Quasi-Homogeneous Wave Front Set and Fundamental Solutions for the Schrödinger Operator

By Tsutomu SAKURAI

Department of Pure and Applied Sciences, College of General Education, University of Tokyo, Komaba, Meguro-ku, Tokyo 153

(Introduced by A. Kaneko)

(Received February 26, 1982)

0. Introduction

Let $P=D_t+|D_x|^2=i^{-1}\partial/\partial t-\Delta$ be the Schrödinger operator, where $D_x=(D_1, \cdots, D_n)$ denotes $i^{-1}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. We consider the initial value problem:

$$(0.1)\qquad \begin{cases} P\cdot u=0 & \text{in } \boldsymbol{R}_t\times\boldsymbol{R}_x^n, \\ u|_{t=0}=\delta(x) & \text{on } \boldsymbol{R}_x^n. \end{cases}$$

This has a unique solution E_0 in $C^\infty(\boldsymbol{R}_t:\mathcal{S}'(\boldsymbol{R}_x^n))$. In fact, E_0 is given by the following oscillatory integral:

$$(0.2)\qquad E_0(t, x)=(2\pi)^{-n}\int e^{i(\langle x,\xi\rangle-t|\xi|^2)}d\xi,$$

where $\langle x, \xi\rangle=x_1\xi_1+\cdots+x_n\xi_n$. The distribution E_0 is analytic in $t\neq 0$. That is,

$$(0.3)\qquad E_0(t, x)=(\sqrt{4\pi ti})^{-n}\exp(-|x|^2/4ti),\ t\neq 0,$$

where $\sqrt{ai}=\sqrt{|a|}\,e^{i(\pi/4)\operatorname{sgn}(a)}$ for $a\in\boldsymbol{R}$. Now set formally

$$(0.4)\qquad E_+(t, x)=iY(t)E_0(t, x),$$

where Y denotes the Heaviside function, and operate P also formally. Then we obtain

$$(0.5)\qquad P\cdot E_+=\delta(t)E_0(t, x)+iY(t)(P\cdot E_0)(t, x)=\delta(t)\delta(x).$$

However, the general theory on the wave front set (see e.g. Hörmander [5]) does not tell us that the product (0.4) can be well defined, since it can be seen immediately that

$$WF(Y_t)=\{(0, x, \pm\lambda dt)\in T^*\boldsymbol{R}^{n+1}\backslash 0\,;\ x\in\boldsymbol{R}^n, \lambda>0\},$$

and

$$(0, 0, -dt)\in WF(E_0).$$

In this paper, employing the quasi-homogeneous wave front set introduced by Lascar in [8], we show that such a product as in (0.4) becomes well defined in the space of distributions. In addition to this, using the expression (0.2), (0.4), we give the precise estimate of the singularities of E_+. And more generally in section 3, combining with a technique of the asymptotic expansion, we construct micro-parametrices for a class of quasi-homogeneous pseudo-differential operators with real principal symbols.

During the preparation of this paper, Parenti-Segala [10] also constructed micro-parametrices and proved the propagation and reflection of singularities for more general class of operators. See also Ohtsuka [9], who studied the reflection of singularities for the Schrödinger operator. However, our construction is different in some points and we believe that ours is more natural at least concerning the Schrödinger operator. Especially, our construction clarifies some refined structure of singularities of fundamental solutions for quasi-homogeneous differential operators, as an extension of the classical Duhamel's principal. Also, the precise estimate of the quasi-homogeneous wave front set obtained here (Theorem 2.1) seems to be unknown in the literature.

1. Quasi-homogeneous wave front set

Let us fix some notation. $x=(x_1, \cdots, x_n)$ denotes an element of $\boldsymbol{R}^n$, $\xi=\xi_1, \cdots, \xi_n)$ an element of $\boldsymbol{R}_n$; the dual space of $\boldsymbol{R}^n$. A multi-weight on $\boldsymbol{R}_n$ is a n-tuple $M=(\mu_1, \cdots, \mu_n)$ of real numbers satisfying $\min\{\mu_j\}=1$. For $\xi=(\xi_1, \cdots, \xi_n)\in \boldsymbol{R}_n\backslash 0$ and $\lambda>0$ we put $\lambda^M\xi=(\lambda^{\mu_1}\xi_1, \cdots, \lambda^{\mu_n}\xi_n)$. A subset $\Gamma\subset\boldsymbol{R}_n\backslash 0$ (resp. $V\subset\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)$) is called M-cone, if it is invariant under the transformation: $\xi\longmapsto\lambda^M\xi$ (resp. $(x, \xi)\longmapsto(x, \lambda^M\xi)$) for all $\lambda>0$. $[\xi]_M$ denotes a corresponding weight function on $\boldsymbol{R}_n$ defined as follows: For a given $\xi^0\in\boldsymbol{R}_n\backslash 0$, we can uniquely take $\lambda_0>0$ and $\omega^0\in S^{n-1}=\{\xi\in\boldsymbol{R}_n\,;\,|\xi|^2=\sum|\xi_j|^2=1\}$ so that $\xi^0=\lambda_0^M\omega^0$. For such λ_0, ω^0 we define $[\xi^0]_M=\lambda_0$, and by convention $[0]_M=0$. From the very definition $[\xi]_M$ has the weighted homogeneity $[\lambda^M\xi]_M=\lambda[\xi]_M$ and satisfies, with a positive constant C,

$$C^{-1}\sum_{j=1}^n|\xi_j|^{1/\mu_j}\leqq[\xi]_M\leqq C\sum_{j=1}^n|\xi_j|^{1/\mu_j}.$$

S_M^m denotes a class of symbols of M-pseudo-differential operators. That is, by $p(x, \xi)\in S_M^m$ we mean:

(i) $p(x, \xi)\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}_n)$.

(ii) For all multi-indices α, β the estimate

$$|\partial_x^\beta\partial_\xi^\alpha p(x, \xi)|\leqq C_{\alpha,\beta}(1+[\xi]_M)^{m-\langle M,\alpha\rangle}$$

is valid with some constant $C_{\alpha,\beta}$.

Moreover, we say that $p(x, \xi)$ belongs to a class of symbols of the classical M-pseudo-differential operators if $p(x, \xi)$ has the following asymptotic expansion:

$$p(x, \xi)\sim p_m(x, \xi)+\sum_{j=1}^\infty p_{m_j}(x, \xi),$$

where $m-1\geq m_1\geq m_2\geq\cdots\rightarrow-\infty$ and each term p_{m_j} has the weighted homogeneity:

$$p_{m_j}(x, \lambda^M\xi)=\lambda^{m_j}p_{m_j}(x, \xi),\ \lambda>0.$$

This subclass of symbols is denoted by $S^m_{M,cl}$ and in such an expansion the top term p_m is called the principal symbol. The operation P of $p(x, \xi)\in S^m_M$ on $u\in \mathcal{S}(\boldsymbol{R}^n)$ is defined by:

$$\begin{aligned} P\cdot u(x)&=p(x, D_x)u(x) \\ &=(2\pi)^{-n}\int e^{i\langle x,\xi\rangle}p(x, \xi)\hat{u}(\xi)d\xi, \end{aligned}$$

where $\hat{u}$ is the Fourier transform of u:

$$\hat{u}(\xi)=\int e^{-i\langle y,\xi\rangle}u(y)dy.$$

In what follows, we only use operators of the classical type.

We employ the following definition of the quasi-homogeneous wave front set. It corresponds to Proposition 3.6 of Lascar [8].

Definition 1.1. If $u\in\mathcal{D}'(\boldsymbol{R}^n)$, then $WF_M(u)$ denotes the closed subset in $\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)$ defined in the following way: For $(x^0, \xi^0)\in\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)$,

$$(x^0, \xi^0)\notin WF_M(u)$$

means that there exist a $\phi\in C_0^\infty(\boldsymbol{R}^n)$ satisfying $\phi(x^0)\neq 0$ and a neighborhood $U\subset \boldsymbol{R}_n\backslash 0$ of ξ^0 such that for every integer N we have, with a suitably chosen constant C_N,

$$|\widehat{\phi u}(\lambda^M\xi)|\leq C_N\lambda^{-N}, \tag{1.1}$$

for $\lambda>0$, uniformly in $\xi\in U$.

Remark 1.2. It is clear that $WF_M(u)$ is a closed M-cone in $\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)$. This is a natural extension of the wave front set in C^∞-category. In fact, when we take $M=(1, \cdots, 1)$ it is reduced to the usual wave front set.

Among basic techniques in pseudo-differential operators we have partitions of unity in the dual space. Therefore, we present the following lemma though the proof is routine.

Lemma 1.3. *Let $\{\Gamma^j_M\}^N_{j=1}$ be an arbitrary open M-conic covering of $\boldsymbol{R}_n\backslash 0$. Then we can construct a corresponding "pseudo-differential partition of unity" $\{\phi_j\}^N_{j=0}$. Here $\phi_0\in C_0^\infty(\boldsymbol{R}_n)$ which is equal to 1 in a neighborhood of the origin, and $\phi_j\in S^0_{M,cl}$ so that* $\operatorname{supp}\phi_j\subset\Gamma^j_M$, *$j=1, 2, \cdots, N$, and they satisfy $\sum^N_{j=0}\phi_j=1$ on $\boldsymbol{R}_n$.*

Next, we state the micro-local inversion theorem for M-pseudo-differential operators, which is Proposition 3.3 of Lascar [8].

THEOREM 1.4. *Let $P=p(x, D_x)\in S^m_{M,cl}$. Then for every $u\in\mathcal{D}'(\boldsymbol{R}^n)$ we have*

$$WF_M(P\cdot u)\subset WF_M(u)\subset WF_M(P\cdot u)\cup\Sigma_M(P), \tag{1.2}$$

where

$$\Sigma_M(P)=\{(x, \xi)\in\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)\,;\; p_m(x, \xi)=0\}.$$

Outline of Proof. Suppose $p(x, \xi)\sim p_m(x, \xi)+\sum p_{m_j}(x, \xi)$. If $p_m(x^0, \xi^0)\neq 0$, then we can construct a left microparametrix $q(x, \xi)\in S^{-m}_{M\,cl}$ such that $q(x, D_x)\cdot p(x, D_x)=$ identity$+r(x, D_x)$. Here $r(x, \xi)$ is of order $-\infty$, in some M-conic neighborhood of (x^0, ξ^0).

Now we shall give one of the main results of this paper concerning the product of two distributions. This is absent from the results of Lascar [8] or Parenti-Segala [10].

THEOREM 1.5. *Let $u_1, u_2\in\mathcal{D}'(\boldsymbol{R}^n)$. If*

$$(x^0, 0)\notin\{(x, \xi^1+\xi^2)\in\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)\,;\; (x, \xi^j)\in WF_M(u_j),\ j=1, 2\},$$

then, on some neighborhood $U\subset\boldsymbol{R}^n$ of x^0, we can naturally define the product u_1u_2 as a distribution. Moreover, for every $\phi\in C_0^\infty(U)$ we have

$$\begin{aligned}WF_M(\phi u_1u_2)\subset WF_M(u_1)\cup WF_M(u_2)\cup\{&(x, \xi^1+\xi^2)\,;\\ &x\in\operatorname{supp}\phi \text{ and } (x, \xi^j)\in WF_M(u_j),\ j=1, 2\}.\end{aligned} \tag{1.3}$$

First, we prove the following elementary inequality.

LEMMA 1.6. *Let Γ^1_M, Γ^2_M be closed M-cones in $\boldsymbol{R}_n\backslash 0$, satisfying*

$$0\notin\Gamma^1_M+\Gamma^2_M. \tag{1.4}$$

Then there exists a positive constant C such that for every $\eta=\xi^1+\xi^2$, with $\xi^j\in\Gamma^j_M$, $j=1, 2$ we have

$$[\eta]_M\geq C([\xi^1]_M+[\xi^2]_M). \tag{1.5}$$

Proof. Take $\omega^j\in S^{n-1}$ so that $\xi^j=\lambda_j^M\omega^j$ with $\lambda_j=[\xi^j]_M$, $j=1, 2$. Consider

$$[\eta]_M/([\xi^1]_M+[\xi^2]_M)=[\lambda_1^M\omega^1+\lambda_2^M\omega^2]_M/(\lambda_1+\lambda_2).$$

This becomes homogeneous of degree 0 with respect to (λ_1, λ_2) and approaches 1 as either of λ_j approaches 0, thus can be regarded as a continuous function on $\tilde{\Gamma}_M^1 \times \tilde{\Gamma}_M^2 \times \{(\lambda_1, \lambda_2) \in \boldsymbol{R}^2 ;\ \lambda_1 \geq 0,\ \lambda_2 \geq 0 \text{ and } \lambda_1 + \lambda_2 = 1\}$, where $\tilde{\Gamma}_M^j = \Gamma_M^j \cap S^{n-1}$, $j=1, 2$. Therefore, the minimum is attained on this compact set, which, however, cannot be 0 by the hypothesis (1.4).

Proof of Theorem 1.5. We define the product in the dual space as a convolution. Choose a neighborhood $U \subset \boldsymbol{R}^n$ of x^0 sufficiently small. Then we can find closed M-cones Γ_M^1, Γ_M^2 in $\boldsymbol{R}_n \backslash 0$ satisfying (1.4), such that for every $\phi \in C_0^\infty(U)$, $\widehat{\phi u_1}$ resp. $\widehat{\phi u_2}$ are rapidly decreasing in the complement of Γ_M^1 resp. Γ_M^2. Taking account of the inequality (1.5) we can easily show that $\widehat{\phi u_1} * \widehat{\phi u_2}(\xi)$ has polynomial growth. The estimate (1.3) follows immediately if one divides the integral by a pseudo-differential partition of unity. The proof of the theorem is complete.

This result, combined with Theorem 1.4, justifies the product (0.4). We present another typical example of such a product.

Example 1.8. Consider $P = D_t + D_x^2$ on $\boldsymbol{R}^2$. If $u \in \mathcal{D}'(\boldsymbol{R}^2)$ satisfies $P \cdot u \in C^\infty(\boldsymbol{R}^2)$, then we have, with $M = (2, 1)$,

$$WF_M(u) \subset \boldsymbol{R}^2 \times \{(-k^2, \pm k) \in \boldsymbol{R}_2 \backslash 0 ;\ k > 0\}.$$

Thus we can define, for every integer l, the l-th power of u:

$$u^l = \overbrace{u \cdot u \cdot \cdots \cdot u}^{l} \in \mathcal{D}'(\boldsymbol{R}^2).$$

Moreover, we obtain

$$WF_M(u^l) \subset \boldsymbol{R}^2 \times \{(-k^2, \theta k) \in \boldsymbol{R}_2 \backslash 0 ;\ k > 0,\ |\theta| \leq \sqrt{l}\}.$$

If we consider products with the δ-function of one variable, then follows

Corollary 1.9. *Consider the space* $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$. *Let* $M = (\nu, \mu_1, \cdots, \mu_n)$ *be a corresponding weight on the dual space and put* $M' = (\mu_1', \cdots, \mu_n')$, *where* $\mu_j' = \mu_j / \min\{\mu_i\}$, $j = 1, 2, \cdots, n$. *If* $u \in \mathcal{D}'(\boldsymbol{R}_t \times \boldsymbol{R}_x^n)$ *satisfies*

$$(0, 0) \times (\pm 1, 0, \cdots, 0) \notin WF_M(u),$$

then there exists a neighborhood U *of the origin in* $\boldsymbol{R}_x^n$ *such that, for every integer* γ, *the restriction* $u_0^{(\gamma)} = D_t^\gamma u|_{t=0}$ *is well defined as a distribution on* U. *And we have for every* $\phi \in C_0^\infty(U)$,

$$WF_{M'}(\phi u_0^{(\gamma)}) \subset \{(x, \xi) \in \boldsymbol{R}^n \times (\boldsymbol{R}_n \backslash 0) ;\ x \in \operatorname{supp} \phi \text{ and } (0, x, \tau, \xi) \in WF_M(u) \text{ for some } \tau\}. \tag{1.6}$$

The two following assertions are proved by using pseudo-differential partitions of unity.

PROPOSITION 1.10. *Under the same notation as in Corollary* 1.9, *assume* $u\in\mathcal{D}'(\boldsymbol{R}_t\times\boldsymbol{R}^n)$ *has a proper support in* $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$ *under the projection* $\boldsymbol{R}_t\times\boldsymbol{R}^n_x\ni(t,x)\longmapsto x\in\boldsymbol{R}^n_x$. *Then for the integration along fibres*

$$f(x)=\int u(t,x)dt\in\mathcal{D}'(\boldsymbol{R}^n_x), \tag{1.7}$$

which is defined by the duality

$$\langle f,\varphi\rangle_x=\langle u,1\otimes\varphi\rangle_{t,x}\quad \textit{for every } \varphi\in C_0^\infty(\boldsymbol{R}^n_x),$$

we have

$$WF_{M'}(f)\subset\{(x,\xi)\in\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)\,;\,(t,x,0,\xi)\in WF_M(u)\ \text{for some}\ t\}. \tag{1.8}$$

THEOREM 1.11. *Let* $u_1\in\mathcal{E}'(\boldsymbol{R}^n)$, $u_2\in\mathcal{D}'(\boldsymbol{R}^n)$. *Then we have for the convolution* u_1*u_2

$$WF_M(u_1*u_2)\subset\{(x^1+x^2,\xi)\in\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)\,;\,(x^j,\xi)\in WF_M(u_j),\ j=1,2\}. \tag{1.9}$$

REMARK 1.12. Theorem 1.11 is a special case of the estimate (2.4) in Parenti-Segala [10].

2. Fundamental solutions for the Schrödinger operator

Now let us return to the study of the Schrödinger operator $P=D_t+|D_x|^2$ in $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$. As mentioned in the introduction,

$$E_0(t,x)=(2\pi)^{-n}\int e^{i(\langle x,\xi\rangle-t|\xi|^2)}d\xi \tag{2.1}$$

is the unique solution in $C^\infty(\boldsymbol{R}_t:\mathcal{S}'(\boldsymbol{R}^n_x))$ of the following:

$$\begin{cases} P\cdot u=0 & \text{in}\ \ \boldsymbol{R}_t\times\boldsymbol{R}^n_x,\\ u|_{t=0}=\delta(x) & \text{on}\ \ \boldsymbol{R}^n_x.\end{cases} \tag{2.2}$$

Then in view of Theorem 1.4, 1.5, the products

$$E_\pm(t,x)=\pm iY(\pm t)E_0(t,x)\qquad (Y\text{: Heaviside function}) \tag{2.3}$$

are well defined and satisfy

$$P\cdot E_\pm=\delta\qquad \text{in}\ \ \boldsymbol{R}_t\times\boldsymbol{R}^n_x. \tag{2.4}$$

We obtain the following description of the singularities of $E_\pm$. That is, the quasi-homogeneous wave front set of $E_\pm$ agree with the union of *half bicharacteristics* in the corresponding direction.

THEOREM 2.1. *Let* $M=(2,1,\cdots,1)$. *We have*

$$(2.5)\pm \qquad WF_M(E_\pm)=\{(0, sk, -|k|^2, k)\in \boldsymbol{R}^{n+1}\times(\boldsymbol{R}_{n+1}\backslash 0)\,;\; k\in \boldsymbol{R}_n\backslash 0,\ \pm s>0\}\cup(\{0\}\times(\boldsymbol{R}_{n+1}\backslash 0)).$$

Proof. We shall show this for E_+. Consider

$$(2.6) \qquad E_+(t,x)=(2\pi)^{-n}iY(t)\int e^{i(\langle x,\xi\rangle-t|\xi|^2)}d\xi.$$

Note that E_0 is analytic in $t\neq 0$ and apply Theorem 1.4 to (2.4). Then we obtain

$$WF_M(E_+)\subset\{(0,x,-|k|^2,k)\in \boldsymbol{R}^{n+1}\times(\boldsymbol{R}_{n+1}\backslash 0)\,;\; x\in \boldsymbol{R}^n,\ k\in \boldsymbol{R}_n\backslash 0\}\cup(\{0\}\times(\boldsymbol{R}_{n+1}\backslash 0)).$$

Next, we show that each element $(-|k|^2, k)$ in the dual space appears in $WF_M(E_+)$ only on the half line $\{(0, sk)\in \boldsymbol{R}^{n+1}\,;\, s\geqq 0\}$.

For a given $\varepsilon>0$, we choose $\phi_\varepsilon\in C^\infty([0,\infty))$ so that $\phi_\varepsilon=0$ in $[0,\varepsilon/2]$ and $\phi_\varepsilon=1$ in $[\varepsilon,\infty)$, and we set for a fixed $k^0\in \boldsymbol{R}_n\backslash 0$

$$\chi_1^\varepsilon(\xi)=\phi_\varepsilon(\xi/|\xi|-k^0/|k^0|)\phi_\varepsilon(|\xi|),$$
$$\chi_2^\varepsilon(\xi)=1-\chi_1^\varepsilon(\xi).$$

Now we divide the integral in (2.6) into two parts

$$E_+(t,x)=(2\pi)^{-n}iY(t)\left[\int e^{i(\langle x,\xi\rangle-t|\xi|^2)}\chi_1^\varepsilon(\xi)d\xi + \int e^{i(\langle x,\xi\rangle-t|\xi|^2)}\chi_2^\varepsilon(\xi)d\xi\right]$$
$$=(2\pi)^{-n}iY(t)[I_1^\varepsilon+I_2^\varepsilon].$$

Obviously, we have

$$(2.7) \qquad WF_M(I_2^\varepsilon)\subset\{(0,x,-|k|^2,k)\in \boldsymbol{R}^{n+1}\times(\boldsymbol{R}_{n+1}\backslash 0)\,;\; x\in \boldsymbol{R}^n,\ |k/|k|-k^0/|k^0||\leq\varepsilon\}.$$

To study the singularities of

$$(2.8) \qquad I_1^\varepsilon=\int e^{i(\langle x,\xi\rangle-t|\xi|^2)}\chi_1^\varepsilon(\xi)d\xi,$$

we need the following lemma:

LEMMA 2.2. *Assume* $x^0\in \boldsymbol{R}^n\backslash 0$ *satisfies* $x^0/|x^0|=k^0/|k^0|$. *Then for every* $\varepsilon>0$

there exists a neighborhood $U\subset \boldsymbol{R}^n$ *of* x^0 *such that we have*

(2.9) $$I_1^\varepsilon\in C^\infty(\overline{\boldsymbol{R}}_t^+\times U),$$

where $\overline{\boldsymbol{R}}_t^+=\{t\in\boldsymbol{R}\,;\,t\geq 0\}$.

Proof of Lemma 2.2. Choose a small neighborhood $U\subset\boldsymbol{R}^n$ of x^0 so that we have with some constant $C>0$

$$|x-2t\xi|\geq C,$$

for every $x\in U$, $\xi\in\operatorname{supp}\chi_1^\varepsilon$, and $t\geq 0$. Then we define the differential operator L whose formal adjoint is

$$^tL=|x-2t\xi|^{-2}\textstyle\sum_{j=1}^n(x_j-2t\xi_j)\partial/\partial\xi_j,$$

so that

$$e^{i(\langle x,\xi\rangle-t|\xi|^2)}={}^tL\cdot e^{i(\langle x,\xi\rangle-t|\xi|^2)},$$

and substitute this in (2.8). After N-fold integration by parts we obtain

$$I_1^\varepsilon(t,x)=\int e^{i(\langle x,\xi\rangle-t|\xi|^2)}((L)^N\cdot\chi_1^\varepsilon)(t,x,\xi)d\xi.$$

Put $F_N(t,x,\xi)=((L)^N\cdot\chi_1^\varepsilon)(t,x,\xi)$, then F_N satisfies for all multi-indices $\alpha,(\gamma,\beta)$ and $x\in U$, $t\geq 0$

(2.10) $$|\partial_t^\gamma\partial_x^\beta\partial_\xi^\alpha F_N(t,x,\xi)|\leq C(1+|\xi|)^{\gamma-|\alpha|-N},$$

where C is a constant independent of t,x. Since we can take an integer N arbitrarily, (2.9) follows from the estimate (2.10).

End of Proof of Theorem 2.1. From the preceding lemma and (2.7) we can conclude that away from the origin

(2.11) $$WF_M(E_+)\subset\{(0,sk,-|k|^2,k)\in\boldsymbol{R}^{n+1}\times(\boldsymbol{R}_{n+1}\backslash 0)\,;\,k\in\boldsymbol{R}_n\backslash 0,\ s>0\}.$$

Finally, we must show the opposite inclusion.

Assume that for some x^0, the set:

$$\{(0,x^0,-|k|^2,k)\in\boldsymbol{R}^{n+1}\times(\boldsymbol{R}_{n+1}\backslash 0)\,;\,k/|k|=x^0/|x^0|\}$$

is not contained in $WF_M(E_+)$. Since E_+ is invariant under the action of the orthogonal group $O(n)$ on x-variables, it follows that E_+ is a smooth function in some neighborhood of $\{(0,x)\in\boldsymbol{R}^{n+1}\,;\,|x|=|x^0|\}$. Choose $\varphi\in C_0^\infty(\boldsymbol{R}^{n+1})$ which is equal to 1 on a neighborhood of

$$\operatorname{sing\,supp}E_+\cap\{(t,x)\in\boldsymbol{R}^{n+1}\,;\,|x|<|x^0|\},$$

and whose support is contained in $\{(t,x)\in\boldsymbol{R}^{n+1}\,;\,|x|<|x^0|\}$. Then for $\varphi E_+\in\mathcal{E}'(\boldsymbol{R}^{n+1})$

we have, with $\Phi\in C_0^\infty(\boldsymbol{R}^{n+1})$,

$$P\cdot(\varphi E_+)=\delta+\Phi. \tag{2.12}$$

We apply Theorem 3.6.1 of Hörmander [2] to (2.12) and obtain

$$\operatorname{sing\,supp} E_+=\{0\}. \tag{2.13}$$

This implies hypo-ellipticity of P, which is a contradiction. Now the proof of the theorem is complete.

Use $E_\pm$ as left parametrices, and apply Theorem 1.11. Then we obtain a proof of the following result on the propagation of singularities, which is partial but more natural compared with that of Lascar [8] or Parenti-Segala [10].

Corollary 2.3. *Let $P=D_t+|D_x|^2$ and $M=(2,1,\cdots,1)$. Then for every $u\in\mathcal{D}'(\boldsymbol{R}_t\times\boldsymbol{R}_x^n)$, the subset of $\Sigma_M(P)$:*

$$WF_M(u)\setminus WF_M(P\cdot u)$$

is invariant under the flow of the vector field in the hyperplane $t=constant$:

$$H_p^M=\langle\xi,\nabla_x\rangle=\textstyle\sum_{j=1}^n\xi_j\partial/\partial x_j.$$

3. Construction of micro-parametrices for a class of quasi-homogeneous pseudo-differential operators

In this section, we construct right parametrices for a operator of the form:

$$P=D_t+q(t,x,D_x)\quad\text{in}\quad \boldsymbol{R}_t\times\boldsymbol{R}_x^n. \tag{3.1}$$

Here we assume:

(A-I) $q\in C^\infty(\boldsymbol{R}_t:S_{cl}^m(\boldsymbol{R}^n))$ with an integer $m\geq 2$.
That is, $q(t,x,\xi)$ has the asymptotic expansion:

$$q(t,x,\xi)\sim\textstyle\sum_{j=0}^\infty q_{m-j}(t,x,\xi),$$

where $q_{m-j}\in C^\infty(\boldsymbol{R}\times\boldsymbol{R}^n\times(\boldsymbol{R}_n\setminus 0)$ is homogeneous of degree $m-j$ with respect to ξ.

(A-II) The principal symbol q_m of q is real.

We take a corresponding weight $M=(m,1,\cdots,1)$ and define the M-Hamiltonian vector field H_p^M following Lascar [8]:

$$H_p^M=\textstyle\sum_{j=1}^n[(\partial q_m/\partial\xi_j)\partial/\partial x_j-(\partial q_m/\partial x_j)\partial/\partial\xi_j]. \tag{3.2}$$

Let Φ_s denote the flow of H_p^M, then we put

$$C^\pm=\{(\rho,\rho')\in\Sigma_M(P)\times\Sigma_M(P);\ \rho=\Phi_s(\rho')\ \text{ for some }\ \pm s>0\}. \tag{3.3}$$

Here C^+ and C^- are the forward and backward characteristic relations. We obtain a result on the existence of micro-parametrices (c. f. Theorem 6 of Parenti-Segala [10]).

THEOREM 3.1. *Let P and M be as above. If $\rho^0=(t^0, x^0, \tau^0, \xi^0)\in\Sigma_M(P)$ satisfies $H_P^M(\rho^0)\neq 0$. Then there exist a M-conic neighborhood $V_M\subset \boldsymbol{R}^{n+1}\times(\boldsymbol{R}_{n+1}\backslash 0)$ of ρ^0 and distribution kernels $E_\pm(t, x, s, y)\in\mathcal{D}'(\boldsymbol{R}^{n+1}\times\boldsymbol{R}^{n+1})$ such that*

(i) $\operatorname{supp} E_\pm\subset\{(t, x, s, y)\in\boldsymbol{R}^{n+1}\times\boldsymbol{R}^{n+1}\,;\ \pm(t-s)\geqq 0\}$,

(ii) $WF'_{(M,M)}(E_\pm)\cap(V_M\times V_M)\subset(\Delta\cup C^\pm)\cap(V_M\times V_M)$,

(iii) $WF'_{(M,M)}(P\cdot E_\pm-I)\cap(V_M\times V_M)=\emptyset$,

where Δ denotes the diagonal set of $(\boldsymbol{R}^{n+1}\times(\boldsymbol{R}_{n+1}\backslash 0))^2$ and for $E\in\mathcal{D}'(\boldsymbol{R}^{n+1}\times\boldsymbol{R}^{n+1})$, $WF'_{(M,M)}(E)$ is defined by:

$$WF'_{(M,M)}(E)=\{(t, x, \tau, \xi, s, y, -\sigma, -\eta)\,;\ (t, x, s, y, \tau, \xi, \sigma, \eta)\in WF_{(M,M)}(E)\}.$$

Outline of Proof. We choose a neighborhood V_M of ρ^0 in the form:

$$V_M=T\times U\times\Gamma_M,$$

where $T=[t^0-\varepsilon, t^0+\varepsilon]$ for some $\varepsilon>0$, $U\subset\boldsymbol{R}^n$ is a neighborhood of x^0, and $\Gamma_M\subset\boldsymbol{R}_{n+1}\backslash 0$ is a M-conic neighborhood of (τ^0, ξ^0). We put

$$\Gamma'=\{\xi\in\boldsymbol{R}_n\backslash 0\,;\ (\tau, \xi)\in\Gamma_M \quad\text{for some } \tau\},$$

which is a conic neighborhood of ξ^0 in $\boldsymbol{R}_n\backslash 0$. As in Hörmander [3] we construct a phase function $\phi(s, x, y, \xi)\in C^\infty(T\times U\times U\times\Gamma')$ so that

(1) ϕ is homogeneous of degree 1 with respect to ξ and

$$\phi(s, x, y, \xi)=\langle x-y, \xi\rangle+O(|x-y|^2|\xi|).$$

(2) $q_m(s, x, \nabla_x\phi)=q_m(s, y, \xi)$.

We consider the following oscillatory integral:

$$\text{(3.4)}\qquad E_0(t, x, s, y)$$
$$=(2\pi)^{-n}\int e^{i(\phi(s,x,y,\xi)-(t-s)q_m(s,y,\xi))}a(s, x, y, \xi)d\xi,$$

with an amplitude function $a\in C^\infty(\boldsymbol{R}_s\,;S^0(\boldsymbol{R}_n))$, which we define afterwards so that we have

$$WF'_{(M,M)}(P\cdot E_0)\cap(V_M\times V_M)=\emptyset.$$

Due to the asymptotic formula, we have

(3.5) $$P\cdot E_0(t, x, s, y)$$
$$=(2\pi)^{-n}\int e^{i(\phi(s,x,y,\xi)-(t-s)q_m(s,y,\xi))}h(t, x, s, y, \xi)d\xi.$$

Here h is given by:

(3.6) $$h(t, x, s, y, \xi)\sim -q_m(s, y, \xi)a(s, x, y, \xi)$$
$$+\sum_\alpha(\partial_\xi^\alpha q)(t, x, \nabla\phi)[D_z^\alpha(e^{i\phi_z}a(s, z, y, \xi)|_{z=x}]/\alpha!,$$

where $\phi_z=\phi(s, z, y, \xi)-\phi(s, x, y, \xi)-\langle\nabla_x\phi(s, x, y, \xi), z-x\rangle$. Now we define the differential operator L whose formal adjoint is ${}^tL=g(s, x, y, \xi)+K(s, y, \xi, \nabla_\xi)$ so that

(3.7) $$(t-s)e^{i(\phi(s,x,y,\xi)-(t-s)q_m(s,y,\xi))}$$
$$=(g(s, x, y, \xi)+K(s, y, \xi, \nabla_\xi))e^{i(\phi(s,x,y,\xi)-(t-s)q_m(s,y,\xi))}.$$

That is,

$$g(s, x, y, \xi)=|\nabla_\xi q_m(s, y, \xi)|^{-2}\langle\nabla_\xi q_m(s, y, \xi), \nabla_\xi\phi(s, x, y, \xi)\rangle,$$

$$K(s, y, \xi, \nabla_\xi)=i|\nabla_\xi q_m(s, y, \xi)|^{-2}\langle\nabla_\xi q_m(s, y, \xi), \nabla_\xi\rangle.$$

Substitute the formal Taylor expansion of h with respect to t around s in (3.5). Then, on account of (3.7) and integration by parts, we obtain formally (3.5) with h replaced by $\tilde{h}(s, x, y, \xi)$ with the asymptotic sum:

(3.8) $$\tilde{h}(s, x, y, \xi)\sim -q_m(s, y, \xi)a(s, x, y, \xi)$$
$$+\sum_{\gamma,\alpha} L^\gamma\cdot[(\partial_t^\gamma\partial_\xi^\alpha q)(s, x, y, \nabla\phi)(D_z^\alpha(e^{i\phi_z}a)|_{z=x})]/\gamma!\,\alpha!.$$

We shall now choose the amplitude function $a(s, x, y, \xi)\sim\sum a_\nu(s, x, y, \xi)$, where a_ν is homogeneous of degree $-\nu$, so that $a_0(s, y, y, \xi)=1$ and in (3.8) the sum of all terms of order $m-1-\nu$ with respect to ξ vanishes. This means, a_ν solves a sequence of transport equations:

(3.9) $$\langle(\nabla_\nu q_m)(s, x, \nabla\phi), D_x\rangle a_\nu+\tilde{q}_{m-1}(s, x, y, \xi)a_\nu=r_\nu, \quad m\neq 2,$$

(3.9)′ $$\langle(\nabla_\xi q_2)(s, x, \nabla\phi), D_x\rangle a_\nu+[\tilde{q}_1(s, x, y, \xi)+g(s, x, y, \xi)(\partial_t q_2)(s, x, \nabla\phi)]a_\nu=r_\nu,$$

where

$$\tilde{q}_{m-1}(s, x, y, \xi)=q_{m-1}(s, x, \nabla\phi)+\sum_{|\alpha|=2}(\partial_\xi^\alpha q_m)(s, x, \nabla\phi)\times(\partial_x^\alpha\phi)(s, x, y, \xi)/i\alpha!,$$

r_ν is determined by $a_0, \cdots, a_{\nu-1}$ and independent of $\partial_t^\gamma q_{m-j}$ for $j+(m-1)\gamma>\nu$.

Summing up, we shall now define E_0 by (3.4), then for every integer N we can write the symbol of $P\cdot E_0$ in the following way:

$$P\cdot E_0=(2\pi)^{-n}\int e^{i(\phi-(t-s)q_m)}[b_N+c_N]d\xi.$$

Here

(1) $b_N(s, x, y, \xi) \in C^\infty(R_s : S^{-N}(U \times \Gamma'))$.

(2) $c_N(t, x, s, y, \xi) \in C^\infty(T \times U \times T \times U \times \Gamma')$, and satisfies for all multi-indices α, β, β' and $(t, x, s, y, \xi) \in T \times U \times T \times U \times \Gamma'$

$$|\partial_x^\beta \partial_y^{\beta'} \partial_\xi^\alpha c_N(t, x, s, y, \xi)| \leq C|t-s|^N |\xi|^{2-\alpha},$$

where C is a constant independent of t, x, s, y.

Again with the integration by parts, since N is chosen arbitrarily, we conclude that

$$WF'_{(M,M)}(P \cdot E_0) \cap (V_M \times V_M) = \emptyset .$$

And the condition $a_0(s, y, y, \xi) = 1$ ensures the pseudo-differential operator

$$E_0(s, x, s, y) = A(s, x, D_x), \quad s \in T \tag{3.10}$$

being elliptic on $U \times \Gamma'$, where s is considered as a C^∞-parameter. Now we construct $A^{-1}(s, x, D_x)$ and set

$$E_\pm(t, x, s, y) = \pm i Y(\pm(t-s))(E_0 \cdot A^{-1})(t, x, s, y). \tag{3.11}$$

Then, by similar arguments as in section 2, we can verify that $E_\pm$ satisfy all requirements of the theorem.

If P satisfies the assumptions of Theorem 3.1, then its formal asjoint ${}^tP$ has the same properties. Thus we obtain another proof of a part of the results in Lascar [8] on the propagation of singularities.

COROLLARY 3.2. *Let P satisfy* (3.1), (A-I, II) *and let $M=(m, 1, \cdots, 1)$ be the corresponding weight. Then for every $u \in \mathcal{D}'(\boldsymbol{R}_t \times \boldsymbol{R}_x^n)$, the subset of $\Sigma_M(P)$:*

$$WF_M(u) \setminus WF_M(P \cdot u)$$

is invariant under the flow of H_p^M.

Remark 3.3. If we use a of the asymptotic sum:

$$a(s, x, y, \xi) \sim \sum_{\nu=0}^\infty a_{m_\nu}(s, x, y, \xi)$$

with $0 = m_0 \geq m_1 \geq \cdots \to -\infty$, then by the same arguments we can construct micro-parametrices for P with $q \in C^\infty(\boldsymbol{R}_t : S^m)$ for every real $m \geq 2$. Moreover, if the principal symbol of q is independent of t, it suffices to assume $m > 1$.

Example 3.4. Consider $P = D_t + D_1 D_2$ in $\boldsymbol{R}_t \times \boldsymbol{R}_x^2$, and set

$$E_0(t, x) = (2\pi)^{-2} \int e^{i(\langle x, \xi \rangle - t\xi_1 \xi_2)} d\xi . \tag{3.12}$$

Then we obtain two fundamental solutions for P:

$$E_{\pm}(t, x)=\pm iY(\pm t)E_0(t, x). \tag{3.13}$$

This situation should be compared with the case that $P'=D_1D_2$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^2$, which has four distinguished fundamental solutions:

$$E_1(t, x)=-\delta(t)Y(x_1)Y(x_2),$$

$$E_2(t, x)=\delta(t)Y(-x_1)Y(x_2),$$

$$E_3(t, x)=-\delta(t)Y(-x_1)Y(-x_2),$$

$$E_4(t, x)=\delta(t)Y(x_1)Y(-x_2).$$

Acknowledgements

I should like to express my hearty gratitude to Professor A. Kaneko for his valuable suggestions and encouragements.

References

[1] Boutet de Monvel, L., Propagation des singularités des solutions d'équations analogues à l'équation de Schrödinger, *Lecture Notes in Math.*, **459**, Springer, 1974, pp. 1–44.
[2] Hörmander, L., *Linear Partial Differential Operators*, Springer, 1963.
[3] ———, On the singularities of solutions of partial differential equations, *Proc. Intern. Conf. on Functional Analysis and related topics*, Univ. of Tokyo Press, 1969, pp. 31–40.
[4] ———, On the existence and the regularity of solutions of linear pseudo-differential equations, *L'Enseignement Math.* **17** (1971), 99–163.
[5] ———, Fourier Integral Operators I, *Acta Math.* **127** (1971), 79–183.
[6] Kaneko, A., *Linear Parial differential Operators with Constant Coefficients*, (in Japanese) Iwanami, Tokyo 1976.
[7] Kumano-go, H., *Pseudo-Differential Operators*, (in Japanese) Iwanami, Tokyo 1974.
[8] Lascar, R., Propagation des singularités des solutions d'équations pseudo-différentielles quasi homogènes, *Ann. Inst. Fourier*, **27, 2** (1977), 79–123.
[9] Ohtsuka, Y., *On the reflection of singularities for the Schrödinger operator*, (in Japanese) Master's thesis, Sophia Univ., 1980.
[10] Parenti, C. and Segala, F., Propagation and reflaction of singularities for a class of evolution equations, *Comm. in P.D.E.* **6(7)** (1981), 741–782.