On the Extinction Probability of Temporally Inhomogeneous Galton-Watson Process with Continuous Time Parameter

Ken-Ichi TANAKA

Department of Pure and Applied Sciences, College of General Education, University of Tokyo, Komaba, Meguro-ku, Tokyo 153

(Received October 15, 1981)

The generating function $F(t,\tau;s)$ of temporally inhomogeneous Galton-Watson process with continuous time parameter satisfies:

$$\begin{cases} \dfrac{\partial F(t,\tau;s)}{\partial \tau} = f(\tau;s)\dfrac{\partial F(t,\tau;s)}{\partial s}, \\ F(t,t;s)=s; \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{\partial F(t,\tau;s)}{\partial t} = -f(t;F(t,\tau;s)), \\ F(\tau,\tau;s)=s; \end{cases} \tag{2}$$

where $f(t;s)=\sum_{n=0}^{\infty} p_n(t)s^n$ is the infinitesimal generating function of the process. (Cf. [1])

In this note we study the extinction probability

$$P_0(t)=\lim_{\tau\to\infty} F(t,\tau;0)$$

of the process. (N. B. $P_0(t)\equiv 1$ or $P_0(t)<1$ for all t. Cf. [1])

From the equation (2) and Ascoli-Arzela's theorem, it follows that $P_0(t)$ is the smallest non-negative solution of

$$\frac{dP_0(t)}{dt} = -f(t;P_0(t)). \tag{3}$$

(*Remark*: If $P_0(t)<1$, then there exist infinitely many solutions of (3) between $P_0(t)$ and 1 (constant function).)

In [1], it is shown that

i) If $\int_t^{\infty} p_0(v)\exp\{-\int_t^v a(u)du\}dv=\infty$, then $P_0(t)\equiv 1$.

ii) If $\int_t^{\infty} b(v)\exp\{-\int_t^v a(u)du\}dv<\infty$, then $P_0(t)<1$.

Here, $a(t)$ and $b(t)$ are the 1st and 2nd derivatives in s of $f(t;s)$ at $s=1$, respectively. We note that the condition in i) is equivalent to

$$\int_t^{\infty} a(v)\exp\{-\int_t^v a(u)du\}dv=\infty .$$

Now we define

$$a_\varepsilon(t)\equiv\sum_{n=1}^{\infty} n^{1+\varepsilon}p_n(t),\ 0\leqq\varepsilon\leqq1\,, \tag{4}$$

and consider the integral

$$\int_t^{\infty} a_\varepsilon(v)\exp\{-\int_t^v a(u)du\}dv\,. \tag{5}$$

Comparing i) and ii), we may expect that there exists such an ε between 0 and 1 that the convergence (resp. divergence) of the integral (5) implies $P_0(t)<1$ (resp. $P_0(t)\equiv1$). But, in fact, we have the following

PROPOSITION. *For any* $0\leqq\varepsilon<1$ *(resp.* $0<\varepsilon\leqq1$*), the convergence (resp. divergence) of* (5) *does not imply* $P_0(t)<1$ *(resp.* $P_0(t)\equiv1$*).*

Proof) i) If we put

$$f(t;s)=\frac{\alpha}{2}\{s^{n+4}+(n+1-t)s^{n+3}+(t-n)s^{n+5}\}-\alpha\left(t+\frac{7}{2}-\frac{5}{2(t+1)}\right)s$$
$$+\alpha\left(t+\frac{5}{2}-\frac{5}{2(t+1)}\right),\ \text{when } n\leqq t\leqq n+1\,,$$

then

$$P_0(t)\equiv1 \text{ for each } \frac{4}{5}<\alpha<\frac{6}{5}\,.$$

But the integral (5) converges when $\alpha>\frac{4+2\varepsilon}{5}$. (If $\alpha<\frac{4+2\varepsilon}{5}$, then (5) diverges.)

ii) If we put

$$f(t;s)=\beta(s-1)(s-q)g(t;s),\ 0<q\leqq\frac{1}{2},\ \beta>0\,;$$

where

$$g(t\,;s)=\frac{1}{(t+1)\log(t+2)}\sum_{n=0}^{\infty}g_n(t)s^n,\ g_0(t)\equiv 1,\ g_1(t)\equiv\frac{1}{2},\ g_2(t)\equiv\frac{1}{3},$$

$$g_{n-1}(t)\equiv\begin{cases}\dfrac{t}{n^2}, & \text{if } 0\leqq t\leqq n\,,\\ \dfrac{1}{n}, & \text{if } n\leqq t\,,\end{cases}\qquad n>3\,,$$

then, obviously, $P_0(t)\leqq q<1$. But the integral (5) diverges when $\beta<\frac{\varepsilon}{1-q}$. (If $\beta>\frac{\varepsilon}{1-q}$, then the integral (5) converges.)

Remark: In these examples, the mean values grow in the polynomial order. This implies that these examples are somewhat pathological ones.

References

[1] Čistyakov, V. P., Markova, N. P.: On some theorems for inhomogeneous branching processes. *Dokl. Acad. Nauk SSSR* **147** (1962), 317-320 (in Russian).
[2] Sevast'yanov, B. A.: *Branching processes.* "Nauka" 1971, in Russian.