Category Theory not based upon Set Theory

By Haruo TSUKADA

Department of Mathematics, Faculty of Science, University of Tokyo
Hongo, Bunkyo-ku, Tokyo, 113

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Contents

0. Introduction

At the present time, if we want to construct a category theory, we first construct a certain appropriate set theory and prepare many set-theoretical tools, and after that we define category as a model of ABSTRACT CATEGORY THEORY (in which the scope of quantifiers is one category)*). More precisely,

*) The important exceptions are Lawvere's [2] and [3].

we define the formula **CATEGORY** (C), for a term (i. e. a set) C.

But this way of construction of a category theory is very unsatisfactory to categorists who regard categories as fundamental objects of mathematics instead of sets. In other words, the following *DOGMA of Set Theory* dominates the category theory of this type.

The DOGMA of Set Theory

[S1] All materials we recognize are sets.

[S2] $\in$ is the only atomic predicate symbol other than $=$.

[S3] All small relations, operations, etc. are constructed set-theoretically. (by determining their elements)

Taking all these facts into consideration, we propose a new category theory which has the following *DOGMA*; the PAN-CATEGORISM.

The DOGMA of Category Theory~PAN-CATEGORISM

[C1] All materials we recognize are categories.

[C2] $\lhd$ is the only atomic predicate symbol other than $=$.

[C3] All small relations, operations, etc. are constructed category-theoretically. (The constructions we can do in set theory can be performed similarly.)

Here, $\lhd$ is a predicate symbol where $x \lhd C$ means "x is *a triangle of a category C*."

In the present paper, we describe a theory $\mathcal{T}_{\text{Cat}}$ as a realization of this PAN-CATEGORISM, in which *all terms are categories*, and which intuitively includes Bourbaki's set theory.

We proceed as follows.

In Chapter 1, we define *equalitarian theory with ordered pair*. We define category in this very general frame in Chapter 2. At this step, we cannot define the formula **CATEGORY** (C) for a term C. In fact, we have no notion of x being a morphism or an element or a triangle of a term C. Therefore, we define the formula $\mathbf{CATEGORY}_x(\varphi)$ for a formula φ and a variable x. (2.3) $\varphi[t]$ will be written "t is a triangle of φ_x" in case φ_x is a category.

In Chapter 3, we describe our category theory $\mathcal{T}_{\text{Cat}}$. $\mathcal{T}_{\text{Cat}}$ is an equalitarian theory with ordered pair which has a new predicate symbol $\lhd$. We interprete $\mathbf{CATEGORY}_x(x \lhd C)$ as "C is *a category*." And we require an axiom $\forall C$ $[\mathbf{CATEGORY}_x(x \lhd C)]$, which is called AXIOM OF PAN-CATEGORISM. Then, to realize [C3], we state category-theoretical version of axioms and a scheme which are similar to those in Bourbaki's set theory.

Finally, in Chapter 4, we prove that $\mathcal{T}_{\text{Cat}}$ is consistent if $\mathcal{T}_{\text{Set}}$ (a set theory which is equivalent to Bourbaki's set theory) is consistent.

Remark: 1) We can also prove that $\mathcal{T}_{\text{Set}}$ is consistent if $\mathcal{T}_{\text{Cat}}$ is consistent.

2) Our method of realizing PAN-CATEGORISM is applicable to any other first order theory.

1. Equalitarian theories with ordered pair

1.1. Equalitarian theories

Throughout this paper we assume the contents of Chapter 1 of [1]. The notions of signs (logical signs, letters, and specific signs (substantific signs, relational signs)), assemblies, mathematical theories, terms, relations, implicit and explicit axioms, schemes, theorems are all due to Chapter 1 of [1].

We use "variable" instead of "letter", "operation symbol (function symbol)" instead of "substantific sign", "predicate symbol" instead of "relational sign", "formula" instead of "relation". We write $\mathbf{R}[x_1,\cdots,x_n]$ instead of $\mathbf{R}\{x_1,\cdots,x_n\}$. Variables are denoted by $x, y, z, \cdots, x_1, x_2, \cdots$, etc. Formulas are denoted by $\varphi, \psi, \theta, \lambda, \cdots$, etc.

Recall that *an equalitarian theory* is a theory with a binary predicate symbol $=$ and with Scheme 1～Scheme 7 below:

1.1.1. Scheme 1. *If φ is a formula, $(\varphi \vee \varphi) \Longrightarrow \varphi$ is an axiom.*

1.1.2. Scheme 2. *If φ and ψ are formulas, $\varphi \Longrightarrow (\varphi \vee \psi)$ is an axiom.*

1.1.3. Scheme 3. *If φ and ψ are formulas, $(\varphi \vee \psi) \Longrightarrow (\psi \vee \varphi)$ is an axiom.*

1.1.4. Scheme 4. *If φ, ψ and θ are formulas,*

$$(\varphi \Longrightarrow \psi) \Longrightarrow ((\theta \vee \varphi) \Longrightarrow (\theta \vee \psi)) \text{ is an axiom.}$$

1.1.5. Scheme 5. *If φ is a formula, t a term, and x a variable,*

$$(t \mid x)\varphi \Longrightarrow (\exists x)\varphi \text{ is an axiom.}$$

1.1.6. Scheme 6. *Let x be a variable, t and s terms, and $\varphi[x]$ a formula. Then $(t=s) \Longrightarrow (\varphi[t] \Longleftrightarrow \varphi[s])$ is an axiom.*

1.1.7. Scheme 7. *Let φ and ψ be formulas and x a variable. Then $((\forall x)(\varphi \Longleftrightarrow \psi)) \Longrightarrow (\tau_x(\varphi)=\tau_x(\psi))$ is an axiom.*

1.2. Equalitarian theories with ordered pair

An equalitarian theory with ordered pair is an equalitarian theory in Bourbaki's sense, with an additional binary operation symbol $\supset$ and an additional axiom; Axiom 1.

1.2.1. Axiom 1. $(\forall x)(\forall y)(\forall z)(\forall w)((\supset xy)=(\supset zw) \Longrightarrow (x=z \wedge y=w))$.

Remark: If t and s are terms, the notation (t, s) is used instead of $\supset ts$.

1.2.2. Definition. The formula $(\exists x)(\exists y)(t=(x, y))$ is written "t is *an (ordered) pair.*' When t is a pair,

$$\tau_x((\exists y)(t=(x, y))),\ \tau_y((\exists x)(t=(x, y)))$$

are denoted by $\mathbf{pr}_1 t$, $\mathbf{pr}_2 t$, respectively.

The term $(x,(y,z))$ is also denoted by (x,y,z). The formula $(\exists x)(\exists y)(\exists z)(t=(x,y,z))$ is written "t is *a triple.*" When t is a triple,

$$\tau_x((\exists y)(\exists z)(t=(x,y,z))),\ \tau_y((\exists x)(\exists z)(t=(x,y,z))),\ \tau_z((\exists x)(\exists y)(t=x,y,z)))$$

are denoted by $\mathbf{pr}_1 t$, $\mathbf{pr}_2 t$, $\mathbf{pr}_3 t$, respectively.

2. Category

Throughout this chapter, $\mathcal{T}$ denotes *a theory stronger than an equalitarian theory with ordered pair.* In this chapter, we prepare some notions which is used in next chapter for the description of the theory $\mathcal{T}_{\mathrm{Cat}}$. They are the notion that a formula in $\mathcal{T}$ is a category, the notions of opposite category, the category of all units,$\cdots$, etc. Notice that we define the formula $\mathbf{CATEGORY}_x(\varphi)$, for *a formula* φ *and a variable* x. (2.3) And $\mathbf{CATEGORY}_x(\varphi)$ is written "φ is *a category with respect to* x." We cannot define the formula $\mathbf{CATEGORY}(C)$ for *a term* C, because we have no means to talk about morphism or element or triangle of C.

2.1. Composition

2.1.1. DEFINITION. Let $\varphi[x]$ be a formula in $\mathcal{T}$, and let y, u, v, w_1, w_2 be variables not appearing in φ. The formula

$$\begin{aligned}&\forall y(\varphi[y] \Longrightarrow y \text{ is a triple})\\ &\wedge \forall u\forall v\forall w_1\forall w_2[(\varphi[(u,v,w_1)]\wedge\varphi[(u,v,w_2)]) \Longrightarrow w_1=w_2]\end{aligned}$$

is written $\mathbf{COMP}_x(\varphi)$, or "φ_x is *a composition,*" or "φ is *a composition with respect to* x."

2.1.2. DEFINITION. When $\mathbf{COMP}_x(\varphi)$, $\exists w\,\varphi[(u,v,w)]$ is written "(u,v) is *a composable pair of* φ_x," or "(u,v) is *composable in* φ_x."

If (u,v) is a composable pair of φ_x, the term $\tau_w(\varphi[(u,v,w)])$ is denoted by $u \underset{\varphi_x}{\star} v$, and is called *the composite of* v *and* u *by* φ_x.

2.1.3. *Example.* Let p, q, r be three terms.

1) The formula $x=(p,q,r)$ is a composition with respect to x.
2) The formula $x=(p,p,p)\vee x=(p,q,p)\vee x=(q,p,p)\vee x=(q,q,q)$ is a composition with respect to x.

2.1.4. DEFINITION. Let $\varphi[x]$ be a formula in $\mathcal{T}$. The formula

$$\begin{aligned}&\varphi[(t,t,t)]\\ &\wedge \forall u\forall v[\{\exists w\varphi[(u,t,w)] \Longrightarrow \tau_w(\varphi[(u,t,w)])=u\}\\ &\qquad \wedge\{\exists w\varphi[(t,v,w)] \Longrightarrow \tau_w(\varphi[(t,v,w)])=v\}],\end{aligned}$$

where u, v, w are variables not appearing in φ, is written $t \underset{x}{\circledcirc} \varphi$.

If $\mathbf{COMP}_x(\varphi)$, $t \underset{x}{\circledcirc} \varphi$ is also written "t is *a unit of* φ_x."

Remark: If $\mathbf{COMP}_x(\varphi)$, $t \underset{x}{\circledcirc} \varphi$ is identical with the formula

$$\begin{aligned}&\varphi[(t, t, t)]\\ &\wedge \forall u \forall v[\{(u, t) \text{ is composable in } \varphi_x \Longrightarrow u \underset{\varphi_x}{\star} t = u\}\\ &\qquad \wedge \{(u, t) \text{ is composable in } \varphi_x \Longrightarrow t \underset{\varphi_x}{\star} v = v\}].\end{aligned}$$

2.1.5. *Example.* The composition in 2.1.3. 2) has q as unit, i.e.

$$q \underset{x}{\circledcirc} (x=(p, p, p) \vee x=(p, q, p) \vee x=(q, p, p) \vee x=(q, q, q))$$

is a theorem.

2.2. Associative composition

2.2.1. Definition. Let $\varphi[x]$ be a formula in $\mathcal{T}$. Let $\mathbf{ASS}_x(\varphi)$ be the formula

$$\begin{aligned}&\mathbf{COMP}_x(\varphi)\\ &\wedge \forall p \forall q \forall r \forall u[\exists s(\varphi[(p, q, s)] \wedge \varphi[(s, r, u)])\\ &\qquad \Longleftrightarrow \exists t(\varphi[(q, r, t)] \wedge \varphi[(p, t, u)])]\\ &\wedge \forall p \forall q \forall r[(p, q) \text{ and } (q, r) \text{ are composable pairs of } \varphi_x\\ &\qquad \Longrightarrow \exists u(\varphi[(p \underset{\varphi_x}{\star} q, r, u)] \wedge \varphi[(p, q \underset{\varphi_x}{\star} r, u)])],\end{aligned}$$

where p, q, r, s, t, u do not appear in φ.

$\mathbf{ASS}_x(\varphi)$ is also written "φ_x is *an associative composition*," or "φ is *an associative composition with respect to* x."

2.2.2. Definition. When $\mathbf{ASS}_x(\varphi)$,
(p, q) and (q, r) are composable pairs of $\varphi_x \Longrightarrow (p \underset{\varphi_x}{\star} q) \underset{\varphi_x}{\star} r = p \underset{\varphi_x}{\star} (q \underset{\varphi_x}{\star} r)$.
The term $\tau_u(\varphi[(p \underset{\varphi_x}{\star} q, r, u)] \wedge \varphi[(p, q \underset{\varphi_x}{\star} r, u)])$ is denoted by $p \underset{\varphi_x}{\star} q \underset{\varphi_x}{\star} r$.

2.3. Category

Now, we come to the definition of category. We first define the notion $\mathbf{CATEGORY}_{x,y}(\varphi, \psi)$.

2.3.1. Definition. Let $\varphi[x]$, $\psi[y]$ be two formulas in $\mathcal{T}$, where x does not appear in ψ and y does not appear in φ, and let u, v, w, e be variables which do not appear in φ nor ψ. The formula

$$\begin{aligned}&\mathbf{ASS}_x(\varphi)\\ &\wedge \forall u \forall v \forall w[\varphi[(u, v, w)]\\ &\qquad \Longrightarrow \{\exists e(\psi[e] \wedge \varphi[(e, u, u)]) \wedge \exists e(\psi[e] \wedge \varphi[(u, e, u)])\\ &\qquad\quad \wedge \exists e(\psi[e] \wedge \varphi[(e, v, v)]) \wedge \exists e(\psi[e] \wedge \varphi[(v, e, v)])\}]\\ &\wedge \forall e[\psi[e] \Longleftrightarrow e \underset{x}{\odot} \varphi]\end{aligned}$$

is written $\mathbf{CATEGORY}_{x,y}(\varphi, \psi)$, or "$(\varphi_x, \psi_y)$ is *a category*," or "(φ, ψ) is *a category with respect to* x, y."

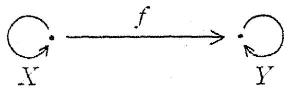
2.3.2. *Example.* Let $\varphi[x]$ be the formula

$$x=(X, X, X) \vee x=(Y, Y, Y) \vee x=(f, X, f) \vee x=(Y, f, f),$$

and let $\psi[y]$ be the formula

$$y=X \vee y=Y.$$

Then $X \neq Y \neq f \neq X \Longrightarrow \mathbf{CATEGORY}_{x,y}(\varphi, \psi)$.



2.3.3. DEFINITION. Let $\varphi[x]$ be a formula in $\mathcal{T}$. The formula

$$\mathbf{CATEGORY}_{x,y}(\varphi, y \underset{x}{\odot} \varphi)$$

is written $\mathbf{CATEGORY}_x(\varphi)$, or "$\varphi_x$ is *a category*," or "φ is *a category with respect to* x."

Remark: This definition makes sense because two formulas: $\mathbf{CATEGORY}_{x,y}(\varphi, y \underset{x}{\odot} \varphi)$ and $\mathbf{CATEGORY}_{x,y'}(\varphi, y' \underset{x}{\odot} \varphi)$ are identical.

As $\mathbf{CATEGORY}_{x,y}(\varphi, \psi) \Longleftrightarrow (\mathbf{CATEGORY}_x(\varphi) \wedge \forall y[\psi[y] \Longleftrightarrow y \odot \varphi])$ is a theorem in $\mathcal{T}$, it does not lose generality to use only the categories of the form $(\varphi, y \underset{x}{\odot} \varphi)$. So, we use the notation $\mathbf{CATEGORY}_x(\varphi)$ mainly. But, sometimes it is essentially convenient to use the notation $\mathbf{CATEGORY}_{x,y}(\varphi, \psi)$. (c. f. Scheme 8 (3.4))

2.3.4. *Example.* Let $\psi_7[X, Y, Z, E, f, g, h, x]$ denote the formula in $\mathcal{T}$ below:

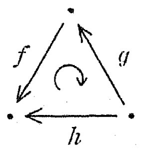
$$\begin{aligned}&x=(X, X, X) \vee x=(Y, Y, Y) \vee x=(Z, Z, Z) \vee x=(E, E, E)\\ &\vee x=(f, Y, f) \vee x=(Z, f, f) \vee x=(g, X, g) \vee x=(Y, g, g)\\ &\vee x=(h, X, h) \vee x=(Z, h, h) \vee x=(f, g, h).\end{aligned}$$

Then $\forall X \forall Y \forall Z \forall E \forall f \forall g \forall h(X, Y, Z, f, g, h$ are pairwise non-equal $\Longrightarrow \mathbf{CATEGORY}_x(\psi_7))$ is a theorem.

Remark: $X=Y=Z=f=g\neq h \Longrightarrow \neg\,\mathbf{CATEGORY}_x(\phi_7)$.

If $\mathbf{CATEGORY}_x(\phi_7)$, ϕ_7 is called "the septet category (7-morphism-category) made of X, Y, Z, E, f, g, h."

2.3.5. DEFINITION. Let φ_x be a category in $\mathcal{T}$. $\varphi[t]$ is written "t is *a triangle of* φ_x." $\varphi[(f, g, h)]$ is also denoted by the diagram below:



in φ_x

The formula

$$\exists v\exists w\varphi[(t, v, w)] \vee \exists u\exists w\varphi[(u, t, w)] \vee \exists u\exists v\varphi[(u, v, t)] \vee t \underset{x}{\odot} \varphi$$

is written $t \underset{x}{\sqsubseteq} \varphi$ or "t is *a morphism of* φ_x."

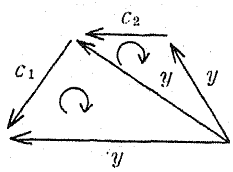
2.3.6. CRITERION. *Let* φ_x *be a category in* $\mathcal{T}$. *Then*

$$\forall y\forall c_1\forall c_2[(c_1 \underset{x}{\odot} \varphi \wedge c_2 \underset{x}{\odot} \varphi \wedge \varphi[(c_1, y, y)] \wedge \varphi[(c_2, y, y)]) \Longrightarrow c_1=c_2],$$

$$\forall y\forall d_1\forall d_2[(d_1 \underset{x}{\odot} \varphi \wedge d_2 \underset{x}{\odot} \varphi \wedge \varphi[(y, d_1, y)] \wedge \varphi[(y, d_2, y)]) \Longrightarrow d_1=d_2],$$

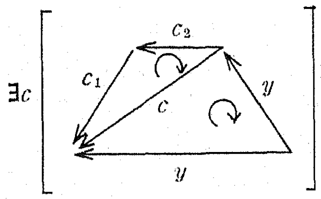
are theorems.

Proof: Suppose $c_1 \underset{x}{\odot} \varphi \wedge c_2 \underset{x}{\odot} \varphi \wedge \varphi[(c_1, y, y)] \wedge \varphi[(c_2, y, y)]$, then



in φ_x.

By the associativity of φ_x,



therefore $c_1=c_1 \star c_2=c_2$. This proves the first half of the criterion. The proof of the second half is similar. ■

2.3.7. DEFINITION. Let φ_x be a category in $\mathcal{T}$, and let $y \underset{x}{\sqsubseteq} \varphi$. Then

$$\tau_c(c \underset{x}{\odot} \varphi \wedge \varphi[(c, y, y)]) \text{ is denoted by } \mathbf{cod}_{\varphi_x}(y),$$

$$\tau_d(d \underset{x}{\odot} \varphi \wedge \varphi[(y, d, y)]) \text{ is denoted by } \mathbf{dom}_{\varphi_x}(y).$$

2.3.8. DEFINITION. The formula $c=\mathbf{cod}_{\varphi_x}(y) \wedge d=\mathbf{dom}_{\varphi_x}(y)$ is denoted by

$$c \xleftarrow{\;y\;} d \quad \textit{in } \varphi_x .$$

2.3.9. CRITERION. *Let* φ_x *be a category in* $\mathcal{T}$. *Then*

$\forall u \forall v[(u, v)$ is composable in $\varphi_x \Longleftrightarrow \mathbf{cod}_{\varphi_x}(u)=\mathbf{dom}_{\varphi_x}(v)]$,

$\forall y[y \underset{x}{\odot} \varphi \Longleftrightarrow y=\mathbf{cod}_{\varphi_x}(y)=\mathbf{dom}_{\varphi_x}(y)]$,

$\forall u \forall v \forall w[\varphi[(u, v, w)] \Longrightarrow \{\mathbf{cod}_{\varphi_x}(w)=\mathbf{cod}_{\varphi_x}(u) \wedge \mathbf{dom}_{\varphi_x}(w)=\mathbf{dom}_{\varphi_x}(v)\}]$,

are theorems. ■

2.3.10. DEFINITION. Let φ_x be a category in $\mathcal{T}$. The formula

$$\forall x(x \underset{x}{\sqsubseteq} \varphi \Longrightarrow x \underset{x}{\odot} \varphi)$$

is written $\mathbf{Disc}_x(\varphi)$, or "φ_x is *discrete.*"

2.3.11. CRITERION. *Let* φ_x *be a category in* $\mathcal{T}$. *Then*

$$\mathbf{Disc}_x(\varphi) \Longrightarrow \forall u \forall v[(u \underset{x}{\sqsubseteq} \varphi \wedge v \underset{x}{\sqsubseteq} \varphi \wedge u \neq v) \Longrightarrow (u, v) \text{ is not composable in } \varphi_x]$$

is a theorem.

Proof: Suppose $\mathbf{Disc}_x(\varphi)$, and suppose $u \underset{x}{\sqsubseteq} \varphi \wedge v \underset{x}{\sqsubseteq} \varphi$. Then $u \underset{x}{\odot} \varphi \wedge v \underset{x}{\odot} \varphi$, so

$$(u, v) \text{ is composable in } \varphi_x \Longrightarrow u=u \underset{\varphi_x}{\star} v=v \Longrightarrow u=v.$$

This proves the criterion. ■

2.3.12. CRITERION. *Let* φ_x *be a category in* $\mathcal{T}$. *Then*

$$\mathbf{Disc}_x(\varphi) \Longrightarrow \forall u \forall v \forall w[\varphi[(u, v, w)] \Longrightarrow w \underset{x}{\odot} \varphi \wedge u=v=w]$$

is a theorem. ■

2.3.13. CRITERION. *Let* $\psi[e]$ *be a formula in* $\mathcal{T}$, *and let* x *be a variable not appearing in* ψ. *Then* $(\exists e(x=(e, e, e) \wedge \psi[e]))_x$ is a category, *and*

$$\mathbf{Disc}_x(\exists e(x=(e, e, e) \wedge \psi[e]))$$

is a theorem. ■

2.3.14. DEFINITION. Let φ_x, ψ_x be two categories in $\mathcal{T}$. The formula

$$\forall y(\varphi[y] \Longrightarrow \psi[y]) \wedge \forall z(z \underset{x}{\odot} \varphi \Longrightarrow z \underset{x}{\odot} \psi)$$

is written $\varphi \underset{x}{\subset} \psi$, or "$\varphi_x$ is *a subcategory of* ψ_x."

2.3.15. Criterion. *Let* $\varphi_x, \psi_x, \theta_x$ *be three categories in* $\mathscr{T}$, *then*

$$\varphi \underset{x}{\subset} \varphi,$$
$$(\varphi \underset{x}{\subset} \psi \wedge \psi \underset{x}{\subset} \theta) \Longrightarrow \varphi \underset{x}{\subset} \theta,$$

are theorems. ■

Now, we shall define the opposite category, the category of all units, the discretization category, the category of all morphisms from X into Y, the product category, the category of all composable pairs. And finally, we shall define functor, the composition of functors.

2.3.16. Definition. Let φ_x be a category in $\mathscr{T}$. The formula

$$x \text{ is a triple } \wedge \varphi[(\mathbf{pr}_2 x, \mathbf{pr}_1 x, \mathbf{pr}_3 x)]$$

is a category with respect to x in $\mathscr{T}$ and is denoted by φ_x^{op}.
φ_x^{op} is called *the opposite (dual) category of* φ_x.

2.3.17. Criterion. *Let* φ_x *be a category in* $\mathscr{T}$, *then*

$$\forall y[(\varphi_x^{\mathrm{op}})_x^{\mathrm{op}}[y] \Longleftrightarrow \varphi_x[y]],$$
$$\mathbf{Disc}_x(\varphi) \Longrightarrow \forall y[\varphi[y] \Longleftrightarrow \varphi_x^{\mathrm{op}}[y]],$$
$$\forall y[y \underset{x}{\odot} \varphi \Longleftrightarrow y \underset{x}{\odot} \varphi_x^{\mathrm{op}}],$$

are theorems. ■

2.3.18. Definition. Let φ_x be a category in $\mathscr{T}$. The formula

$$\exists e(x=(e, e, e) \wedge e \underset{x}{\odot} \varphi)$$

is a discrete category with respect to x in $\mathscr{T}$ (2.3.13), and is denoted by $|\varphi|_x$.
$|\varphi|_x$ is called *the category of all units in* φ_x.

2.3.19. Criterion. *Let* φ_x *be a category in* $\mathscr{T}$, *then*

$$|\varphi| \underset{x}{\subset} \varphi$$

is a theorem. ■

2.3.20. Definition. Let φ_x be a category in $\mathscr{T}$. The formula

$$\exists e(x=(e, e, e) \wedge e \underset{x}{\sqsubseteq} \varphi)$$

is a discrete category with respect to x in $\mathscr{T}$ (2.3.13), and is denoted by $\breve{\varphi}_x$.
$\breve{\varphi}_x$ is called *the discretization of* φ_x.

2.3.21. CRITERION. *Let φ_x be a category in $\mathscr{T}$, then*

$$\forall y[y \underset{x}{\sqsubseteq} \varphi \Longleftrightarrow y \underset{x}{\sqsubseteq} \check{\varphi}_x]$$

is a theorem. ■

2.3.22. DEFINITION. Let φ_x be a category in $\mathscr{T}$, and let X, Y be units of φ_x. The formula

$$\exists e[x=(e, e, e) \wedge \mathbf{dom}_{\varphi_x}(e)=X \wedge \mathbf{cod}_{\varphi_x}(e)=Y]$$

is a discrete category with respect to x in $\mathscr{T}$ (2.3.13), and is denoted by $\varphi_x(X, Y)$, or $\mathrm{Hom}_{\varphi_x}(X, Y)$, etc.

2.3.23. CRITERION. *Let φ_x be a category in $\mathscr{T}$, then*

$$\varphi_x(X, Y) \underset{x}{\subset} \check{\varphi}_x$$

is a theorem. ■

2.3.24. DEFINITION. Let φ_x, ψ_x be two categories in $\mathscr{T}$. The formula

$$\exists y \exists z[x=((\mathbf{pr}_1 y, \mathbf{pr}_1 z), (\mathbf{pr}_2 y, \mathbf{pr}_2 z), (\mathbf{pr}_3 y, \mathbf{pr}_3 z)) \wedge \varphi[y] \wedge \psi[z]]$$

is a category with respect to x in $\mathscr{T}$, and is denoted by $\varphi \underset{x}{\times} \psi$, which is called *the product of φ_x and ψ_x.*

2.3.25. CRITERION. *Let φ_x, ψ_x be two categories in $\mathscr{T}$, then*

$$\forall y[y \underset{x}{\odot} (\varphi \underset{x}{\times} \psi) \Longleftrightarrow (\mathbf{pr}_1 y \underset{x}{\odot} \varphi \wedge \mathbf{pr}_2 y \underset{x}{\odot} \psi)]$$

is a theorem. ■

2.3.26. DEFINITION. Let φ_x be a category in $\mathscr{T}$. The formula

$$\begin{aligned}&\exists u_1 \exists u_2 \exists v_1 \exists v_2 \exists w_1 \exists w_2\\ &[x=((u_1, u_2), (v_1, v_2), (w_1, w_2)) \wedge (\varphi \underset{x}{\times} \varphi)[x]\\ &\wedge (u_1, u_2), (v_1, v_2), (w_1, w_2) \text{ are composable pairs of } \varphi_x.]\end{aligned}$$

is a category with respect to x in $\mathscr{T}$, and is denoted by $\varphi \underset{x}{\bigstar} \varphi$.

2.3.27. DEFINITION. Let $\varphi_x, \psi_x, \alpha_x$ be three categories in $\mathscr{T}$. The formula

$$\begin{aligned}&\alpha \underset{x}{\subset} (\varphi \underset{x}{\times} \psi)\\ &\wedge \forall z(z \underset{x}{\sqsubseteq} \varphi \Longrightarrow \exists! w(w \underset{x}{\sqsubseteq} \psi \wedge (z, w) \underset{x}{\sqsubseteq} \alpha))\\ &\wedge \forall f \forall g \forall h(\varphi[(f, g, h)] \Longrightarrow \psi[(\alpha(f), \alpha(g), \alpha(h))])\\ &\wedge \forall e(e \underset{x}{\odot} \varphi \Longrightarrow \alpha(e) \underset{x}{\odot} \psi),\end{aligned}$$

where $\alpha(a)$ denotes $\tau_b((a, b) \underset{x}{\sqsubseteq} \alpha)$, is written,

"α_x is *a functor from* φ_x *into* ψ_x."

We also write $\varphi_x \xrightarrow{\alpha_x} \psi_x$.

2.3.28. *Example.* Let φ_x be a category in $\mathcal{T}$. The formula

$$\exists u \exists v \exists w[x=((u, u), (v, v), (w, w)) \wedge \varphi[(u, v, w)]]$$

is a category with respect to x in $\mathcal{T}$, which is called *the diagonal category of* φ_x. Furthermore, it is a functor from φ_x into φ_x. The above formula is also denoted by I_{φ_x}.

We have $\forall y[y \underset{x}{\sqsubseteq} \varphi \Longrightarrow I_{\varphi_x}(y)=y]$, so the functor is also called *the identity functor on* φ_x.

2.3.29. Definition. Let $\varphi_x \xrightarrow{\alpha_x} \psi_x \xrightarrow{\beta_x} \theta_x$ be true. Then the formula

$$\begin{aligned}&\exists u_1 \exists u_2 \exists u_3 \exists v_1 \exists v_2 \exists v_3 \exists w_1 \exists w_2 \exists w_3\\ &[x=((u_1, w_1), (u_2, w_2), (u_3, w_3))\\ &\quad \wedge \alpha[((u_1, v_1), (u_2, v_2), (u_3, v_3))]\\ &\quad \wedge \beta[((v_1, w_1), (v_2, w_2), (v_3, w_3))]]\end{aligned}$$

is a category with respect to x in $\mathcal{T}$. And furthermore, it is a functor from φ_x into θ_x.

This functor is denoted by $\beta \underset{x}{\bigcirc} \alpha$, which is called *the composite functor of* α *and* β.

2.3.30. Criterion. *Let* $\varphi_x \xrightarrow{\alpha_x} \psi_x$ *be true. Then*

$$\forall y[y \underset{x}{\sqsubseteq} \varphi \Longrightarrow (\alpha \underset{x}{\bigcirc} I_\varphi)(y)=\alpha(y)],$$
$$\forall y[y \underset{x}{\sqsubseteq} \varphi \Longrightarrow (I_\psi \underset{x}{\bigcirc} \alpha)(y)=\alpha(y)],$$

are theorems. ■

2.3.31. Definition. Let $\varphi_x \underset{\beta_x}{\overset{\alpha_x}{\rightleftarrows}} \psi_x$ be true. Then the formula

$$\forall y[y \underset{x}{\sqsubseteq} \varphi \Longrightarrow (\beta \underset{x}{\bigcirc} \alpha)(y)=y] \wedge \forall y[y \underset{x}{\sqsubseteq} \psi \Longrightarrow (\alpha \underset{x}{\bigcirc} \beta)(y)=y]$$

is written "φ_x and ψ_x are *isomorphic* with α_x and β_x as *isomorphisms*."

3. PAN-CATEGORISM~Category theory $\mathcal{T}_{\mathrm{Cat}}$

$\mathcal{T}_{\mathrm{Cat}}$ is an equalitarian theory with ordered pair, with an additional binary predicate symbol $\lhd$ and additional axioms and a scheme: Axiom 2~Axiom 6, Scheme 8.

If t, C are terms, the formula $\lhd tC$ is denoted by $t \lhd C$.

As the symbol $\lhd$ is introduced, we can define the formula which is read as "C is *a category*," for a term C.

Using $\lhd$, we now state our Axiom of PAN-CATEGORISM.

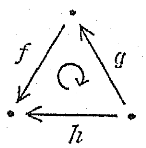
3.1. Axiom of PAN-CATEGORISM

3.1.1. AXIOM 2. (Axiom of PAN-CATEGORISM)

$\forall C[\mathbf{CATEGORY}_x(x \lhd C)]$.

Remark: If we read $\mathbf{CATEGORY}_x(x \lhd C)$ (which does not contain the variable x) as "C is *a category*," Axiom 2 says that "*all terms are categories*," i. e. claims PAN-CATEGORISM. From now on, category also means term.

3.1.2. DEFINITION. $t \lhd C$ is written "t is *a triangle of* C." $(f, g, h) \lhd C$ is also denoted by the diagram below:



in C

$f \underset{x \lhd C}{\star} g$ is denoted by $f \underset{C}{\star} g$ or "*the composite of* g *and* f *in a category* C."

$f \underset{x \lhd C}{\star} g \underset{x \lhd C}{\star} h$ is denoted by $f \underset{C}{\star} g \underset{C}{\star} h$.

$e \underset{x}{\odot} (x \lhd C)$ is written $e \odot C$ or "e is *a unit of* C."

$y \underset{x}{\sqsubseteq} (x \lhd C)$ is written $y \sqsubseteq C$ or "y is *a morphism of* C."

$\mathbf{cod}_{x \lhd C}(y)$ is denoted by $\mathbf{cod}_C(y)$.

$\mathbf{dom}_{x \lhd C}(y)$ is denoted by $\mathbf{dom}_C(y)$.

The formula $c = \mathbf{cod}_C(y) \wedge d = \mathbf{dom}_C(y)$ is denoted by

$$c \xleftarrow{\;y\;} d \quad \textit{in } C.$$

$\mathbf{Disc}_x(x \lhd C)$ is written $\mathbf{Disc}(C)$ or "C is *a discrete category*."

3.1.3. DEFINITION. The formula $x \lhd C \underset{x}{\subset} x \lhd D$, (2.3.14) which does not contain the variable x, is written $C \subset D$, or "C is *a subcategory of* D."

3.1.4. PROPOSITION.

$$\forall C(C \subset C).$$

$$\forall C \forall D \forall E[(C \subset D \wedge D \subset E) \Longrightarrow C \subset E].$$

The proof is immediate from 2.3.15. ■

3.2. Axiom of extentionality

3.2.1. AXIOM 3. (Axiom of extentionality)

$$\forall C \forall D[(C \subset D \wedge D \subset C) \Longrightarrow C = D].$$

Remark: This axiom is equivalent to

$$\forall C \forall D[\{\forall x(x \text{ is a triangle of } C \Longleftrightarrow x \text{ is a triangle of } D) \wedge \forall x(x \text{ is a unit of } C \Longleftrightarrow x \text{ is a unit of } D)\} \Longrightarrow C = D]$$

Therefore, the axiom above claims that a term is specified by the category structure it determines.

3.2.2. CRITERION. *Let (φ_x, ψ_x) be a category, and let C be a variable which do not appear in φ nor in ψ. Then the formula*

$$\forall x \forall y[(x \lhd C \Longleftrightarrow \varphi[x]) \wedge (y \odot C \Longleftrightarrow \psi[y])]$$

is single-valued in C.

This is an immediate consequense of Axiom 3. ■

3.2.3. DEFINITION. Let (φ_x, ψ_x) be a category. The formula

$$\exists C \forall x \forall y[(x \lhd C \Longleftrightarrow \varphi[x]) \wedge (y \odot C \Longleftrightarrow \psi[y])]$$

is written $\mathbf{M\text{-}ABLE}_{x,y}(\varphi, \psi)$, or "$(\varphi_x, \psi_y)$ is *materializable*," or "(φ, ψ) is *materializable with respect to* x, y."

If $\mathbf{M\text{-}ABLE}_{x,y}(\varphi, \psi)$, the term

$$\tau_c(\forall x \forall y[(x \lhd C \Longleftrightarrow \varphi[x]) \wedge (y \odot C \Longleftrightarrow \psi[y])])$$

is written $\mathbf{MTRL}_{x,y}(\varphi, \psi)$, or "*the materialization of* (φ_x, ψ_y)," or "*the materialization of* (φ, ψ) *with respect to* x, y."

Let φ_x be a category.

$$\mathbf{M\text{-}ABLE}_x(\varphi) \text{ denotes } \mathbf{M\text{-}ABLE}_{x,y}(\varphi, y \underset{x}{\odot} \varphi).$$

$$\mathbf{MTRL}_x(\varphi) \text{ denotes } \mathbf{MTRL}_{x,y}(\varphi, y \underset{x}{\odot} \varphi).$$

3.2.4. PROPOSITION. *Obviously,*

$$\forall C[C=\mathbf{MTRL}_x(x \lhd C)]. \blacksquare$$

3.3. Axiom of the existence of category containing arbitrary category as triangle and unit

3.3.1. AXIOM 4. (Axiom of the existence of category containing arbitrary category as triangle and unit)

$$\forall x \forall y \exists L[(x \text{ is a triple} \Longrightarrow x \lhd L) \wedge y \odot L].$$

3.3.2. DEFINITION. The category

$$\tau_L((x \text{ is a triple} \Longrightarrow x \lhd L) \wedge y \odot L)$$

is denoted by $\underset{x,y}{\mathbb{C}}$.

3.3.3. PROPOSITION. $\forall x[x \text{ is a triple} \Longrightarrow \exists L(x \lhd L)] \wedge \forall y[\exists L(y \odot L)]$. $\blacksquare$

3.4. Bourbaki's scheme

3.4.1. SCHEME 8. (Bourbaki's scheme)

Let $\varphi[x]$, $\psi[y]$ be two formulas where x, y are distinct variables and x does not appear in ψ, y does not appear in φ. Let C, D, z, w be variables distinct from x, y where C, D, z do not appear in ψ, C, D, w do not appear in φ.

Then the formula

$$\forall z \forall w \exists C \forall x \forall y[(\varphi \Longrightarrow x \lhd C) \wedge (\psi \Longrightarrow y \odot C)]$$
$$\Longrightarrow \forall D[\mathbf{CATEGORY}_{x,y}(\varphi', \psi') \Longrightarrow \mathbf{M\text{-}ABLE}_{x,y}(\varphi', \psi')]$$

is an axiom, where φ' denotes the formula $\exists z[z \lhd D \wedge \varphi]$, and ψ' denotes the formula $\exists w[w \odot D \wedge \psi]$.

Remark: Intuitively, Bourbaki's scheme asserts the following:

If for each z, w, there exists a category C such that each x satisfying the formula $\varphi[x, z]$ is a triangle of C and that each y satisfying the formula $\psi[y, w]$ is a unit of C, *then* for any category D, all x satisfying $\varphi[x, z]$ where z is a triangle of D and all y satisfying $\psi[y, w]$ where w is a unit of D make a materializable category in case they make a category.

3.5. Applications of Bourbaki's scheme

3.5.1. CRITERION OF SUBCATEGORIES.

Let K be a term, and let $\theta[x]$, $\lambda[y]$ be two formulas where x, y are distinct variables which do not appear in K. Then

$$\mathbf{CATEGORY}_{x,y}(\theta[x] \wedge x \lhd K, \lambda[y] \wedge y \odot K)$$
$$\Longrightarrow \mathbf{M\text{-}ABLE}_{x,y}(\theta[x] \wedge x \lhd K, \lambda[y] \wedge y \odot K)$$

is a theorem.

Proof: Let $\varphi[x]$ be the formula $\theta[x] \wedge x$ is a triple $\wedge x=z$, and $\psi[y]$ be the formula $\lambda[y] \wedge y=w$, where z, w are distinct variables which are distinct from x, y and do not appear in θ, λ, K. By Axiom 4,

$$\forall x \forall y[(\varphi[x] \Longrightarrow x \lhd \underset{z,w}{\mathbb{C}}) \wedge (\psi[y] \Longrightarrow y \odot \underset{z,w}{\mathbb{C}})].$$

Let C be a variable distinct from x, y, z, w which does not appear in θ, λ. Then

$$\forall z \forall w \exists C \forall x \forall y[(\varphi[x] \Longrightarrow x \lhd C) \wedge (\psi[y] \Longrightarrow y \odot C)].$$

So, by Scheme 8,

$$\mathbf{CATEGORY}_{x,y}(\varphi', \psi') \Longrightarrow \mathbf{M\text{-}ABLE}_{x,y}(\varphi', \psi')$$

is true, where φ' denotes the formula $\exists z[z \lhd K \wedge \varphi]$, and ψ' denotes the formula $\exists w[w \odot K \wedge \psi]$.

We have $\varphi' \Longleftrightarrow x \lhd K \wedge \theta[x]$, and $\psi' \Longleftrightarrow y \odot K \wedge \lambda[y]$. Therefore $\mathbf{CATEGORY}_{x,y}(x \lhd K \wedge \theta, y \odot K \wedge \lambda) \Longrightarrow \mathbf{M\text{-}ABLE}_{x,y}(x \lhd K \wedge \theta, y \odot K \wedge \lambda)$. ■

3.5.2. Criterion of Substitution.

Let K, t, s be three terms, x, y, z, w, be distinct variables where x, y do not appear in K, and z, w do not appear in K nor in t, s. Then

$$\mathbf{CATEGORY}_{z,w}(\exists x(z=t \wedge x \lhd K), \exists y(w=s \wedge y \odot K))$$
$$\Longrightarrow \mathbf{M\text{-}ABLE}_{z,w}(\exists x(z=t \wedge x \lhd K), \exists y(w=s \wedge y \odot K))$$

is a theorem.

Proof: Suppose $\mathbf{CATEGORY}_{z,w}(\exists x(z=t \wedge x \lhd K), \exists y(w=s \wedge y \odot K))$. Then t is a triple. So, $\forall z(z=t \Longrightarrow z \lhd \underset{t,s}{\mathbb{C}})$, $\forall w(w=s \Longrightarrow w \odot \underset{t,s}{\mathbb{C}})$ are true. Therefore

$$\forall x \forall y \exists C \forall z \forall w[(z=t \Longrightarrow z \lhd C) \wedge (w=s \Longrightarrow w \odot C)].$$

So, by Scheme 8,

$$\mathbf{CATEGORY}_{z,w}(\varphi, \psi) \Longrightarrow \mathbf{M\text{-}ABLE}_{z,w}(\varphi, \psi)$$

is true, where φ denotes the formula $\exists x[z=t \wedge x \lhd K]$, and ψ denotes the formula $\exists y[w=s \wedge y \odot K]$. Therefore

$$\mathbf{M\text{-}ABLE}_{z,w}(\exists x(z=t \wedge x \lhd K), \exists y(w=s \wedge y \odot K)). \blacksquare$$

3.5.3. Definition. Let x be a triangle of D, and y be a unit of D. Let $\underset{x,y}{\mathbb{C}}{}_{7D}$ denote the subcategory

$$\mathbf{MTRL}_z(\psi_7[\mathbf{dom}_D(\mathbf{pr}_3 x), \mathbf{cod}_D(\mathbf{pr}_2 x), \mathbf{cod}_D(\mathbf{pr}_1 x), y, \mathbf{pr}_1 x, \mathbf{pr}_2 x, \mathbf{pr}_3 x, z])$$

of D.

This definition makes sense because of the criterion of subcategories. In paticular, if $D=\underset{x,y}{\mathbb{C}}$, $\underset{x,y}{\mathbb{C}_7}{}_D$ is denoted by $\underset{x,y}{\mathbb{C}_7}$.

3.5.4. THEOREM.

$\forall X \forall Y \forall Z \forall E \forall f \forall g \forall h$

$[\mathbf{CATEGORY}_x(\psi_7[X, Y, Z, E, f, g, h, x]) \Longrightarrow \mathbf{M\text{-}ABLE}_x(\psi_7[X, Y, Z, E, f, g, h, x])]$,

where ψ_7 *is the septet category defined in* 2.3.4.

Proof: Suppose $\mathbf{CATEGORY}_x(\psi_7[X, Y, Z, E, f, g, h, x])$. Let $t[z]$ denote the term $\tau_y((z=\mathbf{dom}_{\underset{(f,g,h),E}{\mathbb{C}_7}}(h) \wedge y=X) \vee (z=\mathbf{cod}_{\underset{(f,g,h),E}{\mathbb{C}_7}}(g) \vee y=Y) \vee (z=\mathbf{cod}_{\underset{(f,g,h),E}{\mathbb{C}_7}}(f) \vee y=Z) \vee (z=y=E) \vee (z=y=f) \vee (z=y=g) \vee (z=y=h))$. Then

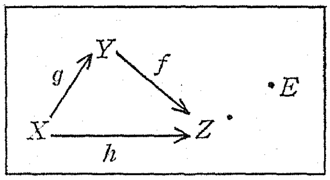
$$\psi_7[X, Y, Z, E, f, g, h] \Longleftrightarrow \exists z[z \lhd \underset{(f,g,h),E}{\mathbb{C}_7} \wedge x=(t[\mathbf{pr}_1 z], t[\mathbf{pr}_2 z], t[\mathbf{pr}_3 z])].$$

Therefore, by the criterion of substitution, $\mathbf{M\text{-}ABLE}_x(\psi_7[X, Y, Z, E, f, g, h])$. The theorem is proved. ■

3.5.5. DEFINITION. If $\mathbf{CATEGORY}_x(\psi_7[X, Y, Z, E, f, g, h, x])$, the term

$$\mathbf{MTRL}_x(\psi_7[X, Y, Z, E, f, g, h, x]),$$

i. e. the corresponding materialization, is denoted by



or $\mathbb{C}_7[X, Y, Z, E, f, g, h]$, etc.

Now, we shall show that various categories defined in 2.3 are materializable if the original categories are materializable:

3.5.6. CRITERION. *Let* φ_x, ψ_x *be two categories, and let* X, Y *be units of* φ_x. *Then*

1) $\mathbf{M\text{-}ABLE}_x(\varphi) \Longrightarrow \mathbf{M\text{-}ABLE}_x(\varphi_x^{\mathrm{op}})$
2) $\mathbf{M\text{-}ABLE}_x(\varphi) \Longrightarrow \mathbf{M\text{-}ABLE}_x(|\varphi|_x)$
3) $\mathbf{M\text{-}ABLE}_x(\varphi) \Longrightarrow \mathbf{M\text{-}ABLE}_x(\breve{\varphi})$
4) $\mathbf{M\text{-}ABLE}_x(\varphi) \Longrightarrow \mathbf{M\text{-}ABLE}_x(\varphi_x(X, Y))$
5) $\{\mathbf{M\text{-}ABLE}_x(\varphi) \wedge \mathbf{M\text{-}ABLE}_x(\psi)\} \Longrightarrow \mathbf{M\text{-}ABLE}_x(\varphi \underset{x}{\times} \psi)$
6) $\mathbf{M\text{-}ABLE}_x(\varphi) \Longrightarrow \mathbf{M\text{-}AELE}_x(\varphi \underset{x}{\bigstar} \varphi)$

are theorems. (2.3.16, 2.3.18, 2.3.20, 2.3.22, 2.3.24, 3.3.26)

We can write above criterion in the form of theorem:

3.5.7. THEOREM.

1) $\forall C[\mathbf{M\text{-}ABLE}_x((x \lhd C)_x^{op})]$
2) $\forall C[\mathbf{M\text{-}ABLE}_x(|x \lhd C|_x)]$
3) $\forall C[\mathbf{M\text{-}ABLE}_x(\overbrace{x \lhd C_x})]$
4) $\forall C \forall X \forall Y[X, Y \text{ are units of } C \Longrightarrow \mathbf{M\text{-}ABLE}_x((x \lhd C)_x(X, Y))]$
5) $\forall C \forall D[\mathbf{M\text{-}ABLE}_x(x \lhd C \underset{x}{\times} x \lhd D)]$
6) $\forall C[\mathbf{M\text{-}ABLE}_x(x \lhd C \underset{x}{\bigstar} x \lhd C)]$

Proof: 1) Let $\varphi[x]$ denote the formula $\exists z(z \lhd C \wedge x = (\mathbf{pr}_2 z, \mathbf{pr}_1 z, \mathbf{pr}_3 z))$, and let $\psi[y]$ denote the formula $\exists w(w \odot C \wedge y = w)$. Then, by the criterion of substitution, $\mathbf{CATEGORY}_{x,y}(\varphi, \psi) \Longrightarrow \mathbf{M\text{-}ABLE}_{x,y}(\varphi, \psi)$. We have

$$\varphi[x] \Longleftrightarrow (x \text{ is a triple } \wedge (\mathbf{pr}_2 x, \mathbf{pr}_1 x, \mathbf{pr}_3 x) \lhd C) \Longleftrightarrow (x \lhd C)_x^{op}[x]$$

$$\psi[y] \Longleftrightarrow y \odot C \Longleftrightarrow y \underset{x}{\odot} (x \lhd C)_x^{op}.$$

Therefore, $\mathbf{CATEGORY}_{x,y}((x \lhd C)_x^{op}, y \underset{x}{\odot} (x \lhd C)_x^{op}) \Longrightarrow \mathbf{M\text{-}ABLE}_{x,y}((x \lhd C)_x^{op}, y \underset{x}{\odot} (x \lhd C)_x^{op})$.

By the fact $\mathbf{CATEGORY}_x((x \lhd C)_x^{op})$, the theorem is proved.

3) By the criterion of substitution.

$\mathbf{M\text{-}ABLE}_x(\exists e(x = (\mathbf{pr}_3 e, \mathbf{pr}_3 e, \mathbf{pr}_3 e) \wedge e \lhd C))$, i. e.

$\mathbf{M\text{-}ABLE}_x(\exists e(x = (e, e, e) \wedge e \sqsubseteq C)$. This proves the theorem.

5) To prove this theorem, we prove first the lemma below:

Lemma: $\forall X_1 \forall Y_1 \forall Z_1 \forall E_1 \forall f_1 \forall g_1 \forall h_1 \forall X_2 \forall Y_2 \forall Z_2 \forall E_2 \forall f_2 \forall g_2 \forall h_2$

$[\{\mathbf{CATEGORY}_x(\psi_7[X_1, Y_1, Z_1, E_1, f_1, g_1, h_1, x])$

$\wedge \mathbf{CATEGORY}_x(\psi_7[X_2, Y_2, Z_2, E_2, f_2, g_2, h_2, x])\}$

$\Longrightarrow \mathbf{M\text{-}ABLE}_x(x \lhd \mathbb{C}_7[X_1, Y_1, Z_1, E_1, f_1, g_1, h_1] \underset{x}{\times} x \lhd \mathbb{C}_7[X_2, Y_2, Z_2, E_2, f_2, g_2, h_2])]$

Proof: The category $x \lhd \mathbb{C}_7[X_1, Y_1, Z_1, E_1, f_1, g_1, h_1] \underset{x}{\times} x \lhd \mathbb{C}_7[X_2, Y_2, Z_2, E_2, f_2, g_2, h_2]$ is a category with 49 morphisms and 121 triangles. The materializability is proved by using Bourbaki's scheme and Axiom 4.

Using this lemma, we have

$$\forall C \forall D \forall x \forall y \forall z \forall w$$

$$[(x \lhd C \wedge y \odot C \wedge z \lhd D \wedge w \odot D) \Longrightarrow \mathbf{M\text{-}ABLE}_p(p \lhd \underset{x,y}{\mathbb{C}_7}{}_C \underset{p}{\times} p \lhd \underset{z,w}{\mathbb{C}_7}{}_D)].$$

The materialization with respect to p is denoted by $\underset{x,y}{\mathbb{C}_7}{}_C \times \underset{z,w}{\mathbb{C}_7}{}_D$.

Then let $x \lhd C$, $y \odot C$,

$$\forall p\forall q[\{(\exists a(a \lhd \underset{x,y}{\mathbb{C}_7}{}_C \wedge$$

$$p=((\mathbf{pr}_1 a, \mathbf{pr}_1 z), (\mathbf{pr}_2 a, \mathbf{pr}_2 z), (\mathbf{pr}_3 a, \mathbf{pr}_3 z))) \wedge z \lhd D) \Longrightarrow p \lhd \underset{x,y}{\mathbb{C}_7}{}_C \times \underset{z,w}{\mathbb{C}_7}{}_D\}$$

$$\wedge \{(\exists b(b \odot \underset{x,y}{\mathbb{C}_7}{}_C \wedge q=(b, w)) \wedge w \odot D \qquad) \Longrightarrow q \odot \underset{x,y}{\mathbb{C}_7}{}_C \times \underset{z,w}{\mathbb{C}_7}{}_D\}].$$

Therefore, $\forall C\forall D\forall x\forall y[(x \lhd C \wedge y \odot C] \Longrightarrow$

$$[\forall z\forall w\forall C'\forall p\forall q$$

$$[\{(\exists a(a \lhd \underset{x,y}{\mathbb{C}_7}{}_C \wedge p=((\mathbf{pr}_1 a, \mathbf{pr}_1 z), (\mathbf{pr}_2 a, \mathbf{pr}_2 z), (\mathbf{pr}_3 a, \mathbf{pr}_3 z))) \wedge z \lhd D) \Longrightarrow p \lhd C'\}$$

$$\wedge \{(\exists b(b \odot \underset{x,y}{\mathbb{C}_7}{}_C \wedge q=(b, w)) \wedge w \odot D \qquad) \Longrightarrow q \odot C'\}]]].$$

So, by Bourbaki's scheme,

$$\forall C\forall D\forall x\forall y[(x \lhd C \wedge y \odot C) \Longrightarrow$$

$$\mathbf{M\text{-}ABLE}_{p,q}(\exists z\exists a(a \lhd \underset{x,y}{\mathbb{C}_7}{}_C \wedge z \lhd D \wedge p=((\mathbf{pr}_1 a, \mathbf{pr}_1 z), (\mathbf{pr}_2 a, \mathbf{pr}_2 z), (\mathbf{pr}_3 a, \mathbf{pr}_3 z))),$$

$$\exists w\exists b(b \odot \underset{x,y}{\mathbb{C}_7}{}_C \wedge w \odot D \wedge q=(b, w)))].$$

I. e. $\forall C\forall D\forall x\forall y[(x \lhd C \wedge y \odot C) \Longrightarrow \mathbf{M\text{-}ABLE}_p(p \lhd \underset{x,y}{\mathbb{C}_7}{}_C \times p \lhd D)]$. The materiali-zation with respect to p is denoted by $\underset{x,y}{\mathbb{C}_7}{}_C \times D$. Then

$$\forall p\forall q[\{(\exists z(z \lhd D \wedge$$

$$p=((\mathbf{pr}_1 x, \mathbf{pr}_1 z), (\mathbf{pr}_2 x, \mathbf{pr}_2 z), (\mathbf{pr}_3 x, \mathbf{pr}_3 z))) \wedge x \lhd C) \Longrightarrow p \lhd \underset{x,y}{\mathbb{C}_7}{}_C \times D\}$$

$$\wedge \{(\exists w(w \odot D \wedge q=(y, w)) \wedge y \odot C \qquad) \Longrightarrow q \odot \underset{x,y}{\mathbb{C}_7}{}_C \times D\}].$$

Furthermore,

$$\forall C\forall D\forall x\forall y\exists C'\forall p\forall q$$

$$[\{\exists z(z \lhd D \wedge p=((\mathbf{pr}_1 x, \mathbf{pr}_1 z), (\mathbf{pr}_2 x, \mathbf{pr}_2 z), (\mathbf{pr}_3 x, \mathbf{pr}_3 z))) \wedge x \lhd C) \Longrightarrow p \lhd C'\}$$

$$\wedge \{(\exists w(w \odot D \wedge q=(y, w)) \wedge y \odot C \qquad) \Longrightarrow q \odot C'\}]$$

So, by Bourbaki's scheme,
$\forall C\forall D[\mathbf{M\text{-}ABLE}_p(p \lhd C \underset{p}{\times} p \lhd D)]$, q. e. d.

2), 3), 6) are obvious because of the facts

$$|x \lhd C|_x \underset{x}{\subset} x \lhd C,$$

$$(x \lhd C)_x(X, Y) \underset{x}{\subset} (\widetilde{x \lhd C})_x,$$

$$x \lhd C \underset{x}{\bigstar} x \lhd C \underset{x}{\subset} x \lhd C \underset{x}{\times} x \lhd C.$$

(2.3.19, 2.3.23) ■

3.5.8. DEFINITION.

1) $\mathbf{MTRL}_x((x \lhd C)_x^{op})$ is written C^{op} or "*the opposite (dual) of C.*"
2) $\mathbf{MTRL}_x(|x \lhd C|_x)$ is written $|C|$ or "*the category of all units in C.*"
3) $\mathbf{MTRL}_x(\overbrace{x \lhd C_x})$ is written $\breve{C}$ or "*the discretization of C.*"
4) $\mathbf{MTRL}_x((x \lhd C)_x(X, Y))$ is denoted by $C(X, Y)$ or $\mathrm{Hom}_C(X, Y)$, etc.
5) $\mathbf{MTRL}_x(x \lhd C \underset{x}{\times} x \lhd D)$ is written $C \times D$ or "*the product of C and D.*"
6) $\mathbf{MTRL}_x(x \lhd C \underset{x}{\bigstar} x \lhd C)$ is denoted by $C \bigstar C$.

3.5.9. Definition. $|C_7[X, Y, Z, E, f, g, h]|$ is denoted by $\{X, Y, Z, E\}$.

3.5.10. Proposition. $\forall X \forall Y \forall Z \forall E[\forall x(x \sqsubseteq \{X, Y, Z, E\} \Longleftrightarrow (x = X \vee x = Y \vee x = Z \vee x = E))]$. ■

3.5.11. Definition. $\{X, X, Y, E\}$ is also denoted by $\{X, Y, E\}$.
$\{X, X, E\}$ is also denoted by $\{X, E\}$.
$\{X, X\}$ is also denoted by $\{X\}$.

3.5.12. Criterion. *Let $\varphi_x, \psi_x, \alpha_x$ be three categories. Then*

$$(\varphi_x \xrightarrow{\alpha_x} \psi_x \wedge \mathbf{M\text{-}ABLE}_x(\varphi) \wedge \mathbf{M\text{-}ABLE}_x(\psi)) \Longrightarrow \mathbf{M\text{-}ABLE}_x(\alpha)$$

is a theorem.

The proof is straightforward by the fact $\alpha \underset{x}{\subset} (\varphi \underset{x}{\times} \psi)$. ■

3.5.13. Definition. The formula $(x \lhd C)_x \xrightarrow{(x \lhd F)_x} (x \lhd D)_x$ (c. f. 2.3.27) is also denoted by

$$C \xrightarrow{F} D,$$

which is written "F is *a functor from C into D.*"

If $C \xrightarrow{F} D \xrightarrow{G} E$, the materialization of the composite functor of F and G (2.3.29) is denoted by $G \bigcirc F$, which is also called *the composite functor of F and G.*

The materialization of the identity functor (2.3.28) on $(x \lhd C)_x$ is denoted by I_C, which is also called *the identity functor on C.*

3.5.14. Definition. Let $\mathbf{Ism}(C, D)$ be the formula below:

$$\exists F \exists G[C \underset{G}{\overset{F}{\rightleftarrows}} D \wedge G \bigcirc F = \mathrm{I}_C \wedge F \bigcirc G = \mathrm{I}_D],$$

which is also written "C and D are *isomorphic.*"

3.5.15. Definition. The term $\tau_z(\mathbf{Ism}(Z, C))$ is denoted by $\mathbf{Type}(C)$.
The discretization of $\tau_z(\mathbf{Ism}(\breve{Z}, \breve{C}))$ is denoted by $\mathbf{Card}(C)$.

3.5.16. THEOREM. The formula $\forall x(\neg x \sqsubseteq X)$ is functional in X.

Proof: Suppose $\forall x(\neg x \sqsubseteq X_1)$, $\forall x(\neg x \sqsubseteq X_2)$.
Then $\forall x(\neg x \lhd X_1)$, $\forall x(\neg x \lhd X_2)$, $\forall x(\neg x \odot X_1)$, $\forall x(\neg x \odot X_2)$. Therefore

$$\forall x(x \lhd X_1 \Longleftrightarrow x \lhd X_2) \wedge \forall x(x \odot X_1 \Longleftrightarrow x \odot X_2),$$

so $X_1 = X_2$. I.e. the formula $\forall x(\neg x \sqsubseteq X)$ is single-valued in X. We have, $\mathbf{MTRL}_x(X \neq X)$ satisfies the formula. ■

3.5.17. DEFINITION. $\tau_x(\forall x(\neg x \sqsubseteq X))$ is denoted by ϕ or 0.
ϕ is written "*empty category*."

3.6. Axiom of power category

3.6.1. AXIOM 5. (Axiom of power category)

$$\forall C \exists D[\forall x(x \sqsubseteq D \Longleftrightarrow x \subset C)].$$

3.6.2. DEFINITION. The discretization of $\tau_D(\forall x(x \sqsubseteq D \Longleftrightarrow x \subset C))$ is written $\mathcal{P}(C)$ or "*the power category of C.*"

3.6.3. *Example.* $\mathcal{P}(\phi) = \{\phi\}$,
$\mathcal{P}(\{\phi\}) = \{\phi, \{\phi\}\}$.

3.7. Axiom of infinity

3.7.1. AXIOM 6. (Axiom of infinity)

$$\exists X[\exists x(x \sqsubseteq X) \wedge \forall x(x \sqsubseteq X \Longrightarrow \exists y(y \sqsubseteq X \wedge x \subset y \wedge x \neq y))].$$

[AXIOMS AND SCHEME OF $\mathcal{T}_{\mathrm{Cat}}$]

3.1.1. AXIOM 2. $\forall C[\mathbf{CATEGORY}_x(x \lhd C)]$.

3.2.1. AXIOM 3. $\forall C \forall D[(C \subset D \wedge D \subset C) \Longrightarrow C = D]$.

3.3.1. AXIOM 4. $\forall x \forall y \exists L[(x \text{ is a triple} \Longrightarrow x \lhd L) \wedge y \odot L]$.

3.4.1. SCHEME 8. *Let $\varphi[x]$, $\phi[y]$ be two formulas where x, y are distinct variables and x does not appear in ϕ, y does not appear in φ. Let C, D, z, w be variables distinct from x, y where C, D, z do not appear in ϕ, C, D, w do not appear in φ. Then*

$$\forall z \forall w \exists C \forall x \forall y[(\varphi \Longrightarrow x \lhd C) \wedge (\phi \Longrightarrow y \odot C)]$$
$$\Longrightarrow \forall D[\mathbf{CATEGORY}_{x,y}(\varphi', \phi') \Longrightarrow \mathbf{M\text{-}ABLE}_{x,y}(\varphi', \phi')],$$

where φ' *denotes* $\exists z[z \lhd D \wedge \varphi]$, ψ' *denotes* $\exists w[w \circledcirc D \wedge \psi]$, *is an axiom.*

3.6.1. AXIOM 5. $\forall C \exists D[\forall x(x \Subset D \Longleftrightarrow x \subset C)]$.

3.7.1. AXIOM 6. $\exists X[\exists x(x \Subset X) \wedge \forall x(x \Subset X \Longrightarrow \exists y(y \Subset X \wedge x \subset y \wedge x \neq y))]$.

4. Consistency of $\mathscr{T}_{\mathrm{Cat}}$ relative to Bourbaki's set theory

4.1. Preliminaries in $\mathscr{T}_{\mathrm{Set}}$

Throughout this chapter, $\mathscr{T}_{\mathrm{Set}}$ denotes a set theory, which is an equalitarian theory with ordered pair, with an additional binary predicate symbol $\in$ and additional axioms and a scheme: Axiom 3'～Axiom 6', Scheme 8' below.

4.1.1. AXIOM 3'. $\forall x \forall y[(x \subset y \wedge y \subset x) \Longrightarrow x = y]$.

4.1.2. AXIOM 4'. $\forall x \exists L[x \in L]$.

4.1.3. AXIOM 5'. $\forall X[\mathbf{Coll}_Y(Y \subset X)]$.

4.1.4. AXIOM 6'. $\exists X[\exists x(x \in X) \wedge \forall x(x \in X \Longrightarrow \exists y(y \in X \wedge x \subset y \wedge x \neq y))]$.

4.1.5. SCHEME 8'. *Let* $\varphi[x]$ *be a formula,* x, y *be distinct variables. Let* X, Y *be variables distinct from* x, y *which do not appear in* φ. *Then*
$\forall y \exists X \forall x[\varphi \Longrightarrow x \in X] \Longrightarrow \forall Y[\mathbf{Coll}_x(\exists y(y \in Y \wedge \varphi))]$ *is an axiom.*

Remark: The definition of $x \subset y$, $\mathbf{Coll}_x(\varphi)$ is those used in [1] Chapter 2. $\mathscr{T}_{\mathrm{Set}}$ and Bourbaki's set theory are equivalent.

4.1.6. DEFINITION IN $\mathscr{T}_{\mathrm{Set}}$. We denote by **CATEGORY**(C), the formula

$$C \text{ is a pair } \wedge \mathbf{pr}_2 C \in \mathscr{P}(\mathbf{pr}_1 C \times \mathbf{pr}_1 C \times \mathbf{pr}_1 C) \wedge \mathbf{pr}_1 C = \mathbf{pr}_1 \langle \mathbf{pr}_2 C \rangle$$

$$\wedge\, \mathbf{CATEGORY}_x(x \in \mathbf{pr}_2 C) \wedge \mathbf{pr}_2 C \neq \Delta_{\mathrm{pr}_1 C \times \mathrm{pr}_1 C \times \mathrm{pr}_1 C}$$

where $\Delta_{x \times x \times x} = \{(z, z, z) | z \in x\}$.

4.1.7. PROPOSITION IN $\mathscr{T}_{\mathrm{Set}}$. The formula

$$(\mathbf{CATEGORY}(C) \wedge x = \mathbf{pr}_2 C) \vee (\neg \mathbf{CATEGORY}(C) \wedge x = \Delta_{C \times C \times C})$$

is single-valued in x for all C.

Proof: Suppose that

$$(\mathbf{CATEGORY}(C) \wedge x_1 = \mathbf{pr}_2 C) \vee (\neg \mathbf{CATEGORY}(C) \wedge x_1 = \Delta_{C \times C \times C}),$$

$$(\mathbf{CATEGORY}(C) \wedge x_2 = \mathbf{pr}_2 C) \vee (\neg \mathbf{CATEGORY}(C) \wedge x_2 = \Delta_{C \times C \times C}).$$

Then $(\mathbf{CATEGORY}(C) \wedge x_1 = x_2 = \mathbf{pr}_2 C) \vee (\neg\mathbf{CATEGORY}(C) \wedge x_1 = x_2 = \Delta_{C\times C\times C})$, i. e. $x_1 = x_2$. ■

4.1.8. DEFINITION IN $\mathcal{T}_{\mathrm{Set}}$. The set

$$\tau_x((\mathbf{CATEGORT}(C) \wedge x = \mathbf{pr}_2 C) \vee (\neg\mathbf{CATEGORY}(C) \wedge x = \Delta_{C\times C\times C}))$$

is written $\mathbf{Tr}\, C$, or "*the set of all triangles of C.*"

4.1.9. PROPOSITION IN $\mathcal{T}_{\mathrm{Set}}$.

$$\forall C[(\mathbf{CATEGORY}(C) \Longrightarrow \mathbf{Tr}\, C = \mathbf{pr}_2 C) \wedge (\neg\mathbf{CATEGORY}(C) \Longrightarrow \mathbf{Tr}\, C = \Delta_{C\times C\times C})].$$ ■

4.1.10. PROPOSITION IN $\mathcal{T}_{\mathrm{Set}}$. $\forall C[\mathbf{CATEGORY}_x(x \in \Delta_{C\times C\times C})]$. ■

4.1.11. COROLLARY. $\forall C[\mathbf{CATEGORY}_x(x \in \mathbf{Tr}\, C)]$. ■

4.1.12. DEFINITION IN $\mathcal{T}_{\mathrm{Set}}$.

$y \underset{x}{\odot} (x \in \mathbf{Tr}\, C)$ is denoted by $y \,\square\, C$.

$\exists u \exists v[(y, u, v) \in \mathbf{Tr}\, C] \vee \exists u \exists v[(u, y, v) \in \mathbf{Tr}\, C] \vee \exists u \exists v[(u, v, y) \in \mathbf{Tr}\, C] \vee y \,\square\, C$

is denoted by $y \mathrm{E} C$.

4.1.13. PROPOSITION IN $\mathcal{T}_{\mathrm{Set}}$.

$$\forall C \forall D[\mathbf{Tr}\, C = \mathbf{Tr}\, D \Longrightarrow \{\mathbf{CATEGORY}(C) \Longleftrightarrow \mathbf{CATEGORY}(D)\}].$$

Proof: Suppose $\mathbf{Tr}\, C = \mathbf{Tr}\, D$, and suppose $\mathbf{CATEGORY}(C) \wedge \neg\mathbf{CATEGORY}(D)$. Then $\mathbf{pr}_2 C = \Delta_{D\times D\times D}$, therefore $\mathbf{pr}_1 \langle \mathbf{pr}_2 C \rangle = \mathbf{pr}_1 \langle \Delta_{D\times D\times D} \rangle$, $\mathbf{pr}_1 C = D$. So, $\mathbf{pr}_2 C = \Delta_{\mathbf{pr}_1 C \times \mathbf{pr}_1 C \times \mathbf{pr}_1 C}$. This contradicts $\mathbf{CATEGORY}(C)$, i. e. $\mathbf{CATEGORY}(C) \Longrightarrow \mathbf{CATEGORY}(D)$. Similarly $\mathbf{CATEGORY}(D) \Longrightarrow \mathbf{CATEGORY}(C)$. ■

4.1.14. PROPOSITION IN $\mathcal{T}_{\mathrm{Set}}$. $\forall C \forall D[\mathbf{Tr}\, C = \mathbf{Tr}\, D \Longrightarrow \{C = D\}]$.

Proof: Suppose $\mathbf{Tr}\, C = \mathbf{Tr}\, D$. *First* suppose $\mathbf{CATEGORY}(C)$. By the proposition 4.1.13, $\mathbf{CATEGORY}(D)$, therefore $\mathbf{pr}_2 C = \mathbf{pr}_2 D$, $\mathbf{pr}_1 C = \mathbf{pr}_1 \langle \mathbf{pr}_2 C \rangle = \mathbf{pr}_1 \langle \mathbf{pr}_2 D \rangle = \mathbf{pr}_1 D$, so $C = D$. *Next*, suppose $\neg\mathbf{CATEGORY}(C)$. By the proposition 4.1.13, $\neg\mathbf{CATEGORY}(D)$, therefore $\Delta_{C\times C\times C} = \Delta_{D\times D\times D}$, so $C = D$. ■

4.1.15. DEFINITION IN $\mathcal{T}_{\mathrm{Set}}$. $\forall x[x \in \mathbf{Tr}\, C \Longrightarrow x \in \mathbf{Tr} D] \wedge \forall x[x \,\square\, C \Longrightarrow x \,\square\, D]$ is denoted by $C \underset{\mathrm{Cat}}{\subset} D$.

4.2. Main theorem

We interprete $x \lhd C$ in $\mathcal{T}_{\mathrm{Cat}}$ as $x \in \mathbf{Tr}\, C$ in $\mathcal{T}_{\mathrm{Set}}$.

4.2.1. MAIN THEOREM. *By the interpretation above, all axioms (implicit and explicit) of $\mathcal{T}_{\mathrm{Cat}}$ become theorems in $\mathcal{T}_{\mathrm{Set}}$.*

Before proving the main theorem above, we prove some lemmas.

Lemma: *The formula* $y \odot C$ *is interpreted as* $y \square C$.
The formula $y \sqsubseteq C$ *is interpreted as* $y \mathrm{E} C$.
The formula $C \sqsubset D$ *is interpreted as* $C \underset{\mathrm{Cat}}{\subset} D$.

The proof is straightforward.

Proof of main theorem:

I) Explicit axioms.

Axiom 2 is interpreted as $\forall C[\mathbf{CATEGORY}_x(x \in \mathbf{Tr}\, C)]$, which is a theorem in $\mathscr{T}_{\mathrm{Set}}$ (4.1.11).

Axiom 3 is interpreted as

$$\forall C \forall D[[\forall x(x \in \mathbf{Tr}\, C \Longleftrightarrow x \in \mathbf{Tr}\, D) \wedge \forall x(x \square C \Longleftrightarrow x \square D)] \Longrightarrow C = D].$$

Equivalently, $\forall C \forall D[\mathbf{Tr}\, C = \mathbf{Tr}\, D \Longrightarrow C = D]$, which is a theorem in $\mathscr{T}_{\mathrm{Set}}$ (4.1.13).

Axiom 4 is interpreted as

$$\forall x \forall y \exists L[(x \text{ is a triple} \Longrightarrow x \in \mathbf{Tr}\, L) \wedge y \square L].$$

This is a theorem in $\mathscr{T}_{\mathrm{Set}}$. In fact, *when* x is a triple, we can take as L, $\{z \mid \phi_7[X, Y, Z, y, f, g, h, z]\}$, where $f = \mathbf{pr}_1\, x$, $g = \mathbf{pr}_2\, x$, $h = \mathbf{pr}_3\, x$, $X = \tau_A((f \neq g \wedge A = (h, g)) \vee (f = g \wedge A = (f, g)))$, $Y = (f, g)$, $Z = \tau_B((f \neq g \wedge B = (f, h)) \vee (f = g \wedge B = (f, g)))$, and *when* x is not a triple, we can take as L, $\{(y, y, y)\}$.

Axiom 5 is interpreted as $\forall C \exists D \forall x[x \mathrm{E} D \Longleftrightarrow x \underset{\mathrm{Cat}}{\subset} C]$. It is enough to prove $\forall C \exists D' \forall x[x \in D' \Longleftrightarrow x \underset{\mathrm{Cat}}{\subset} C]$, i.e. $\forall C[\mathbf{Coll}_x(x \underset{\mathrm{Cat}}{\subset} C)]$. This is clearly a theorem.

Axiom 6 is interpreted as

$$\exists X[\exists x(x \mathrm{E} X) \wedge \forall x(x \mathrm{E} X \Longrightarrow \exists y(y \mathrm{E} X \wedge x \underset{\mathrm{Cat}}{\subset} y \wedge x \neq y))].$$

It is enough to prove

$$\exists X[\exists x(x \in X) \wedge \forall x(x \in X \Longrightarrow \exists y(y \in X \wedge x \underset{\mathrm{Cat}}{\subset} y \wedge x \neq y))].$$

Furthermore it is enough to prove

$$\exists X[\exists x(x \in X) \wedge \forall x(x \in X \Longrightarrow \exists y(y \in X \wedge x \subset y \wedge x \neq y))],$$

which is exactly axiom of infinity in $\mathscr{T}_{\mathrm{Set}}$ (4.1.4).

II) Implicit axioms.

Implicit axiom provided by Scheme 8 is interpreted as

$$\forall z \forall w \exists C \forall x \forall y[(\varphi \Longrightarrow x \in \mathbf{Tr}\, C) \wedge (\psi \Longrightarrow y \square C)]$$

$$\Longrightarrow \forall D[\mathbf{CATEGORY}_{x,y}(\varphi', \psi') \Longrightarrow \exists X \forall x \forall y((x \in \mathbf{Tr}\, X \Longleftrightarrow \varphi') \wedge (y \square X \Longleftrightarrow \psi'))],$$

where φ, ψ are formulas which are the interpretation of certain formulas in $\mathcal{T}_{\mathrm{Cat}}$, and φ' denotes $\exists z(z \in \mathbf{Tr}\, D \wedge \varphi)$, ψ' denotes $\exists w(w \square D \wedge \psi)$. Suppose that $\forall z \forall w \exists C \forall x \forall y[(\varphi \Longrightarrow x \in \mathbf{Tr}\, C) \wedge (\psi \Longrightarrow y \square C)]$. Then $\forall z \exists C \forall x[\varphi \Longrightarrow x \in \mathbf{Tr}\, C]$, $\forall w \exists C \forall y[\psi \Longrightarrow y \square C]$, therefore, $\forall D[\mathbf{Coll}_x(\varphi'')]$, $\forall D[\mathbf{Coll}_y(\psi'')]$, where φ'' denotes $\exists z(z \in D \wedge \varphi)$, ψ'' denotes $\exists w(w \in D \wedge \psi)$. Let $Z = \{x | \varphi'\}$, $W = \{y | \psi'\}$, then

$$\mathbf{CATEGORY}_{x,y}(\varphi', \psi') \Longrightarrow \exists X \exists x \forall y[(x \in \mathbf{Tr}\, X \Longleftrightarrow x \in Z) \wedge (y \square X \Longleftrightarrow y \in W)]$$

is obviously a theorem. ■

Remark: This theorem 4.2.1 claims that $\mathcal{T}_{\mathrm{Cat}}$ is consistent if $\mathcal{T}_{\mathrm{Set}}$ is consistent. On the other hand, we can construct a model of $\mathcal{T}_{\mathrm{Set}}$ in the category theory $\mathcal{T}_{\mathrm{Cat}}$.

References

[1] Bourbaki, N., Theorie des Ensembles, (Hermann) Theory of Sets, (Hermann, Addison-Wesley).

[2] Lawvere, F. W., An Elementary Theory of the Category of Sets, *University of Chicago*, (1966) (mimeographed) Summerized in *Proc. Nat. Acad. Sci., U. S. A.* **52**, 1506–1511 (1964).

[3] Lawvere, F. W., The Category of Categories as a Foundation for Mathematics, *Conference on Categorical Algebra, La Jolla* 1965, Springer-Verlag (1966).