Some Converses of a Theorem of M. Schechter on the Conjugate of a Product of Operators

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§ 1. Definitions and statement of the results

In this note $\mathfrak{L}_0(X, Y)$ denotes the set of densely defined linear operators from X to Y, where X and Y are Banach spaces. $\mathfrak{L}_0(X, X)$ is abbreviated to $\mathfrak{L}_0(X)$. The domain, kernel, and the range of $T \in \mathfrak{L}_0(X, Y)$ are denoted by $\mathfrak{D}(T)$, $\mathfrak{N}(T)$ and $\mathfrak{R}(T)$, respectively. Let $\Phi_-(X, Y)$ be the set of lower semi-Fredholm operators in $\mathfrak{L}_0(X, Y)$, i.e., $T \in \mathfrak{L}_0(X, Y)$ belongs to $\Phi_-(X, Y)$ if and only if T is a closed operator with $\dim Y/\mathfrak{R}(T) < \infty$. Note that $\dim Y/\mathfrak{R}(T) < \infty$ implies the closedness of $\mathfrak{R}(T)$ for closed $T \in \mathfrak{L}_0(X, Y)$ ([1], p 101). The set of bounded linear operators from X to Y is denoted by $\mathfrak{B}(X, Y)$. To be more precise, $T \in \mathfrak{B}(X, Y)$ if and only if $T \in \mathfrak{L}_0(X, Y)$, T is bounded and $\mathfrak{D}(T) = X$. $\mathfrak{B}(X, X)$ is abbreviated to $\mathfrak{B}(X)$.

The conjugate T' of $T \in \mathfrak{L}_0(X, Y)$ is defined as usual: $y' \in \mathfrak{D}(T')$ and $T'y' = x'$ if and only if $y' \in Y'$, $x' \in X'$ and $\langle Tx, y' \rangle = \langle x, x' \rangle$ holds for any $x \in \mathfrak{D}(T)$, where $\langle \cdot, \cdot \rangle$ denotes the canonical bilinear form. Let Z be another Banach space and $S \in \mathfrak{L}_0(Y, Z)$ satisfy $ST \in \mathfrak{L}_0(X, Z)$. Then it is well known that $(ST)'$ is an extension of $T'S'$. A sufficient condition for $(ST)' = T'S'$ is given by M. Schechter [2]:

THEOREM (M. Schechter)
Suppose X, Y be Banach spaces and $T \in \Phi_-(X, Y)$. Then for any Banach space Z and $S \in \mathfrak{L}_0(Y, Z)$, $\mathfrak{D}(ST)$ is dense in X and $(ST)' = T'S'$ holds.

The purpose of this note is to give some converses of the above theorem. Namely we prove the following

THEOREM 1. *Suppose X and Y are Banach spaces and $T \in \mathfrak{L}_0(X, Y)$. Assume that T has the following property:*

For any Banach space Z and $S \in \mathfrak{L}_0(Y, Z)$, $\mathfrak{D}((ST)') \subset \mathfrak{D}(S')$ holds whenever $\mathfrak{D}(ST)$ is dense in X.

Then $\mathfrak{R}(T)$ *is closed and* $\dim Y/\mathfrak{R}(T)<\infty$. *If we further assume* $\mathfrak{N}(T)$ *is closed, then* T *is closed, and hence* $T\in\Phi_-(X, Y)$.

THEOREM 2. *Suppose* X *and* Y *are Banach spaces and* $T\in\mathfrak{B}(X, Y)$ *is not identically zero. Assume that* T *has the following property:*

For any Banach space Z *and* $S\in\mathfrak{L}_0(Y, Z)$, $\mathfrak{D}(ST)$ *is dense in* X.

Then $T\in\Phi_-(X, Y)$.

THEOREM 3. *Suppose* X *and* Y *are separable Hilbert spaces and let* $T\in\mathfrak{L}_0(X, Y)$ *be a closed operator which is not identically zero. If* $\mathfrak{D}(ST)^-=X$ *holds for any closed* $S\in\mathfrak{L}_0(Y, Z)$, *where* Z *is an arbitrary separable Hilbert space, then* $T\in\Phi_-(X, Y)$.

Note that the assumptions in Theorems 1 to 3 are weaker than the cosequence "$\mathfrak{D}(ST)$ is dense in X and $(ST)'=T'S'$" of Schechter's theorem.

§ 2. Proof of Theorem 1

a) First we show that $\mathfrak{R}(T)$ is closed. Suppose $\mathfrak{R}(T)^-\neq\mathfrak{R}(T)$. Then there exist subspaces M and N of Y such that

$$M\neq\{0\},\ \mathfrak{R}(T)^-=\mathfrak{R}(T)\oplus M \quad\text{and}\quad Y=\mathfrak{R}(T)\oplus M\oplus N,$$

where $\oplus$ denotes the algebraic direct sum. The last equality shows that there exists a projection S from Y onto M whose kernel is $\mathfrak{R}(T)\oplus N$. S satisfies

$$0\neq S\in\mathfrak{L}_0(Y),\ \mathfrak{D}(ST)=\mathfrak{D}(T) \quad\text{and}\quad ST=0.$$

Therefore $\mathfrak{D}(S')\supset\mathfrak{D}((ST)')=Y'$ by the assumption. This implies the boundedness of S', since it is an everywhere defined closed linear operator, and hence S is bounded. This in turn implies $S(\mathfrak{R}(T)^-)\subset(S\mathfrak{R}(T))^-=\{0\}$, which contradicts $M\neq\{0\}$.

b) Secondly we prove $\dim Y/\mathfrak{R}(T)<\infty$. Part a) of this section shows that $Y/\mathfrak{R}(T)$ is a Banach space with respect to the quotient norm. If $\dim Y/\mathfrak{R}(T)=\infty$, there exists an everywhere defined unbounded linear operator $S\in\mathfrak{L}_0(Y/\mathfrak{R}(T))$. (Such an operator can be easily constructed by using a Hamel basis of $Y/\mathfrak{R}(T)$.) Let S_0 be the composite of the natural surjection $Y\to Y/\mathfrak{R}(T)$ with S. Then $S_0\in\mathfrak{L}_0(Y, Y/\mathfrak{R}(T))$, $\mathfrak{D}(S_0T)=\mathfrak{D}(T)$ and $S_0T=0$ on $\mathfrak{D}(S_0T)$. Hence

$$\mathfrak{D}(S_0')\supset\mathfrak{D}((S_0T)')=(Y/\mathfrak{R}(T))'$$

by the assumption of the theorem. This implies the boundedness of S_0, and hence S is bounded by the open mapping theorem, which is a contradiction.

c) Lastly we prove that $\mathfrak{R}(T)^{-}=\mathfrak{R}(T)$ implies the closedness of T. Preceding arguments show that there exists a finite dimensional subspace Y_1 of Y for which

$$Y=\mathfrak{R}(T)\oplus Y_1$$

holds. Let $S: Y\to X/\mathfrak{N}(T)$ be defined by

$$Sy=T^{-1}y_1,$$

where $y\in Y$ and $y_1\in\mathfrak{R}(T)$ is the unique element such that $y-y_1\in Y_1$. Then $S\in\mathfrak{L}_0(Y, X/\mathfrak{N}(T))$ and ST is the canonical surjection $X\to X/\mathfrak{N}(T)$. Hence $\mathfrak{D}(S')\supset\mathfrak{D}((ST)')=(X/\mathfrak{N}(T))'$, and hence S is bounded by the same argument as in b). Consequently there exists a constant $M>0$ such that $\|[x]\|\leqq M\|Tx\|$ holds for any $x\in\mathfrak{D}(T)$, where $[x]$ denotes the equivalence class of x mod. $\mathfrak{N}(T)$. To see the closedness of T, let $\{x_n\}$ be a sequence in $\mathfrak{D}(T)$ such that $x_n\to x$ and $Tx_n\to y$ for some $x\in X$ and $y\in Y$, respectively. Then there exists an $x'\in\mathfrak{D}(T)$ such that $Tx'=y$, since $\mathfrak{R}(T)$ is closed. On the other hand the inequality $\|[x_n-x']\|\leqq M\|T(x_n-x')\|$ implies $[x_n-x']\to 0$ as $n\to\infty$. From this it follows that $x-x'\in\mathfrak{N}(T)$ and hence $x\in\mathfrak{D}(T)$ and $Tx=y$.

§ 3. Proof of Theorem 2

For the proof of theorem 2, we prepare the following

LEMMA 1. *Let X be a Banach space and let M be a subspace of X. If M is not closed, then there exists a subspace D of M^- such that $D^-=M^-$ and $(D\cap M)^-\subsetneqq M^-$.*

PROOF. Without loss of generality, we may assume $M^-=X$. If $M\neq X$, we can select two unit vectors $x_0\in M$ and $x_1\in X\setminus M$. Furthermore fix an element $f\in X'$ such that

$$\|f\|=\langle x_0, f\rangle=1,$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $X\times X'$. Define an operator $V\in\mathfrak{B}(X)$ by

$$Vx:=x-\frac{1}{2}\langle x, f\rangle x_1$$

for any $x\in X$ and put $D:=V(M)$. Then D meets the requirements of the lemma. To see this note that the conjugate V' of V is given by

$$V'g=g-\frac{1}{2}\langle x_1, g\rangle f$$

for any $g\in X'$. Therefore if $g\in X'$ annihilates $D, g-1/2\langle x_1, g\rangle f=V'g=0$, which implies

$$||g||=\frac{1}{2}|\langle x_1, g\rangle|\ ||f||\leqq\frac{1}{2}||g||,$$

hence $g=0$. On the other hand for any $y\in D\cap M$, there exists an $x\in M$ such that $y=x-1/2\langle x, f\rangle x_1$. This means $1/2\langle x, f\rangle x_1=y-x\in M$. Hence $\langle x, f\rangle=0$ and $x=y$. Thus $M\cap D\subset \mathrm{Ker}\, f$, and hence $(M\cap D)^-\subset \mathrm{Ker}\, f\subsetneqq X$.

Now we prove Theorem 2.

a) Proof of $\mathfrak{R}(T)^-=\mathfrak{R}(T)$

Suppose $\mathfrak{R}(T)^-\neq\mathfrak{R}(T)$. Then there exists a subspace $D\subset\mathfrak{R}(T)^-$ such that $D^-=\mathfrak{R}(T)^-$ and $(D\cap\mathfrak{R}(T))^-\subsetneqq\mathfrak{R}(T)^-$ by Lemma 1. Let M be an algebraic complement of $\mathfrak{R}(T)^-$ in Y, i.e., a subspace of Y such that $Y=\mathfrak{R}(T)^-\oplus M$. Define $S\in\mathfrak{L}_0(Y)$ by

$$\mathfrak{D}(S)=D+M \quad \text{and} \quad Sy=y \quad \text{for} \quad y\in\mathfrak{D}(S).$$

Then $\mathfrak{D}(ST)=T^{-1}(D)=T^{-1}(D\cap\mathfrak{R}(T))$ is not dense in X. In fact if $\mathfrak{D}(ST)^-=X$, then

$$\mathfrak{R}(T)\subset\overline{T(\mathfrak{D}(ST))}\subset(D\cap\mathfrak{R}(T))^-$$

holds by the continuity of T, which leads to the contradiction $\mathfrak{R}(T)^-\subsetneqq\mathfrak{R}(T)^-$.

b) The proof that $\dim Y/\mathfrak{R}(T)<\infty$

In the sequel, $\mathrm{lin}\, A$ denotes the subspace generated by $A\subset Y$. Suppose $\dim Y/\mathfrak{R}(T)=\infty$. Then the dimension of an algebraic complement N of $\mathfrak{R}(T)$ in Y is infinite. Since $T\neq 0$, there exists a non-zero $x_0\in\mathfrak{R}(T)$. Let M be a topological complement of $\mathrm{lin}\,\{x_0\}$ in $\mathfrak{R}(T)$, i.e., M is a closed subspace of $\mathfrak{R}(T)$ such that $\mathfrak{R}(T)=M\oplus \mathrm{lin}\,\{x_0\}$. Since N is infinite dimensional, there exists a sequence $\{x_n\}_{n\in\boldsymbol{N}}$ of linearly independent unit vectors of N. Let L be an algebraic complement of $\mathrm{lin}\,\{x_n\,;n\in\boldsymbol{N}\}$ in N. Put

$$y_n:=x_0+\frac{1}{n}x_n$$

for $n\in\boldsymbol{N}$ and define a subspace D by

$$D:=\mathrm{lin}\,\{y_n\,;n\in\boldsymbol{N}\}+L+M.$$

Since $\lim y_n=x_0$, $D^-\supset \mathrm{lin}\,\{x_0\}+\mathrm{lin}\,\{x_n\,;n\in\boldsymbol{N}\}+L+M=Y$. Next let $y=\sum\alpha_n y_n+l+m\in D\cap\mathfrak{R}(T)$, where $\alpha_n=0$ except for finite n's, $l\in L$ and $m\in M$. Then

$$y-(\sum\alpha_n)x_0-m=\sum\frac{\alpha_n}{n}x_n+l\in\mathfrak{R}(T)\cap N=\{0\}.$$

Hence $\alpha_n=0$ for any $n\in\boldsymbol{N}$ and $l=0$. Therefore $y=m\in M$, and hence $(D\cap\mathfrak{R}(T))^-\subsetneqq\mathfrak{R}(T)$. Now define an operator $S\in\mathfrak{L}_0(Y)$ by

$$\mathfrak{D}(S)=D \quad\text{and}\quad Sy=y \quad\text{for}\quad y\in\mathfrak{D}(S).$$

Then $(\mathfrak{D}(ST))^-\subsetneqq X$ can be proved as in a) of this section, which is a contradiction.

§ 4. Proof of Theorem 3

For the proof, we prepare the following

LEMMA 2. *Let X, Y be separable Hilbert spaces and let $T\in\mathfrak{L}_0(X, Y)$ be a closed operator with $\mathfrak{R}(T)^-\neq\mathfrak{R}(T)$. Then there exists an infinite dimensional closed linear subspace M of $R(T)^-$ which satisfies $M\cap\mathfrak{R}(T)=\{0\}$.*

PROOF. Let $\mathfrak{G}(T)$ denote the graph space of T, i.e.,

$$\mathfrak{G}(T):=\{(x, y)\,;\, x\in\mathfrak{D}(T), y=Tx\}.$$

Then the mapping $(x, y)\to y$ from $\mathfrak{G}(T)$ into Y has the same range as that of T. Hence we may assume $T\in\mathfrak{B}(X, Y)$. We may also assume that $Y=\mathfrak{R}(T)^-$ and $\mathfrak{N}(T)=\{0\}$, by considering suitable restrictions of T.

Let $T=UP$ be the polar decomposition of T. Then $U\in\mathfrak{B}(X, Y)$ is a surjective isometry and P is an injective positive self-adjoint operator defined on X. Thus it suffices to show the lemma in case $T=P$. Let $P=\int_1^\infty\lambda dE_\lambda$ be the spectral decomposition of P. Then $E_\lambda\neq0$ for any $\lambda>0$, since $E_\lambda=0$ for some $\lambda>0$ implies the invertibility of P and hence $\mathfrak{R}(P)=\mathfrak{R}(P)^-=X$. On the other hand, $E_0=0$ by the injectivity of P. These imply that there exists a sequence $\{I_n\}_{n\in\boldsymbol{N}}$ of disjoint intervals of the form $I_n=(a_n, b_n)$ such that $E_{b_n}-E_{a_n}\neq0$, $0<a_{n+1}<b_{n+1}<a_n$, and $b_n<1/n$ for any $n\in\boldsymbol{N}$. The projection $E_{b_n}-E_{a_n}$ is denoted by P_n. Let $\psi:\boldsymbol{N}\times\boldsymbol{N}\to\boldsymbol{N}$ be the bijection defined by $\psi(i, j)=2^{i-1}(2j-1)$. Choose a unit vector $x_n\in\mathfrak{R}(P_n)$ for any $n\in\boldsymbol{N}$ and put $y_{i,j}:=x_{\psi(i,j)}$ for $i, j\in\boldsymbol{N}$. Then $\{y_{i,j}\}_{i,j}$ is an orthonormal system, and hence

$$z_j:=\sum_{i=1}^{\infty}\frac{1}{i}y_{i,j}$$

exists for any $j\in\boldsymbol{N}$ with $||z_j||^2=\pi^2/6$. Note that $\{z_j\}_j$ is an orthogonal system. Let M be the closed subspace generated by $\{z_j\}_j$. M is clearly infinite dimensional and we claim that $M\cap\mathfrak{R}(P)=\{0\}$. In fact, let $y\in M\cap\mathfrak{R}(P)$. Then there exists a sequence $(\alpha_j)_{j\in\boldsymbol{N}}\in l^2$ such that $y=\sum\alpha_jz_j$. On the other hand there exists an $x\in X$ such that $Px=y$. Let

$$P_{i,j} := P_{\psi(i,j)}$$

for $i, j \in \boldsymbol{N}$ and let $x_{i,j} := P_{i,j}x$. Then

$$P_{i,j}Px = P_{i,j}y = \frac{\alpha_j}{i} y_{i,j}.$$

This implies

$$\frac{|\alpha_j|}{i} \leqq \frac{\|P_{i,j}x\|}{\psi(i,j)} \leqq \frac{\|x\|}{\psi(i,j)},$$

since

$$\|P_{i,j}Px\| = \|PP_{i,j}x\| \leqq \frac{\|P_{i,j}x\|}{\psi(i,j)}$$

holds by the definition of $P_{i,j}$. Therefore

$$|\alpha_j| \leqq \overline{\lim_{i\to\infty}} \frac{i}{\psi(i,j)} \|x\| = 0$$

for any $j \in \boldsymbol{N}$, and hence $y=0$.

LEMMA 3. *Let X, Y and T be as in Lemma 2. Then there exists an invertible operator $V \in \mathfrak{B}(\mathfrak{R}(T)^-)$ for which $D := V(\mathfrak{R}(T))$ satisfies $D^- = \mathfrak{R}(T)^-$ and $D \cap \mathfrak{R}(T) = \{0\}$.*

PROOF. Let M be a subspace of Y whose existence is established by Lemma 2: i.e., M is a closed infinite dimensional subspace of $\mathfrak{R}(T)^-$ satisfying $M \cap \mathfrak{R}(T) = \{0\}$. Take an orthogonal system $\{f_n\}_{n\in\boldsymbol{N}}$ in M such that $\|f_n\| = 1/2^n$ for any $n \in \boldsymbol{N}$. Let $\{e_n\}_{n\in\boldsymbol{N}}$ be an orthonormal basis of $\mathfrak{R}(T)^-$. Then a linear operator $K \in \mathfrak{B}(\mathfrak{R}(T)^-)$ can be defined by putting

$$Ky := \sum_{n=1}^{\infty} \langle y, e_n \rangle f_n$$

for each $y \in \mathfrak{R}(T)^-$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in Y. It is clear that K is a compact operator.

We claim that the operator $V := I - K$ meets the requirement. First let $y \in \mathfrak{N}(V)$. Then

$$\|y\|^2 = \|\sum_{n=1}^{\infty} \langle y, e_n \rangle f_n\|^2 = \sum_{n=1}^{\infty} |\langle y, e_n \rangle|^2 \|f_n\|^2 \leqq \frac{1}{4} \|y\|^2$$

holds, and hence $y=0$. This implies that V is invertible in $\mathfrak{B}(\mathfrak{R}(T)^-)$. Therefore

$D^{-}=\mathfrak{N}(T)^{-}$ holds for $D:=V(\mathfrak{N}(T))$.

Next let $y\in D\cap\mathfrak{N}(T)$. Then there exists a $z\in\mathfrak{N}(T)$ such that

$$y=Vz=z-\sum_{n=1}^{\infty}\langle z, e_n\rangle f_n.$$

This implies

$$z-y=\sum_{n=1}^{\infty}\langle z, e_n\rangle f_n\in M\cap\mathfrak{N}(T)=\{0\}.$$

Hence $z=y$ and $\langle z, e_n\rangle=0$ for any $n\in \boldsymbol{N}$, and hence $y=0$.

Now we prove Theorem 3.

a) Proof of the closedness of $\mathfrak{R}(T)$

Let T_1 be the restriction of T to $\mathfrak{N}(T)^{\perp}$: i.e., $\mathfrak{D}(T_1)=\mathfrak{N}(T)^{\perp}\cap\mathfrak{D}(T)$ and $T_1x=Tx$ for $x\in\mathfrak{D}(T_1)$. It is easy to see that T_1 belongs to $\mathfrak{L}_0(\mathfrak{N}(T)^{\perp}, Y)$ and is an injective closed operator.

Suppose $\mathfrak{R}(T)$ is not closed. Then by Lemma 3 there exists an invertible operator $V\in\mathfrak{B}(\mathfrak{R}(T)^{-})$ for which $D:=V(\mathfrak{R}(T))$ satisfies $D^{-}=\mathfrak{R}(T)^{-}$ and $D\cap\mathfrak{R}(T)=\{0\}$. Since V is invertible, the product $VT_1\in\mathfrak{L}_0(\mathfrak{N}(T)^{\perp}, \mathfrak{R}(T)^{-})$ is also closed and injective. Note that $\mathfrak{R}(VT_1)=V(\mathfrak{R}(T))=D$. Hence we can define an operator $S\in\mathfrak{L}_0(Y, \mathfrak{N}(T)^{\perp})$ as follows: $\mathfrak{D}(S)=D+\mathfrak{R}(T)^{\perp}$ and $Sy=(VT_1)^{-1}y_1$, where $y\in\mathfrak{D}(S)$ and $y_1\in D$ is the unique element for which $y-y_1\in\mathfrak{R}(T)^{\perp}$ holds.

Then it is clear that S is a closed operator and $\mathfrak{D}(ST)=T^{-1}(\mathfrak{R}(T)\cap\mathfrak{D}(S))=\mathfrak{N}(T)$, which contradicts the assumption of the theorem.

b) Proof of $\dim Y/\mathfrak{R}(T)<\infty$

Suppose $\dim Y/\mathfrak{R}(T)=\infty$. Further we assume $\dim\mathfrak{R}(T)=\infty$, since the proof for the case $\dim\mathfrak{R}(T)<\infty$ goes similarly.

Let $\{x_i\}_{i\in N}$ and $\{x_{i,j}\}_{i,j\in N}$ be an orthonormal basis of $\mathfrak{R}(T)$ and $\mathfrak{R}(T)^{\perp}$, respectively. Define $U\in\mathfrak{B}(\mathfrak{R}(T)^{\perp}, Y)$ by

$$U\Big(\sum_{i,j}\alpha_{i,j}x_{i,j}\Big):=\sum_{i,j}\frac{\alpha_{i,j}}{j}\Big(x_i+\frac{1}{j}x_{i,j}\Big),$$

where $\sum_{i,j}|\alpha_{i,j}|^2<\infty$. It is easy to see that U is well defined and

$$\|U\|\leqq\frac{\pi}{\sqrt{3}}.$$

Since $jUx_{i,j}\to x_i$ as $j\to\infty$, $\mathfrak{R}(U)^{-}=Y$. Moreover $\mathfrak{R}(U)\cap\mathfrak{R}(T)=\{0\}$. In fact, let $y=U(\sum_{i,j}\alpha_{i,j}x_{i,j})$ belong to $\mathfrak{R}(T)$, where $\alpha_{i,j}$ satisfy $\sum_{i,j}|\alpha_{i,j}|^2<\infty$. Then y is orthogonal to $x_{i,j}$ for any $i,j\in \boldsymbol{N}$. This implies $\alpha_{i,j}=0$ for any $i,j\in \boldsymbol{N}$, and hence $y=0$. Lastly we note that U is injective.

The preceding arguments show that the operator S defined by

$$\mathfrak{D}(S):=\mathfrak{R}(U),\ Sy:=U^{-1}y \quad \text{for} \quad y\in\mathfrak{D}(S)$$

belongs to $\mathfrak{L}_0(Y, \mathfrak{N}(T)^{\perp})$ and is closed.

In the same way as in a) of this section, we have $\mathfrak{D}(ST)=\mathfrak{N}(T)$, which contradicts the assumption $T\neq 0$.

References

[1] Goldberg, S., *Unbounded Linear Operators: Theory and Applications*, McGrow-Hill, New York, 1966.

[2] Schechter, M., The conjugate of a product of operators, *Journal of Functional Analysis* **6**, 26–28 (1970).