Projective Threefolds with Small Secant Varieties

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Introduction

This note is intended to be a supplement to [2], and we consider here the following problem: Let X be a smooth 3-fold in $\boldsymbol{P}^7$ not contained in any hyperplane. Then, for what X we have $\mathrm{Sec}(X) \neq \boldsymbol{P}^7$, or equivalently, X can be projected isomorphically onto $\boldsymbol{P}^6$?

As was suspected in [2], we have the following answer in char$(k)=0$: *X is isomorphic to (a projection of) one of the following Del Pezzo 3-folds.*

(1) *Veronese 3-fold $X \cong \boldsymbol{P}^3$ embedded by $\mathcal{O}(2)$. $\deg X=8$ in this case.*

(2) *X is the blowing-up of $\boldsymbol{P}^3$ at a point p, embedded by the linear system of proper transforms of quadrics passing through p.* $\deg X=7$.

(3) *X is a hyperplane section of the Segre variety $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ in $\boldsymbol{P}^8$.* $\deg X=6$.

The problem was already studied by Scorza and our answer seems to follow from his results [4]. However, unfortunately, his argument is not easy of access to many mathematicians today because of some language barriers, mathematical and non-mathematical. So we present our own proof, which is similar to his but is different in many aspects. Our method is a variation of the arguments in [2; Theorem 2] concerning the 4-dimensional extremal case.

Recently, we were informed from F. L. Zak that he obtained a classification theory of all the varieties in the (even dimensional) extremal case treated as in [2]. Combined with his results, we hope, our method will work in the odd dimensional near-extremal cases where $X \subset \boldsymbol{P}^N$, $\dim \mathrm{Sec}(X)=N-1$ and $3 \dim X = 2N-5$. We also hope that the results are valid in positive characteristic cases too, probably after some slight modifications. See also (3.38) below.

Acknowledgement. As indicated by the contents, this article is an outcome of the joint efforts of Professor J. Roberts and the author. The latter learned very much from the former about projective techniques, and also the above mentioned paper of Scorza. We owe many ideas to Professor F. L. Zak too. In particular, the lemma (3.25) is a special case of one of his theorems communicated by a letter to Roberts dated July 11, 1980. The author would like to express his hearty thanks to J. Roberts, to F. L. Zak and to Professor R. Hartshorne, who suggested him, among others, the possibility of generalizing the classical result of Severi [5]. He thanks also Miller Institute of the University of California, whose aid made him possible to stay in Berkeley where this research was done.

Notation, Convention and Terminology

Basically we employ the same notation as in [2]. For the sake of simplicity the ground field k is assumed to be the complex number field $\boldsymbol{C}$. Varieties are assumed to be complete unless specifically stated to the contrary. In particular, subvarieties are closed. *Manifold* is a smooth (=non-singular) variety. *Point* means a $\boldsymbol{C}$-rational point. Open set means a Zariski open set. But the convergence of a sequence of points is considered with respect to the strong Hausdorff topology. Vector bundles are confused with the locally free sheaves of their sections. Tensor products of line bundles are denoted additively, while the multiplicative notation is used for the intersection products in the Chow ring. "** *is valid for points* x, y *on* X *in a general position*" means that there exists an open dense subset U of $X\times X$ such that ** is valid for any (x, y) on U. The word "general" should be understood similarly. Finally we show a couple of symbols used in the text.

$[D]$: The line bundle associated with a Cartier divisor D.
$Bs\Lambda$: The intersection of all the members of the linear system Λ. In particular, any fixed component of Λ is a subset of $Bs\Lambda$.
L_T: The pull-back of a line bundle L by a morphism from T. However, when there is no danger of confusion, we often write just L instead of L_T.
$\{Z\}$: The homology class of an algebraic cycle Z.

§1. Examples of threefolds with small secant varieties

(1.1) *Notation.* For any two points p and q on $\boldsymbol{P}^N$, $p*q$ denotes the line passing p and q. For any subvarieties V and W of $\boldsymbol{P}^N$, $V*W$ denotes the closure of the union of all the lines $v*w$ with $v\in V$, $w\in W$ and $v\neq w$. $V*V$ is denoted by $\mathrm{Sec}(V)$. For a manifold X, $\mathrm{Sec}(X)$ is the secant variety of X and is the union of all the secant lines of X and the tangent lines to X. $t_{X,x}$ denotes the embedded tangent space of X at $x\in X$.

(1.2) Let X be a submanifold of $\boldsymbol{P}^N$, $H=\mathcal{O}(1)$, A be the trivial vector bundle with fiber $H^0(\boldsymbol{P}^N, H)$ and let $\varphi: A\rightarrow H$ be the surjective bundle mapping inducing the identity $H^0(\boldsymbol{P}^N, H)=H^0(\boldsymbol{P}^N, A)\rightarrow H^0(\boldsymbol{P}^N, H)$. Let P_1 and P_2 be copies of $\boldsymbol{P}^N$ with fixed isomorphisms $\iota_j: P_j\cong\boldsymbol{P}^N$. The counterparts on P_j of objects on $\boldsymbol{P}^N$ will be indicated by the suffix j. In particular, X_j is the submanifold $\iota_j^{-1}(X)$ in P_j.

The isomorphisms ι_j induce an inclusion of X in $B=X_1\times X_2$. The image Δ will be called the *diagonal*. Let G be the blowing-up of B with center Δ, and let E be the exceptional divisor on G lying over Δ. Let V be (the pull-back of) the vector bundle $\pi_1^*H_1 \oplus \pi_2^*H_2$, where π_j is the projection $B \to X_j$. Let W be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(V_G)$ over G and let W_j be the section of $f: W \to G$ defined by the quotient bundle $\pi_j^*H_j$ of V. Set $D=f^{-1}(E)$. ι_j's induce an isomorphism $H_1 \cong H_2$ on Δ, which gives rise to a section C of $D=\boldsymbol{P}(V_E) \cong E \times \boldsymbol{P}^1 \to E$. C is a fiber of the second projection $D \to \boldsymbol{P}^1$ as well as $D \cap W_j$, and $C \cap W_j = \emptyset$.

Combining $\pi_j^*\varphi_j$ $(j=1, 2)$ together with the given isomorphisms $A_j \cong A$, we get a bundle mapping $\Phi: A \to V$ on B. Unlike φ_j's, Φ is not surjective. In fact, $\mathrm{Supp}(\mathrm{Coker}(\Phi))=\Delta$. Any way, Φ induces a mapping $A \to H^0(B, V) \cong H^0(G, V) \cong H^0(W, L)$, where L is the tautological line bundle $\mathcal{O}(1)$ on $W=\boldsymbol{P}(V_G)$. This defines a linear system Λ on W. By definition of Φ we easily see that $Bs\Lambda = C$. Moreover, if W' is the blowing-up of W with center C and if E_C is the exceptional divisor on W' over C, then the pull-back of Λ on W' is of the form $E_C+\Lambda'$ for some linear system Λ' on W' with $Bs\Lambda' = \emptyset$. So, we obtain a morphism $\rho': W' \to \boldsymbol{P}^N$ such that $(\rho')^*H = L - E_C$, lifting the rational mapping ρ defined by Λ on W. Note that ρ is defined on W_j and the restriction is nothing but the morphism $W_j \to G \to B \to X_j \leftrightarrows X \to \boldsymbol{P}^N$.

Let D' be the proper transform of D on W'. The $D' \cong D$ and $D = D' + E_C$ in $\mathrm{Pic}(W')$. Therefore, by the criterion of Castelnuovo-Moishezon-Nakano, we infer that D' can be blown down to a submanifold with respect to the $\boldsymbol{P}^1$-bundle structure $D' \cong D \cong E \times \boldsymbol{P}^1 \to E$ over E. Let S be the manifold obtained from W' by this blowing-down. Then $W' \to W \to G$ is factored through $p: S \to G$ and p makes S a $\boldsymbol{P}^1$-bundle over G. S is called sometimes the elementary transformation of W with respect to C. Note also that ρ' is factored through a morphism $\sigma: S \to \boldsymbol{P}^N$.

(1.3) PROPOSITION. *For any (x, y) on $B - \Delta \cong G - E$, the image of $p^{-1}(x, y)$ via σ is the line $x*y$ in $\boldsymbol{P}^N$. So* $\mathrm{Im}(\sigma) = \mathrm{Sec}(X)$.

Proof. Let $Y \subset S$ be the fiber over (x, y). Clearly $\sigma(Y)$ is a line in $\boldsymbol{P}^N$ since $\sigma^*H\{Y\} = (L - E_C)Y = 1$. On the other hand, $x = \rho'(Y' \cap W_1)$ and $y = \rho'(Y' \cap W_2)$ where Y' is the inverse image of Y on W'. So $x, y \in \sigma(Y)$. Hence $\sigma(Y) = x*y$. This implies $\mathrm{Im}(\sigma) = \mathrm{Sec}(X)$ by definition of $\mathrm{Sec}(X)$.

Remark. If q is a point on E lying over $x \in \Delta$, then q defines a tangent direction to X at x. It is easy to verify that $\sigma(p^{-1}(q))$ is the line in this direction. In particular, it is contained in $t_{X,x}$.

(1.4) DEFINITION. A line of the form $\sigma(p^{-1}(q))$ for some $q \in G$ is called a *secant line* to X. It is a secant line in the usual sense or a tangent line to X. S is called the *complete secant bundle* of X. For $z \in \mathrm{Sec}(X)$, $\Sigma_z = \sigma(p^{-1}(p(\sigma^{-1}(z))))$ is called the *secant cone* and $Q_z = \Sigma_z \cap X$ is called the *secant locus.* Σ_z is the union of all the secant lines passing z.

Although these definitions are apparently different from those in [2], they are essentially the same. In particular, they coincide with each other for any general point z on Sec(X). Thus, Q_z is the image of $\sigma^{-1}(z)$ via the morphism $S\to G\to B\to X_j\cong X$.

(1.5) Obviously $\dim S=2r+1$ if $\dim X=r$. So $\dim \mathrm{Sec}(X)=2r+1$ if and only if $(\sigma^*H)^{2r+1}\{S\}>0$. Moreover, if this is the case and if ν is the number of secant lines of X passing a general point z on Sec(X), we have $(\sigma^*H)^{2r+1}\{S\}=2\nu\deg(\mathrm{Sec}(X))$ unless X is a hypersurface in $\boldsymbol{P}^N$, because of the following

(1.6) TRISECANT LEMMA. *If a general secant line of X meets X at more than two points, then X is a hypersurface in $\boldsymbol{P}^{r+1}$.*

Proof. If $r=\dim X=1$, this fact is well-known. See e.g. [3; pp. 311-313]. If $r>1$, taking general hyperplane sections we prove the assertion by induction on r.

(1.7) FORMURA (due to A. Holme). $(\sigma^*H)^{2r+1}=d^2-\Sigma_{j=0}^r C_{2r+1,j}H^j s_{r-j}(\Omega_X)\{X\}$, where $d=\deg X=H^r\{X\}$, $C_{n,j}$ denotes the binomial coefficient $n!/j!(n-j)!$, Ω_X is the cotangent bundle of X and s_i is the i-th Segre class. The total Segre class $s(V)$ of any vector bundle V is related to the Chern classes by the formula $s(V)c(V^*)=1$, where V^* is the dual bundle of V.*)

To prove the above formula, we recall the following facts.

(1.8) Let V be a vector bundle of rank v on a manifold M and let $P=\boldsymbol{P}(V)$ be the associated $\boldsymbol{P}^{v-1}$-bundle over M. (Note: A fiber of P over $q\in M$ is the set of hyperplanes of the vector space V_q passing the origin.) Let L be the tautological line bundle $\mathcal{O}_P(1)$. Then $L^{a+v-1}\alpha\{P\}=s_a(V)\alpha\{M\}$ for any $a\geqq 0$ and $\alpha\in H^{2(n-a)}(M)$, where $n=\dim M$.

(1.9) Let M' be the blowing-up of a manifold M with cencer C and let E be the exceptional divisor on M' over C. Then $(E,[-E]_E)\cong(\boldsymbol{P}(N^\vee),\mathcal{O}(1))$ where $N^\vee$ is the conormal bundle of C in M.

(1.10) *Proof of* (1.7). We have $(\sigma^*H)^{2r+1}\{S\}=(L-E_C)^{2r+1}\{W'\}=L^{2r+1}+\Sigma_{i=0}^{2r} C_{2r+1,i}L^i(-E_C)^{2r+1-i}$. By (1.8) we obtain $L^{2r+1}=L^{2r+1}\{W\}=s_{2r}(V)\{G\}=H_1^rH_2^r\{B\}=d^2$. We also have $L^i(-E_C)^{2r+1-i}=-L^i(-E_C)^{2r-i}\{E_C\}=-L^is_{2r-i-1}(N^\vee)\{C\}$, where $N^\vee$ is the conormal bundle of C in W. C is a section of $D=\boldsymbol{P}(V_E)\cong E\times\boldsymbol{P}^1\to E$ and $V_E=H_E\oplus H_E$. Therefore $C\cong E$ and $L_C\cong H_E$. Furthermore, we have an exact sequence $0\to[-D]_C\to N^\vee\to\mathcal{O}_C\to 0$, since the normal bundle of C in D is trivial. Hence $s_i(N^\vee)=s_i([-D])=(-c_1([D]))^i$. Now, using (1.8) and (1.9), we obtain $L^is_{2r-i-1}(N^\vee)\{C\}=H^i(-E)^{2r-i-1}\{E\}=H^is_{r-i}(\Omega_\Delta)\{\Delta\}$, since $[-D]$ is the pullback of $[-E]$ by $D\to E$ and the conormal bundle of Δ in $B=$

*) In many literatures the Segre class is defined by $s(V)c(V)=1$. We employ the present notation in order to avoid signature trouble in the important formula (1.8).

$X_1 \times X_2$ is the cotangent bundle of Δ. In particular, the above number is zero for $i>r$. Combining these calculations we obtain the formula (1.7).

(1.11) *Examples.*

1) Consider the Veronese 3-fold $X \cong \boldsymbol{P}^3$ embedded by $\mathcal{O}(2)$ in $\boldsymbol{P}^9$. By (1.7) we obtain $(\sigma^*H)^7\{S\}=0$. Hence $\dim \mathrm{Sec}(X) \leqq 6$. This fact can be shown by the following observation too. Let z be a general point on $\mathrm{Sec}(X)$ and let $x*y$ be a general secant line passing z, with $x \in X \ni y$. Let Q be the image under the isomorphism $\boldsymbol{P}^3 \cong X$ of the line on $\boldsymbol{P}^3$ passing x and y. Then $HQ=2$ and Q is a plane quadric in $\boldsymbol{P}^9$. It is easy to see that z is on the plane $Q*Q$ and the secant cone Σ_z contains this plane. So $\dim \sigma^{-1}(z) \geqq 1$. Since z is general, this implies $\dim \mathrm{Sec}(X) < \dim S=7$.

2) Let X be the blowing-up of $\boldsymbol{P}^3$ at a point q on $\boldsymbol{P}^3$, embedded in $\boldsymbol{P}^8$ by the linear system of the proper transforms of the quadrics passing q. Then $(\sigma^*H)^7\{S\}=0$ and $\dim \mathrm{Sec}(X) \leqq 6$. This can be shown either by (1.7) or by the observation that X is the projection of the Veronese 3-fold $\cong \boldsymbol{P}^3$ as in 1) from the point q.

3) Let Y be the Segre embedding $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ in $\boldsymbol{P}^8$ and let X be a hyperplane section of Y. Then $(\sigma^*H)^7\{S\}=0$ and $\dim \mathrm{Sec}(X) \leqq 6$. This follows also from the fact $\dim \mathrm{Sec}(Y)=7$ (cf. [*FR*]).

4) Let X be the Segre variety $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ embedded in $\boldsymbol{P}^7$. Then $\deg X=6$ and we get $(\sigma^*H)^7\{S\}=2$ by (1.7). Hence $\mathrm{Sec}(X)=\boldsymbol{P}^7$ and there exists exactly one secant line of X passing any given general point on $\boldsymbol{P}^7$. Correspondingly, the generic projection of X has a unique singular point where two smooth branches intersect normally.

5) In view of the results in [*F*], the preceding facts may be summarized as follows: Let (X, H) be a Del Pezzo threefold with $\deg X \geqq 6$. Then $\dim \mathrm{Sec}(X) \leqq 6$ unless $X \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$. (*Del Pezzo threefold* means a 3-fold whose canonical bundle is linearly equivalent to $-2H$. See [1].)

§ 2. Main result

Our goal is the following

(2.1) THEOREM. *Let X be a smooth 3-fold in $\boldsymbol{P}^7$ such that* $\dim \mathrm{Sec}(X) \leqq 6$ *and that X is not contained in any hyperplane. Then X is a projection of one of the three Del Pezzo 3-folds as in* (1.11; 1, 2 *and* 3).

Our proof goes parallel with that of [2; Theorem 2]. The key is the following

(2.2) MAIN LEMMA. *For any general point z on* Sec(X), *the secant locus Q_z is a smooth plane curve of degree two* (*cf.* (1.4)).

Compare this with [2; Theorem 3]. This lemma will be proved in the next section.

(2.3) LEMMA. $KQ_z=-4$ *for the canonical bundle K of X.*

Proof. We denote the canonical bundle of a given manifold Y by K^Y. Then, under the notation (1.2), we have $K^B=K_1+K_2$ where $K_j=\pi_j{}^*K^{X_j}$, and we also have $K^G=K^B+(r-1)E$ (in our particular case $r=\dim X=3$), $K^W=K^G-2L+\det V$, $K^{W'}=K^W+E_C=K^S+D'$, confusing line bundles with their pull-backs conventionally. Let $R=\sigma^{-1}(z)$ and $R'=(\rho')^{-1}(z)$. $R\cong p(R)\subset G$ is the set of secant lines passing z, and hence $R\cong \boldsymbol{P}^1$ by the Main Lemma. So $K^SR=K^RR=-2$. We have $D'R'=0$, since $\rho'(D')=X$. This implies $R'\cong R$ too. By the remark to (1.3), we infer that $(p^*E)R$ is equal to the number of tangent lines passing z, and hence 2 by virtue of (2.2). On the other hand we have $(\sigma^*H)R=(L-E_C)R'=0$. Thus we obtain $2=ER'=(E_C+D')R'=E_CR'=LR'$. Hence $K^WR'=(K^S+D'-E_C)R'=-4$. We have $H_jR'=2$ and $K_jR'=KQ_z$, because Q_z is the image of R' via the mapping $W'\to W\to G\to B\to X_j\cong X$. So $-4=K^WR'=(K_1+K_2+2E-2L+H_1+H_2)R'=2KQ_z+4$. Thus we prove $KQ_z=-4$.

(2.4) LEMMA. $h^k(X, \mathcal{O}_X)=0$ *for* $k>0$.

To prove this, let $q_k(Y)$ denote $h^k(\tilde{Y}, \mathcal{O})$ for any variety Y, where $\tilde{Y}$ is a nonsingular model of Y. As we saw above, $\rho'(E_C)=\mathrm{Sec}(X)$. Hence $q_k(\mathrm{Sec}(X))\leqq q_k(E_C)=q_k(C)=q_k(E)=q_k(\Delta)=q_k(X)$. Applying Theorem 6.4 in [*FR*] to ρ', we obtain $q_k(\mathrm{Sec}(X))=q_k(W')$. Therefore $q_k(X)\geqq q_k(W')=q_k(G)=q_k(B)\geqq 2q_k(X)$ for $k>0$. So we conclude $q_k(X)=0$.

(2.5) LEMMA. $h^0(X, K+2H)\leqq 1$.

For a proof, refer to [2; Lemma 6.1] and use (2.3).

(2.6) COROLLARY. $h^0(X, K+tH)=0$ *for* $t<2$.

(2.7) LEMMA. $h^0(X, K+2H)>0$.

For a proof, refer to the proof of Lemma 6.5 in [2], and make obvious adjustments.

(2.8) LEMMA. $K=-2H$.

For a proof, see [2; Lemma 6.6].

Remark (due to J. Roberts). This lemma can be proved without (2.4) too.

Indeed, considering the Albanese mapping of X, we infer $q_1(X)=0$ from (2.2). We have $q_3(X)=0$ by virtue of (2.6). So $\chi(X, \mathcal{O})\geqq 1$, while (2.4) gives the equality. But the inequality is enough for similar arguments as in [2] to prove (2.8).

(2.9) *Proof of* (2.1). By (2.8), X is a Del Pezzo 3-fold. So $\Delta(X, H)=1$, and $\deg X=h^0(X, H)-2\geqq 6$ (cf. [1]). By the classification theory of Del Pezzo manifolds in [1] and by (1.11), we complete the proof of Theorem (2.1).

Remark. Theorem 2 in [2] follows from (2.1). Indeed, if Y is a 4-fold in $\boldsymbol{P}^8$ with $\dim(\mathrm{Sec}(Y))=7$, then a general hyperplane section X of Y is a 3-fold such that $\dim(\mathrm{Sec}(X))=6$. Of course, however, such a proof is a detour.

§ 3. Proof of the Main Lemma

Throughout in this section, let X be a threefold as in (2.1) and we use the notations as in (1.2) and (1.4).

(3.1) Lemma. $\dim(\mathrm{Sec}(X))=6$.

Indeed, otherwise, X would be projected isomorphically onto its image in $\boldsymbol{P}^5$, contradicting Theorem 2.4 in [2].

(3.2) Corollary. $\dim Q_z=1$ *and* $\dim \Sigma_z=2$ *for any general point* z *on* $\mathrm{Sec}(X)$.

For a proof, see [2; Lemma 2.2].

(3.3) Definition. Let Σ be the singular locus of $\mathrm{Sec}(X)$ and let $U=\mathrm{Sec}(X)-\Sigma$. For any point u on U, let H_u denote the hyperplane $t_{\mathrm{Sec}(X),u}$ tangent to $\mathrm{Sec}(X)$ at u. Let C_u be the contact locus of H_u to $\mathrm{Sec}(X)$, i.e., the closure of the set $\{v\in U | H_v=H_u\}$. We set further $D_u=C_u\cap X$.

(3.4) Lemma. *C_u is a linear subspace in $\boldsymbol{P}^7$ for any general point u on U.*

For a proof, see [2; § 3].

(3.5) Terracini's Lemma. *There exists an open dense subset U_0 of U such that $H_u=t_{X,x}*t_{X,y}$ for any $u\in U$, $x\in X$ and $y\in X$ with $x\neq y$ and $u\in x*y$.*

For a proof, refer to Lemma 2.1 in [2].

(3.6) Corollary. *$\Sigma_u\subset C_u$ for any general point u on U. Hence* $\dim C_u\geqq 2$.

Indeed, for any general point v on any secant line passing u, we have $H_v=H_v$ by virtue of (3.5).

(3.7) COROLLARY. *$C_u=\mathrm{Sec}(D_u)$ for any general point u on U. Moreover, $Q_v\subset D_u$ for any general point v on C_u.*

Proof. $C_u\supset\mathrm{Sec}(D_u)$ by (3.4). On the other hand, for any general point v on C_u, we have $H_v=H_u$, $C_v=C_u$ and hence $Q_v\subset D_v=D_u$. Moreover, $v\in\mathrm{Sec}(Q_v)\subset\mathrm{Sec}(D_u)$, proving the first assertion too.

(3.8) *Notation.* For any general secant line l of X, $H_u=H_v$ for any general points u, v on l. So this will be denoted by H_l. Similarly we define C_l to be the contact locus of H_l to $\mathrm{Sec}(X)$, and $D_l=C_l\cap X$.

(3.9) Since $\dim Q_v=\dim Q_u$ for any general point v on C_u, we infer from (3.7) that $1+2\dim D_u=\dim Q_u+\dim C_u$. Therefore, by virtue of (3.2) and (3.6), there are only following two possibilities: (1) $\dim C_u=2$ and $\dim D_u=1$, or (2) $\dim C_u=4$ and $\dim D_u=2$.

In case (1), using the Trisecant Lemma (1.6), we easily see that $Q_u=D_u$ and that this is a plane curve of degree two. Similarly as in [2; p. 964], we see that Q_u is irreducible. Now (2.2) follows.

Therefore, from now on, we assume that $\dim C_u=4$ and $\dim D_u=2$ for any general point u on U. Furthermore, we let F_u be the union of components Y of D_u such that $\dim Y=2$ and $Y^*D_u=C_u$. By (3.7), $F_u\neq\emptyset$ and hence $\dim \mathrm{F}_u=2$. Moreover $C_u=\mathrm{Sec}(F_u)$.

(3.10) C_u's form a two-dimensional algebraic family. F_u's form a 2-dimensional algebraic family of divisors on X.

(3.11) *Notation.* For $u, v\in U$, we denote $C_u\cap C_v$ by C_{uv}. Similarly H_{uv} stands for $H_u\cap H_v$, F_{uv} for $F_u\cap F_v$, and C_{uvw} for $C_u\cap C_v\cap C_w$, and so on.

(3.12) LEMMA. *$\dim C_{uv}\leqq 2$ for $u, v\in U$ in a general position.*

Proof. Let U_0 be as in (3.5). Then C_{uv} lies off U_0. So $\dim C_{uv}\leqq\dim(C_u/U_0)\leqq 3$. Moreover, if the equality holds for general v, C_{uv} cannot move while v varies. On the other hand, $y\notin C_v$ for any general point y on F_u, since $C_u=\mathrm{Sec}(F_u)$ implies that F_u is not contained in any hyperplane in C_u. Take a general point x on X and a general point w on x^*y. Then $w\in U_0$ and $y\in C_{uw}$. This contradicts the preceding observation $C_{uv}=C_{uw}$. Thus we prove the lemma.

(3.13) Now we have two possibilities: (1) $\dim C_{uv}=2$ for u, v on U in a general position, or (2) $\dim C_{uv}=1$ for any general u, v. In the sequel we first consider the case (1) and will derive a contradiction. The argument is completed in (3.23).

(3.14) LEMMA. *$\dim F_{uv}\geqq 1$ for any general u, v on U.*

Proof. $F_u\cap C_v\neq\emptyset$ by a dimension argument in C_u. This implies $F_{uv}\neq\emptyset$.

So $\dim F_{uv} \geqq 1$ because they are divisors on the threefold X.

(3.15) LEMMA. $\dim C_{uvw}=1$ *for* u, v, w *on* U *in a general position.*

Proof. By an argument as in (3.12), we infer that C_{uv} moves when v varies. So $\dim C_{uvw} < \dim C_{uv}=2$. On the other hand, $C_{uvw}=C_{uv}\cap C_{uw}\neq\emptyset$ by the assumption (3.13; 1). Furthermore, $\dim(C_u * C_v)=\dim C_u+\dim C_v-\dim C_{uv}=6$ and $C_u * C_v$ is a hyperplane in $\boldsymbol{P}^7$. Hence $C_w \not\subset C_u * C_v$ for w is general. But if C_{uvw} were a point, (3.4) would imply $C_w=C_{uw} * C_{vw}\subset C_u * C_v$. So $\dim C_{uvw}>0$, proving the lemma.

(3.16) LEMMA. *There exists a line* L_u *in* $C_u \cong \boldsymbol{P}^4$ *such that* $L_u \subset C_{uv}$ *for any* v.

Proof. Take general points s, t on U and set $L_u=C_{ust}$, which is a line by (3.15). $C_{us} * C_{ut}$ is a hyperplane in C_u. So C_{uv} is not contained in this hyperplane for any general v on U. If $L_u \not\subset C_{uv}$, then C_{uvs} and C_{uvt} would be different lines on $C_{uv}\cong \boldsymbol{P}^2$, and hence $C_{uv}=C_{uvs} * C_{uvt}\subset C_{us} * C_{ut}$. So $L_u \subset C_{uv}$, proving the lemma.

(3.17) $L_u=C_{vst}$ for any general v, s, t on U. So it is independent of u, and will be denoted by just L. For any general point w on U we have $L * w \subset C_w \subset \mathrm{Sec}(X)$. This implies $L * \mathrm{Sec}(X)=\mathrm{Sec}(X)$. So, $\mathrm{Sec}(X)$ is a cone and any point on L is a vertex of it. If $\pi: \boldsymbol{P}^7 \to \boldsymbol{P}^5$ is the projection from L, then this implies $\dim \pi(\mathrm{Sec}(X))=\dim \mathrm{Sec}(\pi(X))=4$. Moreover we have the following

(3.18) LEMMA. $\dim \pi(X) \leqq 2$.

Proof. It suffices to show $\dim(X\cap(x * L)) \geqq 1$ for any general point x on X. Take general points y and z on X and take general points u and v on $x * y$ and $x * z$ respectively. Then $L\subset C_{uv}$ and $L * x=C_{uv}$. Hence $X\cap(x * L)=D_{uv}\supset F_{uv}$ and (3.14) proves the assertion.

(3.19) LEMMA. $L\subset X$.

Proof. F_{uv} and L are curves in the plane C_{uv}. So $L\cap F_{uv}\neq\emptyset$, proving $L\cap X\neq\emptyset$. We will derive a contradiction assuming that L meets X at only finitely many points.

The projection π is defined by the linear system Λ of hyperplanes containing L. Let P_1 be the blowing-up of $\boldsymbol{P}^7$ with center $B_1=L\cap X$, which is a finite set by assumption, and let E_1 be the exceptional divisor on P_1 over B_1, L_1 and X_1 be the proper transforms of L and X respectively. The pull-back of Λ on P_1 is of the form $E_1+\Lambda_1$, where Λ_1 is a linear system on P_1 such that $Bs\Lambda_1=L_1$. If $L_1\cap X_1=\emptyset$, Λ_1 defines a holomorphic mapping $X_1 \to \boldsymbol{P}^5$, lifting π. If $B_2=L_1\cap X_1\neq\emptyset$, then we let P_2 be the blowing-up of P_1 with center B_2 and let E_2 be the exceptional divisor over B_2. Let L_2 and X_2 be the proper transforms of L_1 and X_1 on P_2. Then the pullback of the linear system Λ_1 on P_2 is of the form

$E_2+\Lambda_2$, where Λ_2 is a linear system on P_2 with $Bs\Lambda_2=L_2$. We repeat similar processes until we reach the situation where $L_k\cap X_k=\emptyset$ for some positive integer k. Let $\pi':X_k\to\boldsymbol{P}^5$ be the morphism defined by Λ_k. Since B_k is a finite set, a prime component Y of E_k is isomorphic to $\boldsymbol{P}^2$ and $[E_k]_Y=\mathcal{O}(-1)$. Hence, $\Lambda_k=\Lambda_{k-1}-E_k$ implies $(\pi')^*\mathcal{O}(1)_Y=\mathcal{O}_Y(1)$. So the image $\pi'(Y)$ is a plane in $\boldsymbol{P}^5$. Therefore, in view of (3.18), we infer that $\pi(X)$ itself is a plane. But this contradicts $\dim \mathrm{Sec}(\pi(X))=4$.

(3.20) LEMMA. $\deg(\det N)=3-\deg X$ *for the normal bundle* N *of* L *in* X.

Proof. Let X' be the blowing-up of X with center L, and let E_L be the exceptional divisor on X' over L. Then $Bs|H-E_L|=\emptyset$ on X', and $(H-E_L)^3=0$ by virtue of (3.18). On the other hand, $H^3=\deg X$, $H^2E_L=0$, $HE_L^2=-1$ and $E_L^3=-c_1(N)$ by (1.8) and (1.9). Our result follows from this calculation.

(3.21) LEMMA. $H_{uvwt}\cong\boldsymbol{P}^3$ *for points* u,v,w,t *on* U *in a general position.*

Proof. Clearly $\dim H_{uv}=5$. All the hyperplanes containing $H_{uv}\cong\boldsymbol{P}^5$ form a one-dimensional family. So $\dim H_{uvw}=4$ by (3.10). The hyperplanes containing $H_{uvw}\cong\boldsymbol{P}^4$ form a two-dimensional linear system. Hence, if $H_t\supset H_{uvw}$ for any general t, H_t is a general member of this linear system. So $X\cap H_t$ is non-singular off $X\cap H_{uvw}$ by Bertini's theorem. On the other hand, Terracini's Lemma (3.5) implies that F_t is in the singular locus of $X\cap H_t$. Therefore $F_t\subset H_{uvw}$. F_t's sweep a dense open subset of X when t moves on U. So we conclude $X\subset H_{uvw}$, a contradiction. Thus we see $H_t\not\supset H_{uvw}$, and so $\dim H_{uvwt}=3$.

(3.22) COROLLARY. $N\cong\mathcal{O}_L(1)\oplus\mathcal{O}_L(1)$.

Proof. Let u,v,w and t be as above. Then $L=C_{uvwt}$ and any of H_u,H_v,H_w and H_t are tangent to X at any point q on L. This implies $t_{X,q}=H_{uvwt}$. So N is isomorphic to the normal bundle of L in $H_{uvwt}\cong\boldsymbol{P}^3$, i.e., $\mathcal{O}_L(1)\oplus\mathcal{O}_L(1)$.

(3.23) Combining (3.20) and (3.22), we obtain $\deg X=1$. This is absurd. Now we see that the possibility (3.13; 1) cannot happen. Thus we have

LEMMA. $\dim C_{uv}=1$ *for any* u,v *on* U *in a general position.*

Next we will show $F_{uv}\neq\emptyset$ for any general points u,v on U.

(3.24) LEMMA. *Let* V *be a subvariety of* $\boldsymbol{P}^N$ *and let* Y *be a subvariety of* V. *Let* z *be a point on* $V*Y$ *and off* V. *Suppose that* V *is smooth at any point on* Y *and that* $\dim V+\dim Y\geqq N$. *Then there exists a point* y *on* Y *such that* $z\in t_{V,y}$.

This is proved by the same argument as in [2; Lemma 4.1].

(3.25) LEMMA. $F_u*X=\mathrm{Sec}(X)$ *for any general point* u *on* U.

Proof. We will show $\mathrm{Sec}(X)=X*Y$ for any component Y of F_u. By virtue of Terracini's Lemma (3.5), we have $t_{X,y}\subset H_u$ for any point y on Y. $\mathrm{Sec}(X)$ is an irreducible divisor on $\boldsymbol{P}^7$ not contained in any hyperplane. So $H_u\not\subset\mathrm{Sec}(X)$. Take a point on H_u off $\mathrm{Sec}(X)$, project things into $\boldsymbol{P}^6$ from this point, and let the images be indicated by $'$. Then X' is the isomorphic image of X, and hence $Y\cong Y'$. Assume that $X*Y\neq\mathrm{Sec}(X)$. Then $X'*Y'\neq\mathrm{Sec}(X')=\boldsymbol{P}^6$. On the other hand, since $(X')*(Y')\supset X'$, this is not contained in the hyperplane H_u'. Take a point on H_u' off $(X')*(Y')$, project things into $\boldsymbol{P}^5$ from this point, and let the images be indicated by $''$. Then Y'' has an open neighborhood in X'' which is isomorphic to its inverse image in X'. Now we can apply (3.24) to the effect that for any point z'' on $\boldsymbol{P}^5$ there exists a point y'' on Y'' such that $t_{X'',y''}\ni z''$. However, $t_{X,y}\subset H_u$ imply $t_{X'',y''}\subset H_u''$ and hence $z''\in H_u''$. This contradiction completes our proof.

(3.26) Corollary. *$Y\cap Q_v\neq\emptyset$ for any general point v on U.*

(3.27) Corollary. *The scheme theoretical intersection $Y\cap F_v$ is the line C_{uv} for any general points u, v on U and any component Y of F_u.*

Proof. Clearly $Y\cap F_v\subset C_{uv}$. On the other hand, $Y\cap F_v\neq\emptyset$ by (3.26) and hence $\dim(Y\cap F_v)\geqq 1$, because they are divisors on the threefold X. Thus we prove (3.27).

(3.28) Corollary. *F_u is irreducible for any general u on U.*

Immediate from (3.27).

(3.29) Lemma. *F_u is a rational surface for any general point u on U.*

Proof. Let S_u be a non-singular model of F_u. S_u is ruled by (3.27). If it is not rational, we have a morphism $\alpha: S_u\to C$ onto an irrational normal curve C. We will derive a contradiction from this.

Let Γ be the space parametrizing C_v's. By (3.10), Γ is a two-dimensional subspace of the Grassmann variety parametrizing $\boldsymbol{P}^4$'s in $\boldsymbol{P}^7$. Let o be the point on Γ corresponding to C_u. We may assume that Γ is smooth at o, since u is general on U. For any general C_v, we see that the proper transform of $F_{uv}\cong\boldsymbol{P}^1$ on S_u is mapped to a point by α. This gives rise to a rational mapping $\varphi:\Gamma\to C$. Moreover, since C is an irrational normal curve, φ can be extended holomorphically at any smooth point on Γ, including o.

Let x and y be points on F_u in a general position. Take a sequence $\{y_j\}$ of points on X converging to y in the strong topology, such that y_j's are general. Then $C_u\cap C_{x*y_j}$ is a line containing x by the lemma (3.23). Therefore, if g_j is the point on Γ corresponding to C_{x*y_j}, we have $\varphi(g_j)=\alpha(x)$. On the other hand, we have $\varphi(o)=\lim\varphi(g_j)$ since $o=\lim g_j$ on Γ. Changing the role of x and y and taking a sequence converging to x, we obtain $\varphi(o)=\alpha(y)$ too. Thus $\alpha(x)=\alpha(y)$.

Since x and y are general, this implies that α is a constant mapping, contradicting the hypothesis. QED.

(3.30) COROLLARY. $h^1(X, \mathcal{O})=0$.

Proof. Let $a: X \to \mathrm{Alb}(X)$ be the Albanese mapping. Let x and y be general points on X. Then, under the notation (3.8), we see that $a(F_{x*y})$ is a point using (3.29). So $a(x)=a(y)$. This implies that a is a constant mapping, because x and y are in a general position. Hence $\mathrm{Alb}(X)=0$, proving the assertion.

(3.31) LEMMA. $F_{uvw}=\emptyset$ *for any* u, v, w *on* U *in a general position.*

Proof. Fix a general point u on U and let L_t denote the line $F_{ut}=C_{ut}$ for the moment. We will derive a contradiction assuming $L_v \cap L_w \neq \emptyset$ for any general v, w.

F_u is not contained in any hyperplane by (3.7), and is almost swept out by L_t's. So, $L_t \not\subset L_v * L_w$ for any general t on U. On the other hand, we have $L_{vt} \neq \emptyset$ and $L_{tw} \neq \emptyset$ by assumption. If the intersection points were different, we would have $L_t = L_{vt} * L_{wt} \subset L_v * L_w$, contradicting the above observation. Thus they are the same point, which must be the intersection point q of L_v and L_w. Hence q is contained in L_t for any general t on U.

For any general point x on X, by virtue of (3.26), there are s, t on U in a general position such that $x \in C_{st}$. Both x and q are on the line $F_{st} \subset X$. Hence $x*q \subset X$. This implies $X*q=X$, because x is general on X. So X is a cone with vertex q, which is absurd since X is non-singular and non-linear. QED.

(3.32) F_u's are linearly equivalent to each other as divisors on X by virtue of (3.30). We let F be the line bundle $[F_u]$. Then $Bs|F|=\emptyset$ by (3.31). Let τ be the morphism defined by $|F|$. Then (3.31) implies $F^3=0$ and $\dim \tau(X)<3$. On the other hand, $F^2H=1$ by (3.27). Therefore, $\dim \tau(X)=2$ and $1=F^2H=(ZH)\deg T$, where $T=\tau(X)$ and Z is a general fiber of τ. So $ZH=\deg T=1$. This implies that T is a plane and $Z \cong \boldsymbol{P}^1$. Moreover, $YH=1$ and $Y \cong \boldsymbol{P}^1$ for any one-dimensional fiber Y of τ, since H is very ample. It is also clear that all but finite fibers of τ are of dimension one. Thus we obtain the following

(3.33) LEMMA. $\dim|F|=2$ *and* $\tau: X \to \boldsymbol{P}^2$ *is a* $\boldsymbol{P}^1$*-bundle over* $\boldsymbol{P}^2$ *off a finite subset of* $\boldsymbol{P}^2$.

(3.34) COROLLARY. F_u *is a* $\boldsymbol{P}^1$*-bundle over* $\boldsymbol{P}^1$ *(hence non-singular) for any general point* u *on* U.

Proof. From (3.10) we infer that F_u is a general member of $|F|$. So (3.33) applies.

(3.35) LEMMA. $F_u \cong \boldsymbol{P}(\mathcal{O}(2) \oplus \mathcal{O}(1))$ *and the restriction of* H *to* F_u *is the tautological bundle on it.*

Proof. (3.34) implies $F_u \cong \boldsymbol{P}(\mathcal{O}(a) \oplus \mathcal{O}(b))$ for some positive integers $a \geqq b$, with the restriction of H being the tautological line bundle. $h^0(F_u, H) \geqq 5$ since F_u is not contained in any hyperplane in $C_u \cong \boldsymbol{P}^4$. F_u cannot be an isomorphic image by a projection from $\boldsymbol{P}^5$, since otherwise F_u would be a Veronese surface $\cong \boldsymbol{P}^2$ (cf. [5]). Hence $5 = h^0(F_u, H) = a + b + 2$. From this we obtain $a = 2$ and $b = 1$.

(3.36) Using (3.7), we see that Q_u is identical to the secant locus of u to F_u. By (3.35) and by an explicit calculation one easily sees that Q_u is actually a smooth plane curve of degree two. Thus we complete the proof of the Main Lemma.

(3.37) *Remark.* As a matter of fact, the case (3.9; 2) where $\dim C_u = 4$ and $\dim D_u = 2$, does not happen at all. This fact was pointed out to the author by J. Roberts. His original proof is different from the one given below.

First, by Terracini's Lemma, we infer that F_u is contained in the singular locus of $H_u \cap X$. So this divisor is of the form $2F_u + E_u$ for some effective divisor E_u. (3.35) implies that, for any general point t on U, F_t is isomorphic to the blowing-up of $\boldsymbol{P}^2$ with center being a point and that the restriction of E_u to F_t is the exceptional curve. Therefore E_u is independent of the choice of u, and will be denoted by E from now on. $FEH = 1$ by the preceding observation. So a general hyperplane section of E is mapped to a line by τ. Hence E contains a component Y whose general hyperplane section is the exceptional curve on some F_v. So $YH^2 = 1$ and Y is a plane in $\boldsymbol{P}^7$. It is easy to see that Y can be blown down to a smooth point on another manifold. Moreover, X must be of type (1.11; 2) and $Y = E$.

Let Λ be the minimal linear system on X containing $H_u \cap X = 2F_u + E$ for any general point u on U. Then $2\tau^* F + E \in \Lambda$ for any general hyperplane F on $\boldsymbol{P}^2$. So we infer $\Lambda = \tau^*|2F| + E$ and $\dim \Lambda = h^0(\boldsymbol{P}^2, \mathcal{O}(2)) - 1 = 5$. On the other hand, any member of Λ is a hyperplane section of $X \subset \boldsymbol{P}^7$ containing $Y \cong \boldsymbol{P}^2$. Hence $\dim \Lambda \leqq 7 - 3 = 4$. This contradiction proves our assertion.

(3.38) After the first version of this note was completed, F. L. Zak communicated to the author that his techniques work not only in even dimensional extremal cases but also in odd dimensional near-extremal cases. In particular, starting from the observation (3.25), one can considerably simplify the argument concerning the case (3.9; 2). His method is very different from Scorza's, whose approach we follow basically. So we retain the present note as it was, hoping that Zak's theory will appear elsewhere, where possibly higher dimensional cases also will be considered.

In another letter he communicated the fact (3.37), pointing out an error in the first version of this note. He found several proofs of (3.37) independently of J. Roberts and the author.

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