Fredholm Determinant of Unimodal Linear Maps

By Yoichiro TAKAHASHI

Department of Mathematics, College of General Education, University of Tokyo, Komaba, Meguro-ku, Tokyo 153

(Received October 22, 1981)

0. Introduction

The notion of the Fredholm determinant $D(z)$ of a (piecewise) C^1-map f of an interval was introduced in [11], where several important conclusions are derived from the hypothesis that the Fredholm theory of compact integral operators could be applicable to the Perron-Frobenius operator $\mathcal{L}$ of the map f. The hypothesis is, of course, not purely mathematical. The Perron-Frobenius operators are never compact by nature, although the compactness is a necessary requirement for the Fredholm theory (e. g. [9]). In fact, the Perron-Frobenius operators satisfy the following, rather strange property stated in Theorem 1 in Section 1: each complex number of modulus less than one is an eigenvalue with infinite degree of multiplicity.

In [11] the Fredholm determinant $D(z)$ is defined as a limit of the Fredholm determinants of the weighted structure matrices of Markov maps f_p which approximate f and it is proved that $D(z)$, in generic cases, coincides with the inverse $1/Z(z)$ of the power series $Z(z)$ that is the Artin-Mazur-Ruelle zeta function under a special choice of the potential function U, i. e.,

$$D(z)=\exp\left[-\sum_{n=1}^{\infty}\frac{z^n}{n}Q_n(f)\right]. \tag{1}$$

Here

$$Q_n(f)=\sum_{x\in Fix(f^n)}1/|(f^n)'(x)|\,, \tag{2}$$

the set $Fix(f^n)$ is the totality of fixed points of the n-fold iterate f^n of the map f (the precise definition of which will be found in Section 4), and the prime denotes the differentiation with respect to x.

Thus the Fredholm determinant $D(z)$ is an invariant under C^1-conjugacy of maps and the hypothesis of the formal applicability of the Fredholm theory leads to the conclusions, such as the relation $D(1)=0$ implies the existence of an absolutely continuous invariant measure for f, the multiplicity of the zero $z=1$ corresponds to the number of absolutely continuous invariant measures, the location of zeros on the unit circle reflects the degree of mixing properties of f, etc.

On the other hand, it seems quite difficult to show that $D(z)$ is the Fredholm determinant of the operator $\mathcal{L}$ in some adequate sense. There is another reason besides the noncompactness of $\mathcal{L}$. The strongest and strange conclusion of the formal applicability of the Fredholm theory in [11] is as follows: let

$$P(f)=\limsup \frac{1}{n}\log Q_n(f). \tag{3}$$

Then, $P(f)>0$ if and only if there is a stable periodic orbit under f. It is an exact result proved by the analysis of the map f as a dynamical system but it means that the minimal nonnegative zero of $D(z)$ can be larger than one, while the norm of the operator $\mathcal{L}$ is one, since $\mathcal{L}$ is an operator on the space $L^1=L^1(J, dx)$ defined by the formula

$$\mathcal{L}\varphi(x)=\sum_{y\in f^{-1}x}\varphi(y)/|f'(y)| \qquad dx\text{-a.e. } x\in J \tag{4}$$

where dx is the Lebesgue measure on the interval J where f is defined. Furthermore, the value of $P(f)$ corresponds to ergodic properties of f, observable chaos and window phenomenon (Section 5).

It is then suggested that $D(z)$ cannot be the Fredholm determinant of the operator $\mathcal{L}$ on L^1 but on some other space since the Fredholm determinant should be $\det(I-zL)$ in some sense. We shall restrict ourselves to the case of unimodal linear maps for which many results are obtained ([4]) and answer to this problem. In Section 2 we shall show that a subspace A of the space of functions with bounded total variation is invariant under $\mathcal{L}$ and it turns out that $D(z)$ is the Fredholm determinant of the operator $\mathcal{L}$ restricted to A in the sense that, for a suitable neighbourhood U of 0 in C,

$$\{z\in U;\ D(z)=0\}=\{z\in U;\ 1/z \text{ is a point spectrum}\} \tag{5}$$

in Section 4. The restriction that $z\in U$ is necessary because $D(z)$ is of form

$$D(z)=1-az-ac_0z^2-\cdots \tag{6}$$

and the power series $D(z)$ has natural boundary unless it is rational.

In order to verify (1) the Artin-Mazur-Ruelle's zeta function is computed in Section 4. In Section 5, we shall see how the Fredholm determinant $D(z)$ reflects the dynamical properties of the map f by giving typical examples of unimodal linear maps, and in the final Section 6 further examples are given in connection with the critical phenomenon in maps of intervals.

Finally we should note that these results are very simple for beta transformations and are essentially obtained in [5], [15]. Furthermore, although we mentioned nothing on the theorem of Šarkovskii explicitly, its generalization in [16] plays an important role in the proof of Theorem 4 in Section 4. The situation was the same for the beta transformations and the analogous order to the Šarkovskii's takes a simpler form

$$2\mapsto 3\mapsto 4\mapsto\cdots\mapsto 1. \tag{7}$$

As to an other assertion of [11] we shall treat in in [18] and [19].

Acknowledgement. The author expresses his sincere thanks to Y. Oono of Loomis Laboratory of Physics UIUC for the stimulating communications with him.

§ 1. Noncompactness of Perron-Frobenius operators

Let f be a piecewise C^1-map of an interval J onto itself and dx the Lebesgue measure on J.

DEFINITION 1. The Perron-Frobenius operator is the following operator on $L^1=L^1(J, dx)$:

$$\mathcal{L}\varphi(x)=\sum_{y\in f^{-1}x}\frac{\varphi(y)}{|f'(y)|} \qquad \text{a.e. } x\in J. \tag{7}$$

The well-definedness is obvious since $\mathcal{L}$ is the dual of the f-action $\varphi\longmapsto\varphi\circ f$ on the spacs $C=C(J)$ of all continuous functions on J. In other words,

$$\int_J \mathcal{L}\varphi(x)\psi(x)dx=\int_J \varphi(x)\psi(fx)dx, \quad \varphi\in L^1,\ \psi\in C. \tag{8}$$

Remark. The operator $\mathcal{L}$ is a nonnegative operator of norm 1, and a nontrivial nonnegative solution of the equation $\mathcal{L}\varphi=\varphi$ gives the density function of an absolutely continuous invariant measure for f. Consequently, the existance of an absolutely continuous invariant measure is equivalent to that 1 is an eigenvalue of $\mathcal{L}$. Furthermore, the mixing properties follow from the study of the iterates $\mathcal{L}^n$ of $\mathcal{L}$, as it has been the method of analysis of number theoretic transformations, such as continued fraction expansion, beta expansion. For example, the weak Bernoulli property was proved in [5] by observing the convergence of the iterates $\mathcal{L}^n$ in a very strong sense for densely many "good" functions.

THEOREM 1. *Assume that f is a piecewise C^1-map of an interval J onto itself which is not monotone. Let $t\in\boldsymbol{C}$ and $|t|<1$. Then t is an eigenvalue of $\mathcal{L}$ with infinite multiplicity.*

The following lemmas are necessary for the proof.

LEMMA 1. *If f is surjective, then there exists a right inverse $\mathcal{K}$ of $\mathcal{L}$, i.e., $\mathcal{L}\mathcal{K}=I$, which is an isometry of L^1.*

Proof. Let J_0 be a Borel subset of J such that $f: J_0 \to J$ is bijective. Put

$$\mathcal{K}\varphi(x)=\begin{cases} \varphi(fx)|f'(x)| & \text{if } x\in J_0 \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\int_J |\mathcal{K}\varphi(x)|dx=\int_{J_0}|\varphi(fx)f'(x)|dx=\int_J|\varphi(y)|dy\,.$$

Thus, $\mathcal{K}: \boldsymbol{L}^1 \to \boldsymbol{L}^1$ is an isometry. Now it is easy to see

$$\mathcal{L}\mathcal{K}\varphi(x)=\sum_{y\in J_0\cap f^{-1}x}\frac{\varphi(fy)|f'(y)|}{|f'(y)|}=\varphi(x)\,.$$

Remark. Lemma 1 shows that the operator $\mathcal{L}$ is surjective.

LEMMA 2. *Assume that f is not monotone. Then the kernel $\{\varphi; \mathcal{L}\varphi=0\}$ is infinite dimensional.*

Proof. It follows from the assumptions that there exist two disjoint subintervals J_1 and J_2 with the following three properties:

(a) $f: J_i \to J$, $i=1, 2$, are injective.
(b) $f(J_1)=f(J_2)$.
(c) $c_1 \leqq f' \leqq c_2$ on J_1 and $-c_1 \geqq f' \geqq -c_2$ on J_2 for some $c_1, c_2>0$.

Put

$$\theta_0(x)=\begin{cases} f'(x) & \text{if } x\in J_1\cup J_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\theta_0 \in \boldsymbol{L}^1$ and it is clear that $\mathcal{L}\theta_0=0$. It follows from the definition of $\mathcal{L}$ that, for any $\varphi\in \boldsymbol{L}^1$ and $\theta\in \boldsymbol{L}^\infty$,

$$\mathcal{L}(\varphi\circ f.\theta)(x)=\varphi(x).\mathcal{L}\theta(x)\,.$$

Consequently, $\varphi\circ f.\theta_0$ belongs to the kernel of $\mathcal{L}$ for any $\varphi\in \boldsymbol{L}^1$ and so the dimension of the kernel is infinite.

Proof of Theorem 1. Take any $\varphi\in \boldsymbol{L}^1$ such that $\mathcal{L}\varphi=0$ and put

$$\psi=\sum_{n=0}^{\infty} t^n \mathcal{K}^n\varphi\,.$$

Since $|t|<1$ and $\mathcal{K}$ is an isometry, thus $\psi\in \boldsymbol{L}^1$ and

$$\mathcal{L}\psi=\sum_{n=1}^{\infty} t^n \mathcal{L}\mathcal{K}^n\varphi=\sum_{n=0}^{\infty} t^{n+1}\mathcal{K}^n\varphi=t\psi\,.$$

§ 2. Invariant subspace under $\mathcal{L}$

We shall restrict ourselves to the unimodal linear maps in the following in order to obtain an invariant space A and the action of $\mathcal{L}$ on A explicitly.

DEFINITION 2. Let a and b be positive real numbers such that $a+b\geqq 1$. The following map of the unit interval $J=[0,1]$ to itself is called unimodal linear [4]:

$$(9)\qquad fx=\begin{cases}(x+a+b-1)/b & \text{if } 0\leqq x\leqq 1-a\\ (1-x)/a & \text{if } 1-a\leqq x\leqq 1\,.\end{cases}\qquad \text{(Fig. 1)}$$

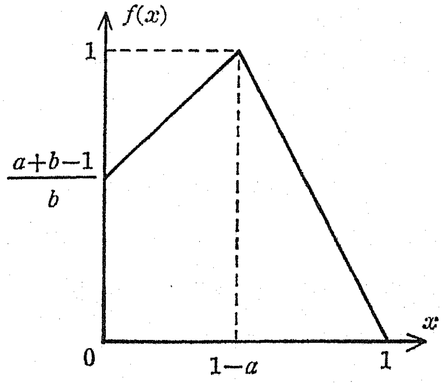


Fig. 1. Unimodal linear map (nondegenerated case)

If a is larger than or equal to 1, then the map f on J does not depend on the parameter b and is an injection on J into itself. Let us call such f degenerated. Otherwise, f is surjective. and the inverse of the map f consists of the following two maps $f_0^{-1}:[(a+b-1)/b,1]\to[0,1-a]$ and $f_1^{-1}:[0,1]\to[1-a,1]$:

$$(10)\qquad f_0^{-1}(x)=bx-(a+b-1),\ f_1^{-1}(x)=1-ax\,.$$

We note that the maps f_0^{-1} and f_1^{-1} are order preserving and reversing, respectively.

For a given t in J let us define

$$(11)\qquad \chi_n^t(x)=1(x\geqq f^n t)=\begin{cases}1 & \text{if } x\geqq f^n t\\ 0 & \text{otherwise}\end{cases}$$

$$(12)\qquad a_n(t)=-a \quad \text{if } f^n t>1-a\,;\quad =b \text{ otherwise}\,.$$

Here $1(\cdot)$ stands for the indicator function of the set indicated by the dot $\cdot$.

LEMMA 3. $\mathcal{L}\chi_n^t = a_n(t)\cdot\chi_{n+1}^t + a$

Proof. If $f^n t \neq 1-a$, then, for $a.e.\ x \in J$,

$$\begin{aligned}\mathcal{L}\chi_n^t(x) &= b\cdot 1(f_0^{-1}x \geqq f^n t)\cdot 1(x \geqq (a+b-1)/b) + a\cdot 1(f_1^{-1}x \geqq f^n t)\\ &= b\cdot 1(f^n t < 1-a)\cdot 1(x \geqq f^{n+1}t) + a\cdot 1(f^n t > 1-a)\\ &\quad + a\cdot 1(f^n t > 1-a)\cdot 1(x \leqq f^{n+1}t)\\ &= [b\cdot 1(f^n t \leqq 1-a) - a\cdot 1(f^n t > 1-a)]\cdot 1(x \geqq f^{n+1}t) + a\\ &= a_n(t)\cdot \chi_{n+1}^t(x) + a\,.\end{aligned}$$

On the other hand, if $f^n t = 1-a$, then, $f^{n+1}t = 1$ and so

$$\begin{aligned}\mathcal{L}\chi_n^t(x) &= b\cdot 1(f_0^{-1}x \geqq 1-a)\cdot 1(x \geqq (a+b-1)/b) + a\cdot 1(f_1^{-1}x \geqq 1-a)\\ &= b\cdot 1(x=1) + a\\ &= a_n(t)\cdot\chi_{n+1}^t + a\,.\end{aligned}$$

Remark. It follows from Lemma 3 by integration that

(13) $$1 - f^n t = a_n(t)\cdot[1 - f^{n+1}t] + a \qquad (n \geqq 0,\ t \in J)\,.$$

Consequently, a real number t admits f-expansion [15]

$$1 - t = a + a\cdot a_0(t) + a\cdot a_0(t)\cdot a_1(t) + \cdots$$

if and only if $a_0(t)\cdot a_1(t)\cdots a_n(t)$ goes to 0 as $n \to \infty$.

The following sequence a_n, $n \geqq 0$, corresponding to the orbit of the point $0+0$ is of great importance:

$$a_n = \lim_{\substack{t\to 0\\ t>0}} a_n(t)\,.$$

In accordance with this, let us define $\chi_n = \lim_{\substack{t\to 0\\ t>0}} \chi_n^t$. Then it is easy to see that $\chi_0(x) = 1(x>0)$ and

$$\chi_n(x) = \begin{cases} 1(x \geqq f^n 0) & \text{if } a_0 a_1 \cdots a_{n-1} < 0\\ 1(x > f^n 0) & \text{if } a_0 a_1 \cdots a_{n-1} > 0\end{cases}$$

and that

$$\mathcal{L}\chi_n(x) = a_n \chi_n(x) + a \qquad \text{for all } x \in J\,.$$

Let $\mathcal{A}$ be the smallest σ-algebra that makes the functions χ_n measurable and $A^\circ = L^1(J, \mathcal{A}, dx)$ be the subspace of $L^1 = L^1(J, dx)$ consisting of all $\mathcal{A}$-

measurable functions.

LEMMA 4. *The subspace A° is invariant under $\mathcal{L}$.*

Proof. Obvious from Lemma 3.

Let us define a map $j: l^1 \to L^{\infty} = L^{\infty}(J, dx)$ by

$$ju = \sum_{n=0}^{\infty} u_n \chi_n \qquad u = (u_n)_{n \geq 0} \in l^1 . \tag{14}$$

and denote the image of j by A. For $\varphi \in A$, put

$$\|\varphi\|_A = \inf\{\|u\|_{l^1} ;\ u \in j^{-1}\varphi\} . \tag{15}$$

LEMMA 5. (i) *$j: l^1 \to A$ is a continuous surjection.*
(ii) *If $\varphi(x)$ and $\psi(x)$ belong to A, so does the product $\varphi(x)\psi(x)$ and*

$$\|\varphi\psi\|_A \leqq \|\varphi\|_A \|\psi\|_A .$$

In other words, $(A, \| \ \|_A)$ is a Banach algebra with unit.

Proof. (i) is evident by the definition of norms. Let

$$\varphi_i(x) = \sum u_{i,n} \chi_n(x) \in A \quad (i=1, 2) ,$$

then,

$$\varphi_1(x)\varphi_2(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{1,n} u_{2,m} \chi_n(x) \chi_m(x)$$

is absolutely convergent and, therefore, equal to

$$\sum_{n=0}^{\infty} [u_{1,n} u_{2,n} + \sum_{m : f^m 0 < f^n 0} (u_{1,n} u_{2,m} + u_{2,n} u_{1,m})] \chi_n(x)$$

Since the coefficients of χ_n in (14) is summable and the absolute sum is dominated by

$$\sum_{n=0}^{\infty} |u_{1,n}| \sum_{m=0}^{\infty} |u_{2,m}| ,$$

thus one obtains (ii).

Now let us define an operator M of l^1 to itself as follows: $v = Mu$ iff $u = (u_n)$, $v = (v_n)$,

$$v_0 = a \sum_{n=0}^{\infty} u_n \quad \text{and} \quad v_n = a_{n-1} u_{n-1} \quad \text{for } n \geqq 1 . \tag{16}$$

LEMMA 6. (i) *$M: l^1 \to l^1$ is a bounded operator such that $jM = Lj$.*

(ii) *The image $\mathcal{L}A$ is contained in A. Moreover, $\mathcal{L}: A \to A$ is a bounded operator.*

Proof. Since $a_n = b$ or $-a$, thus M is bounded on l^1 and the norm $\|M\|$ does not exceed $\max\{2a, a+b\}$. Let $u \in l^1$. Then,

$$jMu(x) = \Big(\sum_{n=0}^{\infty} a \cdot u_n\Big)\chi_0(x) + \sum_{n=1}^{\infty} a_{n-1} u_{n-1} \chi_n(x)$$

$$= \sum_{n=0}^{\infty} u_n \cdot [a_n \chi_{n+1}(x) + a] = \mathcal{L} ju(x),$$

because $\chi_0 = 1$. Hence (i). Now let $\varphi \in A$ and $u \in j^{-1}\varphi$. Then

$$\|\mathcal{L}\varphi\|_A = \|jMu\|_A \leqq \|Mu\|_{l^1} \leqq \|M\| \cdot \|u\|_{l^1}$$

Consequently, $\|\mathcal{L}\varphi\|_A \leqq \|M\| \cdot \|\varphi\|_A$.

Let us denote the orbit $\{f^n 0; n \geqq 0\}$ of 0 by O.

LEMMA 7. (i) *If the set O is finite, then j is injective.*

(ii) *If O is finite, then A is a finite dimensional space.*

Proof. It suffices for both (i) and (ii) to prove that

$$\sum_{n;\, f^n 0 = x,\, a_n < 0} u_n = \sum_{n;\, f^n 0 = x,\, a_n > 0} u_n = 0 \tag{16}$$

for each $x \in 0$ when $ju = 0$. In fact, (i) is then trivial and (ii) follows from the mutual distinctness of $f^n 0$'s when O is infinite.

Now let $u \in l^1$ and $ju = 0$ in L^1. Then, $ju(x) = 0$ for each $x \in J$ since $ju(x)$ is continuous at each point x either from the right or from the left or both. Take any two points $x > y$ in J. Then, $ju(x) - ju(y)$ is equal to the sum of u_n's in such n as either $x \geqq f^n 0 > y$ and $a_n > 0$, or $x > f^n 0 \geqq y$ and $a_n < 0$. Consequently, one obtains (17).

COROLLARY. *The norm $\|\varphi\|_A$ of $\varphi \in A$ is equal to the total variation of the function φ. In order words, the space $(A, \|\ \|_A)$ is the closed linear hull of the indicator functions χ_n, $n \geqq 0$, in the space $BV(J)$ of all functions with bounded variation on J.*

Proof. Obvious from (16).

§ 3. Fredholm determinant of $\mathcal{L}$ on A

Our goal of this section is to show that the power series

$$D(z) = 1 - \sum_{n=0}^{\infty} a c_n z^{n+1}, \quad c_0 = 1, \quad c_n = a_0 a_1 \cdots a_{n-1} \ (n \geqq 1) \tag{18}$$

is the Fredholm determinant $\det(I-zL)$ in some sense. Here a_n's are as before: $a_n=b$ if $f^n t \leqq 1-a$ for any sufficiently small t, and $a_n=-a$ otherwise. Let U be the domain of convergence of the power series (18) and

$$U^{\circ}=\{z\in \boldsymbol{C};\ \sup_m \sum_{n=m}^{\infty} |a_m\cdots a_{m+n-1}z^n|<\infty\}. \tag{19}$$

THEOREM 2. *Let* $\boldsymbol{A}$ *be the* $\mathcal{L}$*-invariant Banach algebra constructed in the previous section and denote the restriction of* $\mathcal{L}$ *to* $\boldsymbol{A}$*, again, by* $\mathcal{L}$*. Assume that the orbit* O *of the point* 0 *under* f *is infinite. Then the following statements are valid*:

(a) *The point spectrum of* $\mathcal{L}$ *on* $\boldsymbol{A}$ *is simple and coincides with the set*

$$\{1/z\,;\ z\in U,\ D(z)=0\}$$

(b) *The resolvent of* $\mathcal{L}$ *in the set* $\{1/z\,;\ z\in U\}$ *coincides with the set*

$$\{1/z\,;\ z\in U^{\circ},\ D(z)\neq 0\}.$$

(c) *The rest of the spectrum of* $\mathcal{L}$ *in* $\{1/z\,;\ z\in U\}$ *is the continuous spectrum.*

THEOREM 3. *The situation is the same as in Theorem 2 but we assume that the orbit* O *is finite. Then the following statements are verified*:

(a) $D(z)$ *is a rational function.*

(b) $\boldsymbol{A}$ *is a finite dimensional space. In particular the Fredholm determinant* $\det(I-zL)$ *is well-defined.*

(c) *If the orbit* O *of* 0 *is periodic with period* p*, then,*

$$\det(I-z\mathcal{L})=(1-a_0a_1\cdots a_{p-1})\cdot D(z).$$

In general, take n^* *such that* $f^{n^*}0$ *is periodic with period* p*. Then,*

$$\det(I-z\mathcal{L})=(1-a_{n^*}a_{n^*+1}\cdots a_{n^*+p-1}z^p)\cdot D(z).$$

Remark. The examples for Theorem 2 will be found in the next section (Examples 4, 5), and the examples for Theorem 3 in case of periodic O will be found in Examples 1-3 and 7. In Example 6 the point 0 falls into the fixed point. Furthermore, there are many cases where 0 falls into a periodic orbit. For example, if $ab^2=1$ and $a>1$, then, $0<f0=f^30<1-a<f^20<1$ and so the point 0 falls into the periodic orbit $\{f0, f^20\}$ of period 2.

The proof will be given after several lemmas. First of all, let us consider the formal solution of the equation $(I-zM)u=w$:

$$u_0-za\sum_{n=0}^{\infty} u_n=w_0,\quad u_n-za_{n-1}u_{n-1}=w_n \quad (n\geqq 1) \tag{20}$$

It follows from (20) with $n\geqq 1$ by induction that

$$u_n = c_n z^n u_0 + \sum_{m=1}^{n} c_n z^n w_m / c_m z^m \quad (n \geqq 1) \tag{21}$$

Replacing (20) with $n=0$ by (21) one obtains formally

$$D(z)u_0 = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a c_n z^{n+1} w_m / c_m z^m . \tag{22}$$

This expression makes sense if the right hand side of (22) is absolutely summable. In particular, if $z \in U^{\circ}$, or if $z \in U$ and $w_n = 0$ except for a finite number of n's.

LEMMA 8. *Let $z \in U$. Then $D(z)=0$ if and only if $1/z$ is an eigenvalue of M on l^1.*

Proof. Put $w=0$ in (20). Then one obtains (21) with $w=0$, which is consistent with (20) for $n=0$ because the both hand sides of (22) vanish and the summability is assured by $z \in U$. Hence (21) with $w=0$ defines a unique eigenvector u corresponding to z.

LEMMA 9. *Let $z \in U^{\circ}$ and $D(z) \neq 0$. Then the inverse $(I-zM)^{-1}$ exists and is a bounded operator on l^1.*

Proof. Since $z \in U^{\circ}$, thus the right hand side of (22) is absolutely convergent. Hence (22) defines a continuous functional $w \to u_0$ on l^1, because $D(z) \neq 0$. Then the well-defined formula (21) gives an bounded operator $w \to u$ of l^1 to itself and u is the unique solution of (20) for w.

LEMMA 10. *Let $z \in U$ and assume that $(I-zM)^{-1}$ exists and is bounded on l^1. Then $z \in U^{\circ}$ and $D(z) \neq 0$.*

Proof. Let $w=(w_n)$ have only a finite number of nonzero w_n's. Then (20)–(22) are meaningful. Since the solvability of (20) is assumed, thus, $D(z) \neq 0$. Solving u_0 from (22), one obtains from (21) that

$$\begin{cases} u_0 = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} a c_k z^{k+1} w_m / c_m z^m D(z) \\ u_n = \sum_{m=0}^{\infty} c_n z^n w_m / c_m z^m D(z) + \sum_{m=n+1}^{\infty} c_n z^n \sum_{k=m}^{\infty} a c_k z^{k+1} w_m / c_m z^m D(z) \quad (n \geqq 1) . \end{cases} \tag{23}$$

In particular, if $w_n = 0$ $(n \neq m)$ and $w_m = 1$, then, for $n \geqq m$,

$$u_n = c_n z^n / c_m D(z) .$$

Consequently, for each $m \geqq 0$,

$$\sum_{n=m}^{\infty} |c_n z^n / c_m z^m| \leq |D(z)| \cdot \|u\|_u \leq |D(z) \cdot \|(I-zM)^{-1}\| .$$

Hence $z \in U^o$.

LEMMA 11. *Let $M=(M_{ij})_{i,j=1,\dots,k}$ be a matrix. Then,*

$$\det(I-zM)=\exp\left\{-\sum \frac{z^n}{n} \operatorname{tr} M^n\right\} = \prod_{i=1}^{k} (1-\sum_{n=2}^{\infty} R_{in} z^n)$$

for any z in a neighbourhood of 0 *in* **C**, *where $R_{i1}=M_{ii}$ and*

$$R_{in} = \sum_{i_1,\dots,i_{n-1}>i} M_{ii_1} M_{i_1 i_2} \cdots M_{i_{n-1} i}$$

Proof. The first identity is a version of $\det \exp M = \exp \operatorname{tr} M$. The second one is proved e.g., in [16] in terms of dynamical systems using the notion of shifts with orbit basis.

Proof of Theorem 2.

Since O is infinite, thus, $\mathcal{L}$ on A is isomorphic to M on l^1 by Lemma 7. Hence, the assertion (a) follows from Lemma 8, (b) from Lemmas 9 and 10, and (c) follows from the fact that $(I-zM)^{-1}$ always has dense domain when $z \in U$ and $D(z) \neq 0$.

Proof of Theorem 3.

Let n^* be the minimal number n such that $f^n 0$ is a periodic point. Then the coefficients ac_n in $D(z)$ form a periodic sequence from $n=n^*$ on. Hence, (a). The second assertion (b) is already obtained as Lemma 7 (ii). Let p be the period of $f^{n^*}0$. Then,

$$\chi_{n+p} = \chi_n \quad \text{for} \quad n \geq n^* .$$

Consequently, the operator $\mathcal{L}$ on A admits a matrix representation

$$L=\left(\begin{array}{cccc|ccc}
a & a_0 & & & & & \\
a & & a_1 & & & 0 & \\
\vdots & & & \ddots & & & \\
a & & & & a_{n^*} & & \\
\hline
a & & & & & a_{n^*+1} & \\
\vdots & & 0 & & & & \ddots \\
a & & & & & & a_{n^*+p-1} \\
a & & & & a_{n^*} & &
\end{array}\right) \tag{24}$$

Applying Lemma 11, one can easily compute $\det(I-zL)$:

$$\begin{aligned}\det(I-zL)&=(1-a_{n^*}\cdots a_{n^*+p-1}z^p)\cdot(1-az-aa_0z^2-\cdots-aa_0\cdots a_{n^*-1}z^{n^*+1}\\ &\quad -aa_0\cdots a_{n^*}z^{n^*+2}-aa_0\cdots a_{n^*}az^{n^*+3}-\cdots)\\ &=(1-a_{n^*}\cdots a_{n^*+p-1}z^p)\cdot D(z)\,.\end{aligned} \tag{25}$$

Remark. The matrix L in (24) is equivalent to the weighted structure matrix S of the map f in the sense that

(i) S acts on the direct sum of subspaces V and W.

(ii) $\dim W=1$ and $SW=\{0\}$.

(iii) There is an isomorphism i of V onto C^{n^*+p} such that $iS=Li$.

(iv) In particular, $\det(I-zS)=z\cdot\det(I-zL)$.

In fact, f is a Markov map because f is monotone on each subinterval $[x_{i-1}, x_i]$ and maps x_i's to themselves if one take as x_i's the orbit $x_0=0<x_1<\cdots<x_k=1-a<\cdots<x_n=1$ of the point $1-a$ $(n=n^*+p+2)$. The weighted matrix S is computed according to its definition:

$$S_{ij}=0 \quad \text{if} \quad f(x_{i-1}, x_i)\cap(x_{j-1}, x_j)=\emptyset \quad \text{and} \tag{26}$$

if $f(x_{i-1}, x_i)\cap(x_{j-1}, x_j)\neq\emptyset$, then,

$$S_{ij}=\text{ the constant value } 1/|f'| \text{ on } (x_{i-1}, x_i)\,.$$

The space $\boldsymbol{A}$ is spanned by the indicator functions of subintervals with end points x_{i-1}, x_i $(i\neq k, k+1)$ and x_{k-1}, x_{k+1} and, therefore, it is isomorphic to the space

$$V=\{u=(u_i)\in \boldsymbol{C}^n\,;\; u_k=u_{k+1}\}$$

On the other hand, the space

$$W=\{u=(u_i)\in \boldsymbol{C}^n\,;\; u_i=0\;\;(i\neq k, k+1),\;\; u_k+u_{k+1}=0\}$$

is contained in the kernel of S and $\boldsymbol{C}^n=W\dot{+}V$.

§ 4. Zeta functions

Let us show that the formula (1) in Introduction is valid in some definite sense. Let $Per(n, f)$ be the set of connected components of periodic points of f with period n. Note that there do appear subintervals consisting of periodic points in some pathological case stated below in Lemma 16. Put

$$Fix(f^n)=\bigcup_{m|n} Per(m, f)\,. \tag{27}$$

DEFINITION 3. The following power series $Z^\circ(z)$ and $Z(z)$ are called Artin-Mazur's and Artin-Mazur-Ruelle's zeta functions [1], [12]:

(28) $$Z^{\circ}(z)=\exp\sum_{n=1}^{\infty}\frac{z^n}{n}Q_n^{\circ}(f) \qquad Q_n^{\circ}(f)=\sum_{x\in Fix(f^n)}1\,,$$

(29) $$Z(z)=\exp\sum_{n=1}^{\infty}\frac{z^n}{n}Q_n(f) \qquad Q_n(f)=\sum_{x\in Fix(f^n)}1/|(f^n)'(x)|\,.$$

Remark. Usually, the Artin-Mazur-Ruelle zeta function is defined for an arbitrary continuous function U in place of $1/|f'(x)|$. The special choice of U is the crusial key to be related to the ergodic theory of maps of intervals.

As is mentioned in Introduction, the inverse $1/Z(z)$ is expected to be $D(z)$. The following theorem answers this almost affirmatively, and, in another direction, makes it possible to compute zeta functions of unimodal linear maps through the orbit of 0.

THEOREM 4. *Let f be a unimodal linear map of the unit interval and*

$$a_n=\begin{cases} b & \text{if } f^n t\leqq 1-a \text{ for any sufficiently small } t>0 \\ -a & \text{otherwise}\end{cases}$$

$$D(z)=1-\sum_{n=0}^{\infty}aa_0a_1\cdots a_{n-1}z^{n+1}$$

(i) *Let f be degenerated. Then*

$$1/Z(z)=\begin{cases}(1-a^2z^2)\cdot D(z) & \text{if } a<1 \\ (1-a^2z^2)^2\cdot D(z) & \text{if } a=1\end{cases}$$

(ii) *Let f be nondegenerated and assume that there is a number p such that*

(30) $$a_{np}=a_{np+1}=\cdots=a_{np+p-2}=b \quad \text{and} \quad a_{np+p-1}=-a\,.$$

Then,

$$1/Z(z)=\begin{cases}(1-a^2b^{2p-2}z^{2p})\cdot D(z) & \text{if } ab^{p-1}<1 \\ (1-a^2b^{2p-2}z^{2p})^2\cdot D(z) & \text{if } ab^{p-1}=1\end{cases}$$

(iii) *If f is nondegenerated and* (30) *is not satisfied for any p, then* (1) *holds, i.e.*

$$1/Z(z)=D(z)\,.$$

THEOREM 5. *The Artin-Mazur's zeta function $Z^{\circ}(z)$ satisfies the statements of Theorem* 4 *if one takes $Z^{\circ}(z)$ in place of $Z(z)$, $D^{\circ}(z)$ in place of $D(z)$ and* $\operatorname{sgn} a_n$ *in place of a_n which appear in $D(z)$ and $Z(z)$.*

Remark. (i) The condition (30) is equivalent to the following (31) and (32)

$$a(1+b+\cdots+b^{p-2})<1 \tag{31}$$

$$ab^{p-1}\geqq 1 \tag{32}$$

It is also equivalent to the existance of a subinterval J_0 of J such that the restriction of the iterate f^p on J_0 is a degenerated unimodal linear map.

The case in (ii) where $ab^{p-1}=1$ and the case in (i) where $a=1$ are the pathological cases due to the linearity of f. (Example 3 in Section 5)

The case in (ii) where $ab^{p-1}>1$ shows window phenomenon, i.e., it shows formal chaos (and Li-Yorke chaos) but does not show observable chaos. (Example 5 in Section 5)

(ii) In the case (ii) except the pathological cases the function $1/Z(z)$ can be understood as the Fredholm determinant of the natural extension $\overline{\mathcal{L}}$ of the operator $\mathcal{L}$ on $\boldsymbol{A}$ to the closed linear hull of the indicator functions χ_n, $n\geqq 0$, and their point-wise limits as $n\to\infty$ in the space of functions with bounded variation. The limit functions are indicator functions of one point sets and so they do not belong to the space $\boldsymbol{L}^1$. They represent the point masses (=measures supported by one point set). In this sense, *the zero of* $1/Z(z)$ *are eigenvalues or generalized eigenvalues of the Perron-Frobenius operator* $\mathcal{L}$.

The proofs of Theorem 4 and 5 will be given after several lemmas.

For a while, let us assume that the orbit O of the point 0 is periodic under f. Put $O=\{x_0=0<x_1<\cdots<x_q=1-a<\cdots<x_{p-1}=1\}$ and $J_i=(x_{i-1}, x_i)$, $i=1,2,\cdots,p-1$. Define real numbers M_{ij} by the formula

$$\mathcal{L}\theta_i=\sum_{j=1}^{p-1} M_{ij}\theta_j \qquad \text{on} \quad O^c, \tag{33}$$

where θ_i is the indicator function of J_i. In other words, the matrix $M=(M_{ij})$ is the other form of the weighted structure matrix S of the Markov map f stated in the final remark in Section 3. It is obvious that the matrix M is a representation on the operator $\mathcal{L}$ on $\boldsymbol{A}$ and so it follows from (25) and Lemma 11 that

$$\begin{aligned}\det(I-zM)&=(1-a_0a_1\cdots a_{p-1}z^p)\cdot D(z)\\&=\exp\left[-\sum_{n=1}^{\infty}\frac{z^n}{n}\operatorname{tr}M^n\right].\end{aligned} \tag{34}$$

Now we are going to be concerned with what contributes to $\operatorname{tr} M^n$.

LEMMA 12. *Let* $i_0, i_1, \cdots, i_n$ *be such that* $i_n=i_0$ *and* $M_{i_0i_1}M_{i_1i_2}\cdots M_{i_{n-1}i_n}\neq 0$. *Then one of the following three holds*:

(a) *There exists a unique fixed point of* f^n *in the open subinterval* $J_{i_0\cdots i_{n-1}}=J_{i_0}\cap f^{-1}J_{i_1}\cap\cdots\cap f^{-(n-1)}J_{i_{n-1}}$.

(b) *There are no fixed points of* f^n *in* $J_{i_0\cdots i_{n-1}}$ *but one and only one of the end point of* $J_{i_0\cdots i_{n-1}}$ *is a fixed point of* f^n *which belongs to* O. *In particular,* p *divides* n.

(c) *Both end points of* $J_{i_0\cdots i_{n-1}}$ *are fixed under* f^n *and belong to* O.

Proof. First of all, claim that f^n has a fixed point in the closure of the open interval J. If $M_{ij}\neq 0$, then, $f^{-1}J_j\cap J_i\neq\emptyset$. Thus, $fJ_i\supset J_j$ since the end points of J_n's belong to O. Consequently, f^n maps $J_{i_0\cdots i_{n-1}}$ onto itself, and so f^n has a fixed point in the closure of $J_{i_0\cdots i_{n-1}}$ in virtue of the intermediate value theorem.

Now suppose that (a) does not hold. Since f^n is linear on $J_{i_0\cdots i_{n-1}}$ thus, either f^n is the identity map there or one and only one of the end points is fixed under f^n. In either case, the fixed end point belongs to O because the end points fall into the periodic orbit O under the iteration of f. As a consequence, p divides n.

LEMMA 13. *Assume that (c) in Lemma 12 holds for* $i_0,\cdots,i_{n-1}$. *Then* p *is even and there exists exactly one fixed point of* $f^{p/2}$ *in the interval* $J_{i_0\cdots i_{n-1}}$ *unless* f *is the identity map of* J.

Proof. First claim that $J_{i_0\cdots i_{n-1}}=J_{i_0}$. In fact, if it were not true, then, there would be a point in that interval which fall into O under some iterate of f, i.e., which is a point of O. (Recall that f^n is the identity map there.)

It then follows that f^p is the identity map on J_{i_0}. Next claim that the union of $f^kJ_{i_0}$, $k\geqq 0$, is a proper subset of J. In fact, if the union were J, then the map f^p would be the identity on J.

Since the union of $f^kJ_{i_0}$ has O as its boundary, thus, p must be even. On the other hand, $f^{p/2}$ cannot be the identity on J_{i_0}. In fact, if it were not the case, then O could not be a single orbit. Consequently, $f^{p/2}$ is the order reversing linear homeomorphism of J_{i_0} and so it has one fixed point in J_{i_0}.

Remark. Under the assumptions of Lemma 13, the set $Per(p,f)$ consists of half open intervals.

Now one can correspond to each finite sequence $(i_0,\cdots,i_{n-1})$ such that $M_{i_0i_1}\cdots M_{i_{n-1}i_0}\neq 0$

the unique fixed point in $J_{i_0\cdots i_{n-1}}$ in case (a) of Lemma 12,
the fixed end point in case (b) of Lemma 12 or
the fixed point of $f^{p/2}$ in Lemma 13 in case of (c) of Lemma 12.

Let us denote the fixed point by $x(i_0,\cdots,i_{n-1})$.

LEMMA 14. *The map* $(i_0,\cdots,i_{n-1})\longmapsto x(i_0,\cdots,i_{n-1})\in Fix(f^n)$ *is*
(i) *injective and*

(ii) *surjective if the case (c) does not happen.*

(iii) *If the case (c) happens, then its image is* $Fix(f^n)\cap O^c$.

Proof. It suffices to prove (i) in case (b) since the injectivity is evident in other cases. Assume that p divides n. Then f^n takes an extremal value at each point x of O since the critical point $1-a$ of f belongs to the finite orbit $f^i x$, $0\leqq i<n$. Consequently, the other neighbouring subinterval of form $J_{i'_0\cdots i'_{n-1}}$ having $x(i_0,\cdots,i_{n-1})$ as an end point in common corresponds to no $(i'_0,\cdots,i'_{n-1})$ such that $M_{i_0i_2}\cdots M_{i_{n-1}i_0}\neq 0$. Hence, (i). By the way, (iii) is automatically proved by the same argument. In fact, the subintervals neighbouring to a subinterval of case (c) cannot be corresponded to $(i_0,\cdots,i_{n-1})$ that contribute to $\operatorname{tr} M^n$, and so the set O is not contained in the image.

Finally, let $x\in Fix(f^n)$ and assume that the case (c) does not happen, or that $x\in O^c$. Then it is evident that x is the unique fixed point of f^n in some subinterval $J_{i_0\cdots i_{n-1}}$ i.e., $x=x(i_0,\cdots,i_{n-1})$ for some $(i_0,\cdots,i_{n-1})$ such that $M_{i_0i_1}\cdots M_{i_{n-1}i_0}\neq 0$. Hence, one gets (ii) and (iii).

COROLLARY. *If* $x(i_0,\cdots,i_{n-1})$ *is defined, then,*

$$M_{i_0i_1}M_{i_1i_2}\cdots M_{i_{n-1}i_0}=1/|(f^n)'(x(i_0,\cdots,i_{n-1}))|\,. \tag{35}$$

Proof. Obvious from the definition of $x(i_0,\cdots,i_{n-1})$ and the fact: $f'=|M_{ij}|$ on J_i.

LEMMA 15. (i) *If the case (c) does not happen, then,*

$$1/Z(z)=\det(I-zM)\,. \tag{36}$$

(ii) *If (c) takes place, then,* $a_0a_1\cdots a_{p-1}=1$ *and*

$$1/Z(z)=(1-z^p)\cdot\det(I-zM)\,. \tag{37}$$

Proof. Immediate from Lemma 11 and Lemma 14 together with Corollary.

Finally we need the following consequence of the results in [16], which enables us to approximate general cases by the cases discussed above.

LEMMA 16. *For a given non-degenerated unimodal linear map f there is a sequence of unimodal linear maps f_n with the following three properties:*

(a) *The orbit of* 0 *under* f_n *is periodic. Moreover, the period* p_n *tends to infinity if the orbit of* 0 *under* f *is not periodic.*

(b) *The set* $Per(k,f_{n+1})$ *and* $Per(k,f)$ *are naturally embedded in* $Per(k,f_n)$ *for each k and n. Under this identification,*

$$Per(k,f)=\bigcap_n Per(k,f_n)\,. \tag{38}$$

(c) *The sign of* $a_i(f_n)$, sgn $a_i(f_n)$, *converges to* sgn $a_i(f)$ *for each* i.

Proof. It is known in [16] (or [8], [13]) that the symbolic structure of unimodal continuous maps is completely determined by the sequence sgn a_i, $i\geqq 0$. (Recall that sgn a_i indicates whether $f^i 0$ is larger or smaller than the critical point of f.) Consequently, the piecewise linearity (or the Schwarzian condition) is sufficient to bring back the results in symbolic dynamics to the results for maps themselves.

Proof of Theorem 4.

First of all, we note that the results from Section 2 on remains valid under the change of the coefficients a and b in the operator $\mathcal{L}$: If one takes

$$\mathcal{L}'\varphi(x)=b'\varphi(f_0^{-1}x)\cdot 1(x\geqq(a+b-1)/b)+a'\cdot\varphi(f_1^{-1}x)\,, \tag{39}$$

then all Lemmas and Theorems remain valid if one define a_n's from a' and b' etc. Let us write new $D(z)$, $Z(z)$, etc. by $D(z\,;f,a',b')$ $Z(z\,;f,a',b')$ etc.

The first assertion (i) of Theorem 4 is trivial. Let us assume that the orbit O is periodic. Then the assertions (ii) and (iii) follow from Lemma 15 since the pathological case (c) in Lemma 12 is exactly the case in the statement (ii) where $ab^{p-1}=1$.

Now it follows from Lemma 16 that, if O is not periodic, then, $Z(z\,;f_n,a,b)$ and $D(z\,;f_n,a,b)$ converge to $Z(z)=Z(z\,;f,a,b)$ and $D(z)=D(z\,;f,a,b)$, respectively, as n goes to ∞. Since the period p_n of the orbit of 0 under f_n tends to infinity, thus the factors of the form $1-const.\,z^{p_n}$ in Theorem 3 (c) and (37) vanish as $n\to\infty$. Consequently, one gets the formula: $1/Z(z)=D(z)$ when the orbit O of 0 under f is not periodic.

Proof of Theorem 5.

In virtue of the argument at the beginning of the proof of Theorem 4, the assertions follow by setting $a=b=1$ in Theorem 5.

§ 5. Formal chaos and observable chaos: Examples

Let us exhibit how the Fredholm determinants or the zeta functions work in the study of one dimensional maps. First of all we shall introduce several notions on dynamical properties of maps.

DEFINITION 4. A piecewise continuous map of an interval J is called to show (strong or weak) formal chaos if there is a compact subset C of J such that the subsystem (C,f) is conjugate with a (strongly or weakly) mixing Markov shift of finite type [17], [10], [20].

Remark. The following statements are mutually equivalent:

(a) f shows formal chaos.

(b) The topological entropy of f is positive.

(c) There are two mutually dijoint subintervals $J_1, J_2 \neq \emptyset$ such that, for some p, $f^p J_i \supset J_1 \cup J_2$ $(i=1,2)$.

The notion of formal chaos and Li-Yorke chaos [6] does not fit for the ergodic theory nor the numerical analysis. It is the reason why the following notion [11] is introduced.

DEFINITION 5. A piecewise continuous map f of an interval J is called to show strong or weak observable chaos if the following two conditions are satisfied:

(a) There exists an invariant Borel probability measure μ for f such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x}=\mu \quad (\delta_x \text{ being the unit point mass}) \tag{40}$$

holds almost everywhere on some open subset of J with respect to the Lebesgue measure dx on J.

(b) The endomorphism (J, μ, f) is strongly or weakly mixing, respectively.

Those invariant measures which satisfy the condition (a) are called asymptotic measures for f. From the point of view of numerical experiment, asymptotic measures are the invariant measures that are "observable". (Nevertheless, the existance of non-observable formal chaos can be observed as a transient phenomenon of orbits under the process of numerical experiments.)

The definitions above are naturally extended to the maps on Riemannian manifolds, where the Riemannian volume plays the role of Lebesgue measure. But it may be better to use the positivity of Kolmogorov-Sinai entropy in place of weak mixing property in (b) of Definition 5.

There may happen that there exists an asymptotic measure with entropy 0 as is pointed out by May [7]. It is called window phenomenon.

LEMMA 17. *If f is a piecewise C^1-map of J, then the condition (a) is equivalent to the following*:

(a′) *There is an open subset J' of J such that*

$$w\text{-}\lim_{n\to\infty}\mathcal{L}^n\varphi=\left(\int_J\varphi(x)dx\right)\mu \tag{41}$$

for any $\varphi\in \boldsymbol{L}^1(J, dx)$ with support in J'.

Proof. Obvious.

It is not difficult to see the following: If $z=1$ is the unique zero of $D(z)$ in a neighbourhood of the unit disc and is a simple zero, then the unimodal linear map f has a unique absolutely continuous invariant measure μ on $(J, \bar{\mathcal{A}})$ with respect to which f is weak Bernoulli, where $\bar{\mathcal{A}}$ is the smallest σ-algebra which contains $\mathcal{A}$ and is invariant under the action of f^{-1}. As a consequence, if the zeros of $D(z)$ in a neighbourhood of the unit disc are the roots of $z^p=1$ and are simple, then f has a unique absolutely continuous invariant measure on $(J, \bar{\mathcal{A}})$ with respect to which f is weakly mixing and f^p is weak Bernoulli.

We do not give a proof of this result since it is a rephrase of the known result [2] in our formulation. It also follows from the fact that any one dimensional map can be realized by Markov shifts (of infinite order) [14]. In our formulation the proof is based on the convergence of $\mathcal{L}^n$ on the space A which implies the uniform convergence of $\mathcal{L}^n\varphi$ in the uniform norm on a suitable dense subset. (The idea was used in [5] for beta transformations.)

Now we shall give the examples which are typical cases of unimodal linear maps. The results obtained by usual method in [4] are stated in A-1), 2) and the results in our formulation [7] are stated in B-0), 1), 2). The latter give another proof of the former.

Example 1. *Tent map*: $f0=0$, i.e., $a+b=1$.

This is the typical case of observable chaos.

A-1) The asymptotic measure μ is unique and is absolutely continuous with respect to the Lebesgue measure.

2) The support of μ is J and (J, μ, f) is mixing (a fortiori, Bernoulli).

B-0) $a_n=b$ for any n. $U=U^\circ=\{z\,;\,|z|<1/b\}$

$D(z)=1-az-abz^2/(1-bz)=1-z$

$1/Z(z)=(1-bz)(1-z)$

1) $P(f)=0$ and $z=1$ is a simple zero of $D(z)$.

2) There are no other zero in the neighbourhood U of 0.

Example 2. (Fig. 2) *Ascending cycle*: $f^p0=0<f0<f^20<\cdots<f^{p-1}0=1$ $(p\geqq 3)$.

It appears when $a(1+b+\cdots+b^{p-2})=1$ and $ab^{p-1}<1$, and it shows observable chaos.

A-1) The asymptotic measure μ is unique and is absolutely continuous.

2) The support of μ is J and (J, μ, f) is mixing.

B-0) $a_{np}=a_{np+1}=\cdots=a_{np+p-2}=b,\ a_{np+p-1}=-a\ (n\geqq 0)$. $U=U^\circ=\{z\,;\,|z|<1/ab^{p-1}\}$.

$D(z)=(1-ab^{p-1}z^p)^{-1}Z(z)^{-1}$

$Z(z)=1-az-abz^2-\cdots-ab^{p-2}z^{p-1}$

1) $P(f)=1$ and $z=1$ is a simple zero of $D(z)$.

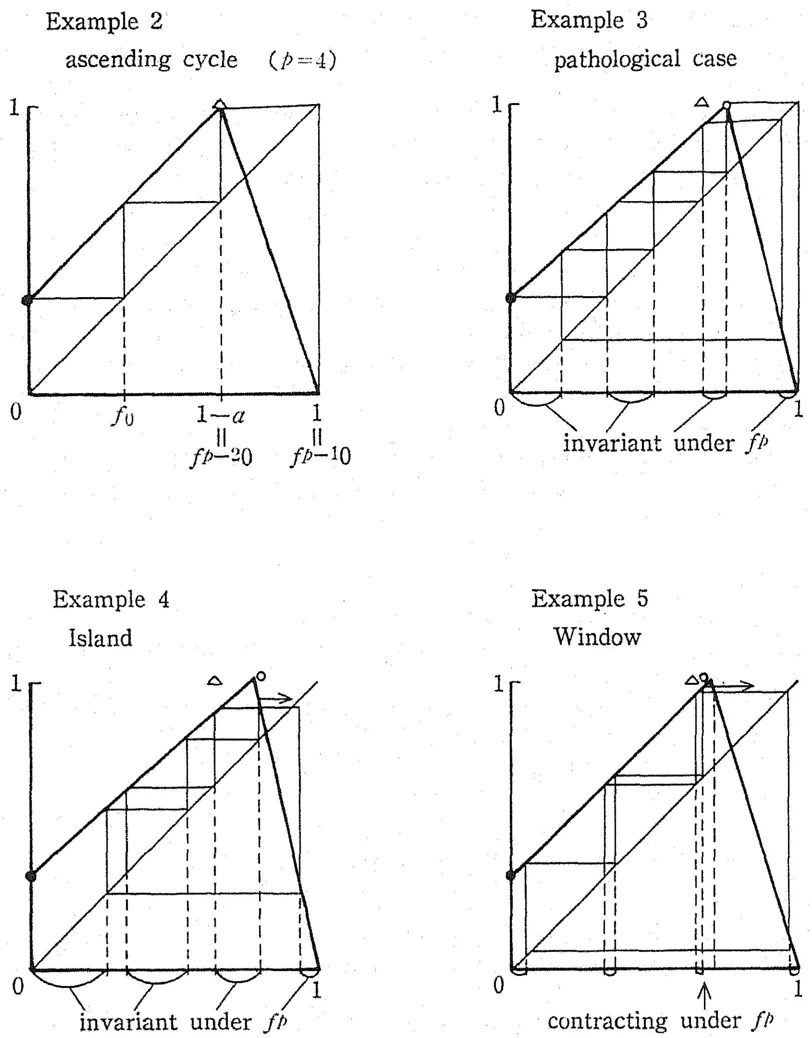
2) There are no other zeros of $D(z)$ in the neighbourhood U of 0.

Example 3. (Fig. 2) *Pathological case* (c): $p=2q=4, 6, 10, 12, \cdots$

$f^{2q}0=0<f^q0<f0<f^{q+1}0<f^20<\cdots<f^{q-2}0<f^{2q-2}0=1-a<f^{q-1}0<f^{2q-1}0=1$

This case appears when $ab^{q-1}=1$ and $a(1+\cdots+b^{q-2})<1<a(1+\cdots+b^{q-1})$.

A-1) The asymptotic measure does not exists.



In these figures, $(a+b-1)/b$ is fixed and a changes.

Fig. 2. Bifurcation: ascending cycle→window→pathological case→island.

2) Any absolutely continuous measure is supported by the union of q disjoint subintervals $[f^i 0, f^{q+i}0]$, $i=0,1,\cdots,q-1$.

B-0) $a_{nq}=a_{nq+1}=\cdots a_{nq+q-2}=b$ and $a_{nq+q-1}=-a$. $U=U^\circ=\{z\,;\,|z|<1\}$.

$D(z)=(1+z^q)^{-1}(1-az-abz^2-\cdots ab^{q-2}z^{q-1})$

$1/Z(z)=(1-z^{2q})(1-z^q)(1-az-abz^2-\cdots-ab^{q-2}z^{q-1})$

1) $z=1$ is not a zero of $1/Z(z)$.

2) The minimal nonnegative zero of $1/Z(z)$ is greater than 1.

Example 4. *Island*: $p=3, 4, 5\cdots$
$0<f^p0<f0<f^{p+1}0<f^2 0<\cdots<f^{p-2}0<1-a<f^{2p-2}0<f^{p-1}0<f^{2p-1}0<1$
This is the weak observable chaos and appears when $ab^{p-1}<1$, $a(a+b)b^{q-2}\geqq 1$ and $a(1+b+\cdots+b^{q-2})<1<a(1+b+\cdots+b^{q-1})$.
A-1) The asymptotic measure μ is unique and is absolutely continuous.
2) μ is supported by p disjoint subintervals $J'=[f^{q-2}0, f_1^{-1}f^{2q-2}0]$ and $[f^i 0, f^{q+i}0]$, $i=0, \cdots, p-3, p-1$. The support of consists of p or $2p$ disjoint subintervals according as the map f' defined below has or has not odd period 1. The system (J, μ, f) is weakly mixing but not mixing.
B-0) $a_{np}=a_{np+1}=\cdots=a_{np+p-3}=b$, $a_{np+p-1}=-a$, $a_{p-2}=b$, $a_{2p-2}=-a$.
The sequence a_{np+-2} corresponds to the map f' which is obtained by rescaling the restriction of f^p to J'.

$$D(z)=(1-az-abz-\cdots-ab^{p-2}z^{p-1})D(z^p; f')$$
$$1/Z(z)=(1-az-abz^2-\cdots-ab^{p-2}z^{p-1})/Z(z^p; f')$$

1) $P(f)=0$ and $z=1$ is a simple zero of $D(z)$.
2) The zeros of $D(z)$ in a neighbourhood of the unit disc are the roots of $z^p=1$.

Remark. The factor $1-az-\cdots-ab^{p-2}z^{p-1}$ is the inverse of zeta function of f restricted on the set

$$J\cap(\bigcup_{n=0}^{\infty} f^{-n}J')^c.$$

It indicates that the restiction is "conjugate" with the map with ascending p-cycle. The conjugacy is, of course, not topological but Borel.

Now let us move the parameters to the direction opposite to Example 4 from the pathological Example 3.

Example 5. (Fig. 2) *Window*: $p=3, 4, 5, \cdots$
$0<f^p0<f0<f^{p+1}0<f^2 0<\cdots<f^{p-2}0<f^{2p-2}0<1-a<f^{p-1}0<f^{2p-1}0<1$
This case is formal chaos but not observable chaos and appears when $ab^{p-1}>1$ and $a(1+b+\cdots+b^{p-2})<1<a(1+b+\cdots+b^{p-1})$.
A-1) The asymptotic measure exists, is unique but is not absolutely continuous.
2) The asymptotic measure is supported by a periodic orbit with period p. Consequently, it is ergodic but not weakly mixing.
B-0) $a_{np+i}=b$ $(i=0, 1, \cdots, p-2)$, $a_{np+p-1}=-a$. $U=U^\circ=\{z; |z|<1/ab^{p-1}\}$

$$D(z)=(1+ab^{p-1}z^p)^{-1}(1-az-abz^2-\cdots-ab^{p-2}z^{p-1})$$
$$Z(z)=(1-ab^{p-1}z^p)(1-az-abz^2-\cdots-ab^{p-2}z^{p-1}).$$

1) $P(f)$ is positive and $z=\exp P(f)$ is a simple zero of $1/Z(z)$.
2) The zeros of $1/Z(z)$ in a neighbourhood of $z=\exp P(f)$ are the root of $z^p=\exp P(f)$.

Example 6. *Absence of odd period*>1: $f0=$ the unique fixed point $1/(1+a)$ of f, i.e., $a(a+b)=1$.

A-1) The asymptotic measure exists uniquely and is absolutely continuous.

2) It is supported by two intervals $[0, 1/(1+a))$ and $(1/(1+a), 1]$. and is weakly mixing.

B-0) $a_0=b,\ a_n=-a\ (n\geqq 1)$. $U=U^\circ=\{z\,;\,|z|<1/a\}$

$D(z)=1-az-abz^2+a^2bz^3/(1+az)=(1-z^2)/(1+az)$

$Z(z)=1/D(z)$

1) $P(f)=0$ and $z=1$ is a simple zero of $D(z)$

2) the zeros in U are 1 and -1.

Example 7. *Oscillating cycle*: $p=3, 5, \cdots$(odd)

$0<1-a<f^{p-2}0<f^{p-4}0<\cdots<f^3 0<f0<f^2 0<\cdots<f^{p-1}0=1$

This is observable chaos and appears when $1-a=b(a-a^2+a^3-\cdots+a^p)$.

A-1) The asymptotic measure exists uniquely and is absolutely continuous.

2) Its support is J and is mixing.

B-0) $a_{np}=b,\ a_{np+i}=-a\ (i=1, \cdots, p-1)$.

$D(z)=(1-a^{p-1}bz^p)^{-1}(1-az-abz^2+a^2bz^3-a^3bz^4+\cdots-a^{p-2}bz^{p-1})$

$1/Z(z)=1-az-abz^2+a^2bz^3-a^3bz^4+\cdots-a^{p-2}bz^{p-1}$.

1) $P(f)=0$ and $z=1$ is a simple zero of $D(z)$.

2) There are no other zeros in a neighbourhood of the unit disc.

§ 6. Further examples: Critical cases

Although we did not give the proof, our theory works for general, piecewise linear maps of intervals. For example, the Fredholm determinant of a β-transformation is

$$D(z)=1-\sum_{n=0}^{\infty} a_n \beta^{-n-1} z^{n+1} \tag{42}$$

where the sequence a_n is defined in the same way as before from the orbit of $0+0$. [5], [15].

There are interesting maps of intervals, for which certain similarity laws hold. The case where a continuous map f has all powers of 2 as its period and no other integers appear as its period is widely known. The similarity law is as follows: f has two mutually disjoint subintervals such that $fJ_1=J_2$ and $fJ_2=J_1$, and f^2 on each J_i is cojugate with f itself. Let us generalize this ([17]).

Let $N_n\geqq 2\ (n\geqq 0)$ and $F(N_n, n\geqq 0)$ be the set of continuous maps f on intervals J defined by the following conditions:

(a) There are N_0 mutually disjoint nonempty subintervals J_i, $i=1, \cdots, N_0$, such that $fJ_i=J_{g(i)}$ for some cyclic permutation g of $1, \cdots, N_0$.

(b) Almost all points of J fall into the union of J_i's, i.e., the union of the inverse images of the union of J_i's under f^n's has full Lebesgue measure in J.

(c) The restriction of $f^N 0$ on each J_i belongs to the set $F(N_{n+1},\ n\geqq 0)$.

DEFINITION 6. A continuous map f on an interval J is called (N_n)-critical if it belongs to the set $F(N_n, n\geqq 0)$. When $N_n=N$ for all n, let us call f N^∞-critical.

In the following let us construct a unimodal piecewise linear continuous map of the unit interval $J=[0.1]$ which is N^∞-critical and compute $D(z)$ and $Z(z)$. It is also obtained that the value of $P(f)$ can take any nonpositive real number including $-\infty$.

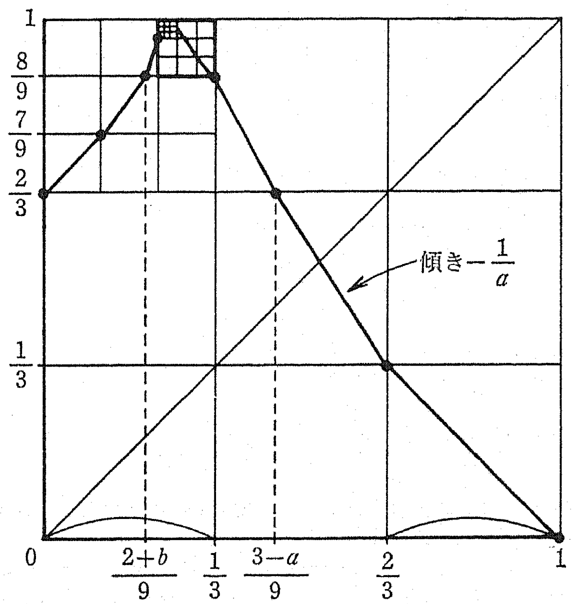


Fig. 3. 2^∞-critical case.

Example 8. *2^∞-critical case.* (Fig. 3).

Let k_n, $n\geqq 0$, be real numbers in $(0,1)$. Then there is a piecewise linear unimodular continuous map f whose zeta function is

$$Z(z)=\prod_{m=0}^{\infty}\left(1-\frac{z^{2m}}{k_m}\right)^{-1}. \tag{43}$$

It then follows that $P(f)=\lim\sup(\log k_m)/2^m\in[-\infty, 0]$. Furthermore, it can be shown for any 2^∞-critical f,

$$D(z)=\prod_{m=0}^{\infty}(1-z^{2m}) \tag{44}$$

The construction of f is as follows. For a given $a, b\in(0,1)$ let $g=g_{a,b}$ be the piecewise linear continuous map of $[0, 2/9]\cup[1/3, 1]$ to $[0, 1]$ whose end points of linear parts are given by

$$g(0)=\frac{2}{3},\ g\left(\frac{1}{9}\right)=\frac{2}{3}+\frac{1}{9},\ g\left(\frac{1+b}{9}\right)=\frac{2}{3}+\frac{2}{9},\ g\left(\frac{2}{9}\right)=\frac{2}{3}+\frac{2}{9}+\frac{2}{27},$$

$$g\left(\frac{1}{3}\right)=\frac{2}{3}+\frac{2}{9},\ g\left(\frac{2-a}{3}\right)=\frac{2}{3},\ g\left(\frac{2}{3}\right)=\frac{1}{3},\ g(1)=0.$$

Next define $g_{a_1,b_1,\cdots,a_n,b_n}$:

$$\left[0, \frac{2}{9}\left(1+\frac{1}{9}+\cdots+\left(\frac{1}{9}\right)^{n-1}\right)\right]\cup\left[\frac{1}{3}-\frac{2}{27}\left(1+\frac{1}{9}+\cdots+\left(\frac{1}{9}\right)^{n-1}\right), 1\right]\to[0, 1]$$

inductively on n by

$$g_{a_1,\cdots,b_n}=\begin{cases} g_{a_1,b_1}(x) & \text{if } x\in\left[0, \frac{2}{9}\right]\cup\left[\frac{1}{3}, 1\right] \\ g_{a_2,\cdots,b_n}(9x-2) & \text{otherwise.}\end{cases}$$

Finally, define f for given k_n by

$$f(x)=\lim g_{k_0,\cdots,k_{2n-1}}(x) \quad \text{if} \quad x\neq\frac{1}{4} \quad \text{and}$$

$$f\left(\frac{1}{4}\right)=1.$$

Then the map f leaves $[0, 1/3]\cup[2/3, 1]$ invariant and any point except the unique fixed point falls into this union under the iteration of f. Similarly a look on the level of size 1/9 shows that $Per(2, f)$ consists exactly of two points and any point except the fixed points of f^2 fall into the union of the intervals $[2i/9, (2i+1)/9]$, $i=0.1, 3, 4$. Consequently, the set $Per(2^m, f)$ consists of exactly 2^m points and, for $n\neq 2^m$, $Per(n, f)=\emptyset$. Furthermore, non-periodic points lie in the basin of the attractor C which is the classical Cantor set.

Now the formula (39) follows from $(f^{2^m})'=-k_m$ on $Per(2^m, f)$.

Next let us compute $D(z)$, which is defined by the orbit of $0+0$. A slight look on the structure of the graph of f shows that the sequence a_n, $n\geqq 0$, is obtained by the following generating rule:

0) $a_0=b$, $a_1=-a$.

1) $(a_0, \cdots, a_{2^{m+1}-1})=(a_0, \cdots, a_{2^m-1}, a_0, \cdots, a_{2^m-3}, a^*_{2^m-2}, a_{2^m-1})$

where $b^*=-a$ and $(-a)^*=b$.

It is not so difficult to obtain from this that

$$-c_n=-\sum_{i=0}^{n-1} a_i=(-1)^{\Sigma e_k(n+1)} \quad (n\geqq 0).$$

Here $e_k(m)\in\{0,1\}$ is the k-th coefficient of binary expansion of $m=\sum e_k(m)\cdot 2^k$. Consequently,

$$D(z)=1-\sum_{n=0}^{\infty} c_n z^{n+1}=\sum(-1)^{\Sigma e_k} z^{\Sigma e_k\cdot 2^k} \quad (e_k=0,1)$$

$$=(1-z)(1-z^2)(1-z^4)\cdots.$$

as is expected from the form of $1/Z(z)$. The function $D(z)$ may be interpreted as Fredholm determinant of the restriction of f on the Cantor set C.

In summary one obtains the following [17].

a-1) The asymptotic measure exists uniquely but it is not absolutely continuous.

2) It is the uniform measure on the Cantor set C, and is ergodic but not weakly mixing.

b-1) $z=1$ is a zero of $D(z)$ with infinite multiplicity.

2) $P(f)\in[-\infty,0]$

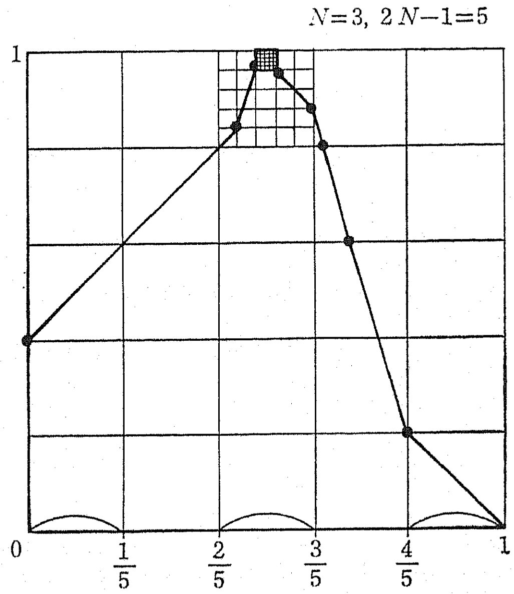


Fig. 4. N^{∞}-critical case.

Example 9. *N^{∞}-critical case* (Fig. 4)

The example can be constructed in a similar way as in Example 8. But one must divide J into $2N-1$ intervals $J_i=[(i-1)/(2N-1), i/(2N-1)]$ and the similarity takes place in the subinterval J_{2N-3}. Note that this is consistent since $2N-1=3$ and $2N-3=1$ for $N=2$. In this case the generating rule for the sequence a_n is as follows:

0) $a_0=\cdots=a_{N-2}=b,\ a_{N-1}=-a$.

1) $(a_0, \cdots, a_{N^m-1})=u_m$ is obtained by

$$u_{m+1}=u_m u_m^* \cdots u_m^* u_m^*,$$

where $u^*=(d_1, \cdots, d_{n-2}, d_{n-1}^*, d_n)$ if $u=(d_1, \cdots, d_n)$ and d^* is as before

A similar argument as above shows the following:

a-1) The asymptotic measure exists uniquely but it is not absolutely continuous.

2) It is the uniform measure on the N-ary Cantor set and it is ergodic but not weakly mixing under f.

b-0) $D(z)=\prod_{m=0}^{\infty} D_N(z^{N^m}; 1, 1)$ and $Z(z)=\prod_{m=0}^{\infty} D_N(z^{N^m}; a_N, b_N)$ where $D_N(z; a, b)$ is the inverse zeta function with the coefficients a and b:

$$D_N(z; a, b)=1-az-abz^2-\cdots ab^{N-2}z^{N-1}.$$

1) $z=1$ is a zero of $D(z)$ with infinite multiplicity.

2) $P(f)\in[-\infty, 0]$.

Reference

[1] M. Artin and B. Mazur, On periodic points, *Ann. Math.* **81** (1965), 82–99.

[2] R. Bowen, Invariant measures for Markov maps of the interval, *Commun. Math. Phys.* **69** (1979), 1–17.

[3] M. J. Feigenbaum, Qualitative universality for a class of nonlinear transformations *J. Stat. Phys.* **19** (1978) 25–52; **21** (1979) 669–706.

[4] S. Ito, H. Nakada and S. Tanaka, On unimodal linear transformations I, II, *Tokyo J. Math.* **2** (1979), 221–259.

[5] S. Ito and Y. Takahashi, Markov subshifts and realization of β-transformations, *J. Math. Soc. Japan* **26** (1974), 33–55.

[6] T. Li and J. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.

[7] R. M. May, Biological populations obeying difference equations, stable cycles, and chaos, J. Theor. Biol. **52** (1975) 511.

[8] J. Milnor and W. Thurston, On iterate maps of intervals, preprint.

[9] S. Mizohata, *Introduction to Theory of Integral Equations*, Asakura Shoten, 1968.

[10] Y. Oono and M. Osikawa, Chaos in C^0-endomorphisms of intervals, *Publ. RIMS Kyoto Univ.* **17** (1981), 165–177.

[11] Y. Oono and Y. Takahashi, Chaos, external noise and Fredholm theory, *Prog. Theor. Physics* **63** (1980), 1804–1807.

[12] D. Ruelle, Encycopedia of Mathematics vol. 5 *Thermodynamical formalism*, Addison-Wesley, 1979.

[13] A. N. Šarkovskii, Coexistence of cycles of a continuous map of a real line to itself, *Ukrain. Mat. Ž.* **16** (1964), 61–71 (in Russian).

[14] S. Shinada, Decomposition theorem of words for dynamics on closed intervals and its application, dissertation for master degree, Tokyo Inst. Technology, 1980 (unpublished).

[15] Y. Takahashi, β-transformations and symbolic dynamics, Proc. Third Japan-USSR Symp. on Prob. Theory=Springer Lecture Notes in Math. 330 (1973), 455–464.
[16] ———, A formula for topological entropy of one-dimensional dynamics, *Sci. Papers Coll. Gen. Educ. Univ. Tokyo* **30** (1980), 11–22.
[17] ———, *Chaos and periodic points of interval dynamics*, Seminar Report Math. Depart. Tokyo Metropolitan Univ. 1980. *Chaos and equilibrium measure* in Lecture Note of Winter School in Ergodic Theory held at Lake Yamanaka in 1980=Seminar in Probability vol 50 (1981), 89–158 (both in Japanese).
[18] ———, Shifts with orbit basis, preprint.
[19] ———, Varational principle for one-dimensional maps, preprint.
[20] H. Totoki, Pseudo-Markov transformation, *Ann. Polonici Math.* **41** (1982), to appear.
[21] J. Hadamard, Essai sur l'étude des fonctions données par leur développement de Taylor, Oeuvres Tom I, Ed. CNRS, Paris VII, 1968, 7–92.