On the Harmonic Functions of Critical or Supercritical Galton-Watson Processes with Continuous Time Parameters

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§ 1. Introduction

In [3], Dubuc determined all the harmonic functions of supercritical Galton-Watson processes with discrete time parameters. For critical ones, see [5]. Here the harmonic function for Galton-Watson process is the function h from $\{1, 2, 3, \cdots\}$ to $\boldsymbol{R}$ such that

$$\sum_{n=1}^{\infty} P_t(m, n)h(n) = h(m), \quad n = 1, 2, 3, \cdots,$$

where $P_t(m, n)$ are the transition probabilities of the process.

In this note we determine the Green functions for the processes and prove the uniqueness of the harmonic functions of the supercritical processes. The uniqueness of the invariant measures of all types of processes is already known, but the proof in this note seems to be very easy.

§ 2. Results

The generating function of a continuous time Galton-Watson process $\{Z(t); 0 \leqq t < \infty\}$ is defined by

$$F(t; s) \equiv E\{s^{Z(t)}/Z(0) = 1\} = \sum_{n=0}^{\infty} P_n(t)s^n,$$

where $P_n(t) \equiv P\{Z(t) = n/Z(0) = 1\}$.

$F(t; s)$ satisfies

$$\begin{cases} \dfrac{\partial F(t; s)}{\partial t} = f(F(t; s)), \\ F(0; s) = s; \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{\partial F(t\,;s)}{\partial t} = f(s)\dfrac{\partial F(t;s)}{\partial s}\,, \\ \\ F(0\,;s) = s\,. \end{cases} \tag{2}$$

Here, $f(s)$ is the infinitesimal generating function of the process:

$$f(s)=\sum_{n=0}^{\infty} p_n s^n \text{ with } p_1<0\,;p_n\geqq 0,\ n\neq 1\,;\text{ and } \sum_{n=0}^{\infty} p_n=0\,.$$

(In the sequel, we put $p_1=-1$ without any loss of generality; and we also assume that $p_0+p_1<0$, the case $p_0+p_1=0$ being trivial. Finally, we assume that $a\equiv f'(1)$ be finite.)

Obviously $f(1)=0$, and it is well-known that if $a\leqq 0$, i.e., the process is critical or subcritical, then $f(s)$ has no zeroes in the interval $[0,1)$, and that if $a>0$, i.e., if the process is supercritical, then $f(s)$ has unique zero q in $[0,1)$. This number q is the extinction probability of the corresponding process, i.e.,

$$q=\lim_{t\to\infty} P\{Z(t)=0/Z(0)=1\}\,.$$

And in the case of critical or subcritical processes, this is equal to 1. It is also known that $F(t;s)\to q$ as $t\to\infty$ for each s in the open unit disk in $\boldsymbol{C}$. (Cf. [1] or [2].)

LEMMA 1. *The function* $f(s)$ *has no zeroes in* $\{s\in \boldsymbol{C}|\ |s|\leqq 1\}$ *other than* 1 *and* q.

Proof)

Remark: If $f(r)=0$, $0<r\leqq 1$, then, obviously, for each $s\neq r$ with $|s|=r$, we have $f(s)\neq 0$, and there are no zeroes on $\{|s|=1\}$ other than 1.

First, we show that the absolute values of the zeroes of $f(s)$ are smaller than or equal to q. In fact, if $f(s)=0$, then

$$\sum_{\substack{n=0\\n\neq 1}}^{\infty} p_n|s|^n \geqq \Big|\sum_{\substack{n=0\\n\neq 1}}^{\infty} p_n s^n\Big| = |s|,\ (\text{We put } p_1=-1.)$$

i.e., $f(|s|)=\sum_{\substack{n=0\\n\neq 1}}^{\infty} p_n|s|^n-|s|\geqq 0.$

From Lemma 2-1-2 in [2], this implies $|s|\leqq q$.

Now we prove the lemma separately.

i) The case of supercritical processes.

Since $q<1$, we can take $r\in(q,1)$. And since $f(s)$ and $g(s)\equiv -s$ have no zeroes on the circle $S\equiv\{|s|=r\}$ and $r>q$, by Lemma 2-1-2 in [2], we have on S:

$$|f(s)-g(s)|=\Big|\sum_{\substack{n=0\\n\neq 1}}^{\infty} p_n s^n\Big|\leqq \sum_{\substack{n=0\\n\neq 1}}^{\infty} p_n|s|^n<|s|=|g(s)|\,.$$

Then, according to Rouché's thorem, $f(s)$ has no zeroes in $\{|s|<r\}$ other than the extinction probability q and the multiplicity of q is equal to 1. Thus the zeroes of $f(s)$ in $\{|s|\leqq 1\}$ are the simple zeroes 1 and q in the whole.

ii) The case of critical or subcritical processes.

First, we note that p_0 is strictly positive in this case. We define

$$w(s)\equiv\frac{f(s)}{1-s}=(\sum_{n=0}^{\infty}s^n)(\sum_{k=0}^{\infty}p_ks^k)=\sum_{n=0}^{\infty}(\sum_{k=0}^{n}p_k)s^n\equiv\sum_{n=0}^{\infty}q_ns^n .$$

The coefficients $\{q_n\}$ satisfy the following:

$$q_0=p_0>0,\ q_1=p_0+p_1<0,\ q_n\uparrow 0 \text{ as } n\uparrow\infty\ (n>0),\ \sum_{n=1}^{\infty}q_n=-f'(1)\geqq 0 .$$

The function $w(s)$ has no zeroes in $\{|s|<1\}$, and hence $f(s)$ has no zeroes in $\{|s|<1\}$, either. In fact,

$$|w(s)|=|\sum_{n=0}^{\infty}q_ns^n|\geqq q_0+\sum_{n=1}^{\infty}q_n|s|^n>q_0+\sum_{n=1}^{\infty}q_n\geqq 0 ,$$

for each s in $\{|s|<1\}$.

Now we remark that the Green functions of the processes are all finite.

LEMMA 2. *We define the Green function by*

$$G(m,n)\equiv\int_0^{\infty}P_t(m,n)dt , \tag{3}$$

where $P_t(m,n)$ is the transition probability $P\{Z(t)=n/Z(0)=m\}$; and we define for $|s|<1$ and $m>0$

$$H_m(s)\equiv\sum_{n=1}^{\infty}G(m,n)s^n . \tag{4}$$

Then we have

$$H_m(s)=\int_0^{s}\frac{q^m-z^m}{f(z)}dz \tag{5}$$

for each $m\geqq 1$, where we assume $c=f'''(1)<\infty$ when the process is critical.

Proof) First, we calculate formally. The branching property implies

$$\sum_{n=0}^{\infty}P_t(m,n)s^n=\{F(t;s)\}^m ,$$

and hence,

$$H_m(s)=\int_0^\infty [\{F(t;s)\}^m-\{F(t;0)\}^m]dt\,. \tag{6}$$

By a formal differentiation of both sides of (6),

$$\begin{aligned}\frac{d}{ds}H_m(s)&=\int_0^\infty \frac{\partial}{\partial s}[\{F(t;s)\}^m-\{F(t;0)\}^m]dt\\&=\int_0^\infty [m\{F(t;s)\}^{m-1}\frac{\partial}{\partial s}F(t;s)]dt\\&=\int_0^\infty [m\{F(t;s)\}^{m-1}\frac{1}{f(s)}\frac{\partial}{\partial t}F(t;s)]dt\\&=\frac{1}{f(s)}\int_0^\infty \frac{\partial}{\partial t}[\{F(t;s)\}^m]dt\\&=\frac{1}{f(s)}[q^m-s^m]\,,\end{aligned} \tag{7}$$

for $|s|<1$. (We have made use of the differential equations (1), (2), and the fact

$$\lim_{t\to\infty} F(t;s)=q \text{ for } |s|<1.)$$

Since it is clear that $H_m(0)=0$, (7) can be rewritten in the integrated form:

$$H_m(s)=\int_0^s \frac{q^m-z^m}{f(z)}dz\,.$$

(*Remark*: Lemma 1 implies that the integrand $\frac{q^m-z^m}{f(z)}$ is analytic in the open unit disc.)

The justification of the above procedure is as follows:

1. The integral (5) converges absolutely:
First, note that we have only to prove this when $m=1$, because

$$|F(t;s)|\leqq 1,$$

$$\{F(t;s)\}^m-\{F(t;0)\}^m=\{F(t;s)-F(t;0)\}\sum_{k=0}^{m-1}\{F(t;s)\}^k\{F(t;0)\}^{m-k-1}\,,$$

and hence

$$|\{F(t;s)\}^m-\{F(t;0)\}^m|\leqq m|F(t;s)-F(t;0)|\,.$$

i) The case of non-critical processes (i.e., when $f'(1)\neq 0$).

According to Corollary 1 to Theorem III-8-1 in [5], we have

$$\lim_{t\to\infty} e^{\beta t}[F(t;s)-q]=A(s)$$

with $\beta=-f'(q)>0$. Therefore, $F(t;s)-F(t;0)$ obviously belongs to $L^1(0,\infty)$ as a function of the variable t.

ii) The case of critical processes.
If $c=f'''(1)<\infty$, then by Theorem 2-5-2 in [2],

$$R(t;s)=1-F(t;s)=\frac{1-s}{\frac{Bt}{2}(1-s)+1}(1+u(t;s)),$$

where $u(t;s)=0\left(\frac{\ln t}{t}\right)$ when $t\to\infty$ uniformly in $|s|<1$.

Thus, we have

$$\begin{aligned}
&F(t;s)-F(t;0)\\
&=R(t;0)-R(t;s)\\
&=\frac{1}{\frac{Bt}{2}+1}(1+u(t;0))-\frac{1-s}{\frac{Bt}{2}(1-s)+1}(1+u(t;s))\\
&=\frac{1}{\left(\frac{Bt}{2}+1\right)\left(\frac{Bt(1-s)}{2}+1\right)}\left[\frac{Bt(1-s)}{2}\{u(t;0)-u(t;s)\}+\{u(t;0)+(1-s)u(t;s)\}+s\right]\\
&=0\left(\frac{\ln t}{t^2}\right)+0\left(\frac{\ln t}{t^3}\right)+0\left(\frac{1}{t^2}\right)\quad\text{when } t\to\infty,
\end{aligned}$$

which implies

$$F(t;s)-F(t;0)\in L^1(0,\infty).$$

2. Here, we prove that $\frac{\partial}{\partial s}F(t;s)\in L^1(0,\infty)$.

i) The case of non-critical processes.
By Theorem III-8-1 in [5],

$$\lim_{t\to\infty} e^{\beta t}\frac{\partial}{\partial s}F(t;s)=A'(s).$$

ii) The case of critical processes.
Making use of the differential equations (1), (2), and Taylor expansion, we have:

$$\frac{\partial F(t;s)}{\partial s}=\frac{1}{f(s)}\frac{\partial F(t;s)}{\partial t}=\frac{1}{f(s)}f(F(t;s))=\frac{1}{f(s)}f''(\theta)(1-F(t;s))^2,\ |\theta|<1,$$

under the condition $b=f''(1)<\infty$. As in the proof of Theorem 2-5-1 in [2], we have

$$1-F(t;s)=O\left(\frac{1}{t}\right)$$

as $t\to\infty$, and hence

$$\frac{\partial F(t;s)}{\partial s}=O\left(\frac{1}{t^2}\right),$$

as $t\to\infty$. Thus, this belongs to $L^1(0,\infty)$, completing the proof.

THEOREM 3. (*Uniqueness of harmonic functions of supercritical or critical processes*)

For supercritical or critical processes, the sequence $\frac{G(m,n)}{G(1,n)}$ *converges to* $\sum_{i=0}^{m-1} q^i$ *as* $n\to\infty$ *for each* $m\geqq 1$, *where* q *is the extinction probability of the process, and we assume* $c=f'''(1)<\infty$ *in the case of critical processes. The limit is equal to* m *when the process is critical.*

Proof)

We put

$$h(s)=\sum_{j=0}^{\infty} h_j s^j \equiv \frac{q-s}{f(s)}.$$

This function is analytic in $\{|s|<1\}$ since, by Lemma 1, $\frac{f(s)}{q-s}$ has no zeroes in $\{|s|<1\}$, and hence we can expand $h(s)$ in power series. Now the formula (7) implies

$$\frac{G(m,n)}{G(1,n)}=\frac{\sum_{k=0}^{m-1} q^k h_{n-m+k-1}}{h_n},$$

for $n>m$, $m\geqq 1$.

On the other hand, we put $f(s)=(q-s)(1-s)e(s)$. Then according to Lemma 1, $e(s)\neq 0$ for each $|s|<1$, and $e(s)$ is analytic in $\{|s|<1\}$. Hence, $d(s)\equiv\frac{1}{e(s)}$ is analytic in $\{|s|<1\}$, and we can put:

$$d(s)=\sum_{i=0}^{\infty} d_i s^i.$$

Remark that $0<\sum_{i=0}^{\infty} d_i<\infty$, since $0<f'(1)<\infty$ in the case of supercritical processes, and since $0<f''(1)<\infty$ in the case of critical ones.

Therefore, the function $h(s)$ can be rewritten as

$$h(s)=\frac{q-s}{(q-s)(1-s)e(s)}=\frac{1}{1-s}\frac{1}{e(s)}=(\sum_{j=0}^{\infty} s^j)(\sum_{i=0}^{\infty} d_i s^i)=\sum_{j=0}^{\infty}(\sum_{i=0}^{j} d_i)s^j .$$

This implies that $h_j=\sum_{i=0}^{j} d_i$, and

$$\frac{h_{j-1}}{h_j}=\frac{\sum_{i=0}^{j-1} d_i}{\sum_{i=0}^{j} d_i}\to 1 \text{ as } j\to\infty, \text{ since } 0<\sum_{i=0}^{\infty} d_i<\infty ;$$

hence $\frac{h_{m-p}}{h_m}=\prod_{j=0}^{p-1}\frac{h_{m-j-1}}{h_{m-j}}\to 1$ as $m\to\infty$.

Finally, we get:

$$\frac{G(m,n)}{G(1,n)}=\sum_{k=0}^{m-1} q^k \frac{h_{n-m+k+1}}{h_n}\to\sum_{k=0}^{m-1} q^k, \text{ as } n\to\infty, \text{ for } m\geqq 1 .$$

(*Remark*: The above proof is not valid in the case of general subcritical processes.)

Remark 4. In the case of a subcritical process, if the radius of convergence R of the infinitesimal generating function $f(s)$ is greater than 1, and if there exists a positive number C in the interval $(1, R]$ such that $f(C)=0$, then the sequence $\frac{G(m,n)}{G(1,n)}$ converges to

$$\sum_{k=0}^{m-1} C^k \text{ as } n\to\infty, \text{ for each } m\geqq 1 .$$

Proof) As is well-known, $\tilde{f}(s)\equiv\frac{1}{C}f(Cs)$ is an infinitesimal generating function of a supercritical process with the extinction probability $\frac{1}{C}$. Moreover, the transition probabilities of the original and the derived processes $P_t(m,n)$ and $\tilde{P}_t(m,n)$ satisfy $P_t(m,n)=C^{m-n}\tilde{P}_t(m,n)$, and the corresponding Green functions $G(m,n)$ and $\tilde{G}(m,n)$ have the relation

$$G(m,n)=C^{m-n}\tilde{G}(m,n) .$$

Hence, according to the preceding theorem,

$$\lim_{n\to\infty}\frac{\tilde{G}(m,n)}{\tilde{G}(1,n)}=\sum_{k=0}^{m-1}\left(\frac{1}{C}\right)^k ,$$

and we get:

$$\lim_{n\to\infty}\frac{G(m,n)}{G(1,n)}=\lim_{n\to\infty}\frac{C^{m-n}\bar{G}(m,n)}{C^{1-n}\bar{G}(1,n)}=C^{m-1}\sum_{k=0}^{m-1}\left(\frac{1}{C}\right)^k=\sum_{k=0}^{m-1}C^k.$$

Remark 5. The formula (7) implies that the ratio $\frac{G(m,n)}{G(m,1)}$ is independent of m when it is greater than n; and the limit

$$\lim_{m\to\infty}\frac{G(m,n)}{G(m,1)}$$

exists, implying the uniqueness of the invariant measures. This is valid for any type of processes. (Here, we need the assumption $c=f'''(1)<\infty$ in the case of critical processes.)

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