q, e_s satisfy

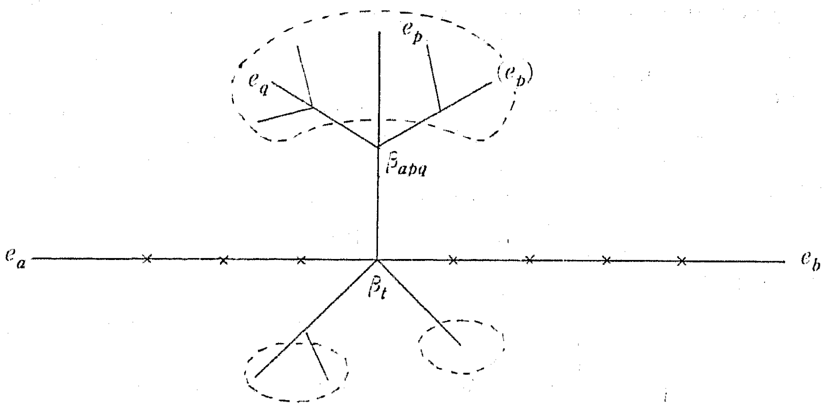
$$|P_T(e_p, e_q)\cap P_T(e_q, e_s)\cap P_T(e_s, e_p)|=1.$$

We have to show that every $e_p\in E$ belongs to the right hand side of (**). This can be seen as follows: $\beta_{abp}\in P_T(e_a, e_b)$ implies the existence of some t such that $\beta_{abp}=\beta_t$. Thus we have $e_p\in E_{ab}(\beta_t)$, Q.E.D.

We now define an equivalence relation relation $\simeq$ in $E_{ab}(\beta_t)$ for each t (fixing e_a, e_b).

DEFINITION. Let $e_p, e_q\in E_{ab}(\beta_t)$. Then we write $e_p\simeq e_q$ either (i) $e_p=e_q$ or (ii) $d(e_a, \beta_t)<d(e_a, \beta_{apq})$.

Since $d(e_a, \beta_t)<d(e_a, \beta_{apq})$ is equivalent to the fact $P_T(e_a, \beta_t)\subset P_T(e_a, \beta_{apq})$, it is easy to see that the relation $\simeq$ is an equivalence relation (see Fig. 4.1).



(Fig. 4.1).

We denote the $\simeq$——equivalence class of e_p in $E_{ab}(\beta_t)/\simeq$ by (e_p).

PROPOSITION 4.3. *The equivalence relation $\simeq$ holds iff the following distance-inequality holds:*

$$e_p \simeq e_q \Longleftrightarrow d_{pq}+d_{ab}<d_{ap}+d_{bq}=d_{aq}+d_{bp}.$$

Proof. ($\Longrightarrow$) $d(e_a, \beta_t)<d(e_a, \beta_{apq})$ implies $\beta_t, \beta_{apq} \in P_T(e_a, e_p)$ and $d(\beta_t, \beta_{apq})>0$. Hence

$$d_{ap}=d(e_a, \beta_t)+d(\beta_t, \beta_{apq})+d(\beta_{apq}, e_p),$$
$$d_{bq}=d(e_b, \beta_t)+d(\beta_t, \beta_{apq})+d(\beta_{apq}, e_q)$$

and we get

$$d_{ap}+d_{bq}=d_{ab}+d_{pq}+2d(\beta_t, \beta_{apq})>d_{ab}+d_{pq}.$$

($\Longleftarrow$) Let us derive a contradiction assuming $\beta_t=\beta_{apq}$.

$$d_{ap}=d(e_a, e_p)=d(e_a, \beta_t)+d(\beta_t, e_p),$$
$$d_{bq}=d(e_b, e_q)=d(e_b, \beta_t)+d(\beta_t, e_q)$$

implies $d_{ap}+d_{bq}=d_{ab}+d_{pq}$ which is not compatible with the inequality above, Q.E.D.

COROLLARY 4.4. $(e_p) \nsim (e)$ *iff* $\beta_t=\beta_{apq}$.

Remark. It is easy to see that the equivalence relation $\simeq$ is also expressed by $\tilde{D}$, the distance matrix of the reduced tree $\tilde{T}$.

$$e_p \simeq e_q \Longleftrightarrow \tilde{d}(e_p, e_q) + \tilde{d}(e_a, e_b) < \tilde{d}(e_a, e_p) + \tilde{d}(e_b, e_q) = \tilde{d}(e_a, e_q) + \tilde{d}(e_b, e_p).$$

DEFINITION. Suppose $e_a, e_b \in E$. Put $P_T(e_a, e_b) \cap B = \{\beta_1, \cdots, \beta_\nu\}$, $\nu = \nu(a, b)$ and $d(e_a, \beta_1) < d(e_a, \beta_2) < \cdots < d(e_a, \beta_\nu)$. Then the subset $\tilde{A}_{ab}(\beta_t)$ of V is defined as follows: Put

$$P_T(\beta_t, e_a) \cap A(\beta_t) = \{v_{ta}\},$$
$$P_T(\beta_t, e_b) \cap A(\beta_t) = \{v_{tb}\}.$$

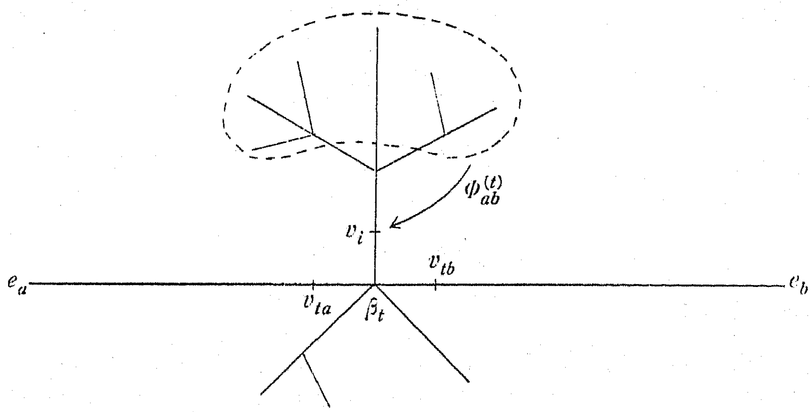
Then we define

$$\tilde{A}_{ab}(\beta_t) = \begin{cases} A(\beta_t) & \text{if } \nu = 1 \\ A(\beta_t) - \{v_{tb}\} & \text{if } \nu > 1 \text{ and } t = 1 \\ A(\beta_t) - \{v_{ta}\} & \text{if } \nu > 1 \text{ and } t = \nu \\ A(\beta_t) - \{v_{ta}, v_{tb}\} & \text{if } 1 < t < \nu. \end{cases}$$

Then we have for every $\simeq$—class (e_p) that $|P_T(\beta_t, e) \cap \tilde{A}_{ab}(\beta_t)| = 1$ for any $e \in (e_p)$. Thus we may define an injective map

$$\phi_{ab}^{(t)} : E_{ab}(\beta_t)/\simeq \longrightarrow \tilde{A}_{ab}(\beta_t)$$

by $\phi_{ab}^{(t)}((e_p)) = v$, where $\{v\} = P_T(\beta_t, e) \cap A_{ab}(\beta_t)$, for $e \in (e_p)$ (see Fig. 4.2).



(Fig. 4.2).

The map $\phi_{ab}^{(t)}$ is a bijection between $E_{ab}(\beta_t)/\simeq$ and $\tilde{A}_{ab}(\beta_t)$, since for every $v \in \tilde{A}_{ab}(\beta_t)$ there exists $e \in E$ such that $v \in P_T(\beta_t, e)$ and one has $\phi_{ab}^{(t)}((e)) = v$. We denote $(\phi_{ab}^{(t)})^{-1}$ by $\tilde{\phi}_{ab}^{(t)}$.

PROPOSITION 4.5. *Let v be a point of $\tilde{A}_{ab}(\beta_t)$. Then*

$$\mu(v)=\underset{\substack{e\in E\\ \beta_t\notin P_T(v,e)}}{\text{Maximum}}\,|P_T(v,e)\cap B|$$

is obtained from the distance matrix $\tilde{D}$ *of* $\tilde{T}$ *as follows:*

$$\mu(v)=\underset{e\in\tilde{\phi}_{ab}^{(t)}(v)}{\text{Maximum}}\,|P_T(v,e)\cap B|$$
$$=\underset{e_p\in\tilde{\phi}_{ab}^{(t)}(v)}{\text{Maximum}}\,\nu(a,p)-t.$$

Moreover t *is obtained as follows:*

$$t=\frac{1}{2}(\tilde{d}_{ab}+\tilde{d}_{ap}-\tilde{d}_{bp}),$$

where e_p *attains the value* $\mu(v)$.

Proof. For $e_p\in\tilde{\phi}_{ab}^{(t)}(v)$, we have a disjoint union $P_T(e_a,e_p)\cap B=(P_T(e_a,\beta_t)\cap B)\amalg(P_T(v,e_p)\cap B)$. Thus comparing the cardinalities it follows:

$$\nu(a,p)=t+|P_T(v,e_p)\cap B|.$$

Hence

$$\mu(v)=\underset{e_p\in\tilde{\phi}_{ab}^{(t)}(v)}{\text{Maximum}}\,|P_T(v,e_p)\cap B|$$
$$=\underset{e_p\in\tilde{\phi}_{ab}^{(t)}(v)}{\text{Maximum}}\,\nu(a,p)-t$$
$$=\underset{e_p\in\tilde{\phi}_{ab}^{(t)}(v)}{\text{Maximum}}\,\tilde{d}_{ab}-1-t.$$

Since $\nu(a,b)$ gives essentially the distance matrix of the tree $\tilde{T}$, it is easy to see $t=\tilde{d}(e_a,\beta_t)$ in $\tilde{T}$. Hence we have

$$t=\frac{1}{2}\{\tilde{d}_{ab}+\tilde{d}_{ap}-\tilde{d}_{bp}\}$$

for $e_p\in\tilde{\phi}_{ab}^{(t)}(v)$, Q.E.D.

Thus the vector $M(\beta_t)$ can be given explicitly by using $\tilde{D}$ for the case $\tilde{A}_{ab}(\beta^t)=A(\beta_t)$. Let us then consider the remaining cases where $\tilde{A}_{ab}(\beta_t)\subsetneqq A(\beta_t)$. We have only to determine the vector components $\mu(v_{ta})$ and $\mu(v_{tb})$.

CASE I. $|E_{ab}(\beta_t)/\simeq|\geqq 2$.

Let $(e),(e')\in E_{ab}(\beta_t)/\simeq$ and $(e)\neq(e')$. Take $e_p\in(e)$, $e_q(e')$. Then $(e_a),(e_b)\in E_{pq}(\beta_t)/\simeq$ implies

$$\mu(v_{ta})=\underset{e_k\in(e_a)}{\text{Maximum}}\,\nu(p,k)-s,$$

$$\mu(v_{tb}) = \underset{e_k \in (e_b)}{\text{Maximum}}\ \nu(q, k) - s'$$

where s, s' are given by $s = \tilde{d}(e_p, \beta_t)$, $s' = \tilde{d}(e_q, \beta_t)$ in $\tilde{T}$ respectively.

CASE II. $|E_{ab}(\beta_t)/\simeq| = 1$.

Let e_p be a representative of the unique class in $E_{ab}(\beta_t)/\simeq$. Then $(e_b) \in E_{ap}(\beta_t)/\simeq$.
Hence

$$\mu(v_{tb}) = \underset{e_k \in (e_b)}{\text{Maximum}}\ \nu(p, k) - s'.$$

We have now $(e_a) \in E_{bp}(\beta_t)/\simeq$. Thus

$$\mu(v_{ta}) = \underset{e_k \in (e_a)}{\text{Maximum}}\ \nu(p, k) - s,$$

where s, s' are given in Case I.
Thus let $\beta = \beta_{abc}$ for given β. Let us give an ordering for $\{\beta_1, \cdots, \beta_{\nu(a,b)}\} = P_T(e_a, e_b) \cap B$ by

$$d(e_a, \beta_1) < d(e_a, \beta_2) < \cdots < d(e_a, \beta_{\nu(a,b)}).$$

Then β determines the number t by $d(e_a, \beta) = d(e_a, \beta_t)$.

Now let us return to the r-dimensional vector $M(\beta) = (\mu(v_1), \mu(v_2), \cdots, \mu(v_r))$. Take $e_i \in E$ which attains the value $\mu(v_i)$ for $i = 1, 2, \cdots, r$. Then we have the following:

PROPOSITION 4.6. *If $\mu(v_i) > 0$, then e_i is a bud of $\tilde{T}$.*

Proof. Put $\phi(e_i) = \beta'$. Since $\mu(v_i) = |P_T(v_i, e_i) \cap B| > 0$, we see that $d(\beta, e_i) > d(\beta', e_i)$ and every e_p with $\phi(e_p) = \beta'$ satisfies $e_p \sim e_i$. Hence by Prop. 3.2, we have $e_i \in S_1$, Q.E.D.

PROPOSITION 4.7. *In the tree T_1, the vector*

$$M_{T_1}(\beta) = (\eta^{(1)}(v_1), \mu^{(1)}(v_2), \cdots, \mu^{(1)}(v_r))$$

satisfies

$$\mu^{(1)}(v_i) = \begin{cases} \mu(v_i) - 1 & \text{if } \mu(v_i) > 0 \\ 0 & \text{if } \mu(v_i) = 0. \end{cases}$$

Proof. If $e_j \in S_1$, then the branch point $\phi(e_i)$ in the tree T is not a branch point in the tree T_1. Hence by Prop. 4.6, $\mu(v_i) > 0$ implies $\mu^{(1)}(v_i) = \mu(v_i) - 1$. Now $\mu(v_i) = 0$ implies $e_i \notin S_1$. Since $P_T(\beta, e_i) \cap B = P_{\tilde{T}}(\beta, e_i) \cap B = \{\beta\}$, we have $v_i = e_i$ and $\mu^{(1)}(v_i) = 0$, Q.E.D.

Corollary 4.8. *In the tree* T_α, *the vector* $M_{T_\alpha}(\beta)=(\mu^{(\alpha)}(v_1), \mu^{(\alpha)}(v_2), \cdots, \mu^{(\alpha)}(v_r))$ *satisfies*

$$\mu^{(\alpha)}(v_i)=\begin{cases}\mu(v_i)-\alpha & \text{if } \mu(v_i)\geqq\alpha\\ 0 & \text{if } \mu(v_i)<\alpha.\end{cases}$$

Proposition 4.9. *Let* $M_{T_\alpha}(\beta)=(\mu^{(\alpha)}(v_1), \mu^{(\alpha)}(v_2), \cdots, \mu^{(\alpha)}(v_r))$ *and suppose that* α *is the least value such that*

$$\mu^{(\alpha)}(v_1)=\mu^{(\alpha)}(v_2)=\cdots=\mu^{(\alpha)}(v_{r-1})=0.$$

Then $e_1, \cdots, e_{r-1}$ *are* ϕ*-equivalent in* E_α *and* $[e_1]_\alpha$ *is a tip of branches in* $\tilde{T}_\alpha$.

Proof. $\mu^{(\alpha)}(v_i)=\mu^{(\alpha)}(v_j)=0$ means $e_i\sim e_j$ in E_α. Also by Prop. 3.2, $\mu^{(\alpha)}(v_1)=\mu^{(\alpha)}(v_2)=\cdots=\mu^{(\alpha)}(v_{r-1})=0$ implies that for any three points e_i, e_j, e_k in E_α satisfying $\beta_{ijk}=\beta=\phi(e_i)$ one has $e_i\sim e_j, e_i\sim e_k, e_j\sim e_k$. Hence $e_1\in S_{\alpha+1}$, Q.E.D.

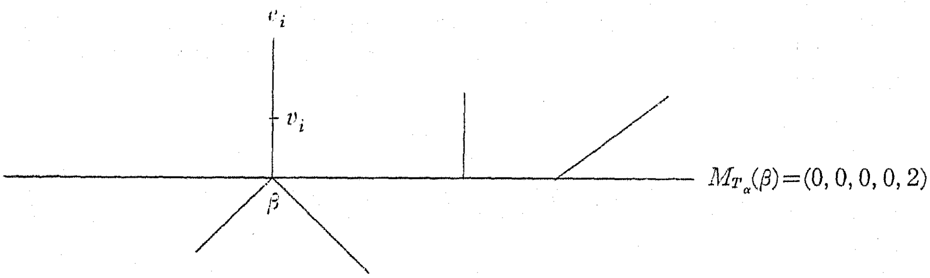
Now let us consider the converse of Prop. 4.9.

Proposition 4.10. *Let* $[e]_\alpha$ *be a tip of branches in* T_α. *Put* $\#[e]_\alpha=r_0$, $\phi(e)=\beta$ $(\beta\in B_\alpha)$ *and* $\#A(\beta)=r$. *Then we have* $r_0=r$ *or* $r_0=r-1$. *In other words, the vector*

$$M_{T_\alpha}(\beta)=(\mu^{(\alpha)}(v_1), \mu^{(\alpha)}(v_2), \cdots, \mu^{(\alpha)}(v_r))$$

satisfies $\mu^{(\alpha)}(v_1)=\cdots=\mu^{(\alpha)}(v_{r-1})=0$.

Proof. Put $[e]_\alpha=\{e_1, e_2, \cdots, e_{r_0}\}$. Since $|P_{T_\alpha}(e_i, \beta)\cap A(\beta)|=1$ for $i=1, 2, \cdots, r_0$, we define a map h from $[e]_\alpha$ to $A(\beta)$ as follows: $h(e_i)=v_i$ (see Fig. 4.3).



(Fig. 4.3)

It is easy to see that the map h is injective because $P_{T_\alpha}(e_i, e_j)\cap B_\alpha=\{\beta\}$ for $i\neq j$. Then we have $r_0\leqq r$. Moreover in the components of $M_{T_\alpha}(\beta)$ we get

$$\mu^{(\alpha)}(v_i)=0 \text{ for } i=1, \cdots, r_0.$$

If $r_0=r$, then $|E_\alpha|=1$. Therefore $\alpha=\lambda$ and $\beta_{\lambda-1}=\{\beta\}$. Suppose now $r_0<r-1$, then

we have

$$\mu^{(\alpha)}(v_i)=0 \quad (1\leqq i\leqq r_0),$$
$$\mu^{(\alpha)}(v_{r_0+1})>0, \quad \mu^{(\alpha)}(v_{r_0+2})>0.$$

Take points e_{r_0+1}, e_{r_0+2} which attain $\mu^{(\alpha)}(v_{r_0+1})$, $\mu^{(\alpha)}(v_{r_0+2})$ respectively. Then $e_{r_0} \nsim e_{r_0+1}, e_{r_0} \nsim e_{r_0+2}$ and satisfies $\beta=\beta_{r_0, r_0+1, r_0+2}$ which contradicts $[e]_\alpha$ to be a tip of branches in T_α. Hence $r_0=r-1$, Q.E.D.

Now we are ready to give the degree $\#A(\beta)$ of $\beta\in B$.

THEOREM 4.11. *Using the notations $T, T_1, T_2, \cdots, T_\lambda$ and $E\supset E_1\supset E_2\supset\cdots\supset E_\lambda$ of the trees. One has the following: for any branch point $\beta\in B$ there exists uniquely an integer α in $[0, \lambda-1]$ and a bud $e\in S_{\alpha+1}$ such that $\#A(\beta)=\#[e]_\alpha+1$. In particular, if $|E_\lambda|=1$, then putting $[e]_{\lambda-1}=E_{\lambda-1}$ and $\phi_{\lambda-1}(e)=\beta$, we have $\#A(\beta)=\#[e]_{\lambda-1}$.*

Proof. Take the vector $M(\beta)=(\mu(v_1), \mu(v_2), \cdots, \mu(v_r)), r=\#A(\beta)$. Take also the points $e_i\in E$ which attain $\mu(v_i)'s$. Put $\mu(v_{r-1})=\alpha$. Suppose $\mu(v_r)>\alpha$, then

$$M_{T_\alpha}(\beta)=(\mu^{(\alpha)}(v_1), \cdots, \mu^{(\alpha)}(v_r))$$

satisfies $\mu^{(\alpha)}(v_i)=0$ for $i=1, \cdots, r-1$ and $\mu^{(\alpha)}(v_r)>0$. Hence by Prop. 4.9 $\{e_1, e_2, \cdots, e_{r-1}\}$ constitutes a tip of branches in $\tilde{T}_\alpha$. Let e_1 be the representative of this class. Then $e_1\in S_{\alpha+1}, \phi(e_1)=\beta$. Thus we have $r=\#A(\beta)=\#[e_1]_\alpha+1$ by the construction of the vector $M_{T_\alpha}(\beta)$. In the case $|E_\lambda|=1$ it is an immediate consequence from Prop. 4.10, Q.E.D.

References

[1] S. L. Hakimi and S. S. Yau, Distance Matrix of a Graph and its Realizability, *Quart. Appl. Math.* **22** (1964).

[2] J.M.S. Simões Pereira, Some Results on the Tree Realization of a Distance Matrix, (Intern. Symp. Centre Intern. Calcul, Rome 1966), Dunod, Paris, 1967.