

# A Fusion Theoretical Approach to Groups of Type $PSL_3(2^n)$ and $PSp_4(2^n)$

by Kensaku GOMI

Department of Mathematics, College of General Education  
University of Tokyo, Komaba, Meguro-ku, Tokyo 153

(Received February 5, 1980)

## Introduction

In this note we consider the problem of finding all finite groups  $G$  in which a Sylow 2-subgroup  $S$  contains precisely two maximal elementary abelian 2-subgroups,  $A$  and  $B$ , and  $S=AB$ . One possible approach to this problem is the application of Gilman and Gorenstein's theorem [4], as the nilpotency class of  $S$  is two. Indeed, the structure of such a group  $G$  is easily determined by the use of their theorem, provided that  $O_{2',2}(G)=O(G)$ . The purpose of this note, however, is not to show that, but to give an almost fusion theoretical proof of the following result:

**THEOREM.** *Let  $G$  be a finite group in which a Sylow 2-subgroup  $S$  contains precisely two maximal elementary abelian 2-subgroups,  $A$  and  $B$ , and  $S=AB$ . Then one of the following holds:*

- (i)  $|S:A|=|S:B|=2$ ;
- (ii)  $A \in \text{Syl}_2(\langle A^G \rangle)$ ;
- (iii)  $B \in \text{Syl}_2(\langle B^G \rangle)$ ;
- (iv)  $O^{2'}(G)/O(O^{2'}(G))=K*L$  (central product),  $K$  is a group with elementary abelian Sylow 2-subgroups, and  $L$  is a perfect central extension of  $PSL_3(q)$  or  $PSp_4(q)$ , where  $q=|S:A|=|S:B|$ .

The main tool used in the proof is Goldschmidt's "2-fusion theorem" [6]. This theorem together with certain side techniques, also due to Goldschmidt, enables one to reduce the problem to the case where the 2-local structure of  $G$  looks like that of  $PSL_3(2^n)$  or  $PSp_4(2^n)$ ,  $2 \leq n$ . In this situation there is a variety of method to identify  $G$ . Probably, the best method is a geometrical method as used in the proof of Theorem 2 of Aschbacher [2]. In this note, however, we shall simply use a result that classifies groups of characteristic 2 type having a Sylow 2-subgroup of nilpotency class two [7, 8]. Therefore, the

proof of the theorem is independent of Gilman and Gorenstein's theorem. Such a proof has the effect of making certain papers on standard component problems, e.g. [9], free from Gilman and Gorenstein's paper and, in fact, this was the main motivation for the present work.

## 1. Fusion Lemmas

In this section we collect some basic results on fusion of  $p$ -elements that we shall need for the proof of the theorem.

**2-FUSION THEOREM [6].** *Let  $G$  be a finite group,  $S$  be a Sylow 2-subgroup of  $G$ , and  $A$  be an elementary abelian subgroup of  $S$ . If  $A$  is strongly closed in  $S$  with respect to  $G$ , then  $\langle A^g \rangle | O(\langle A^g \rangle)$  is a central product of an elementary abelian 2-group and Goldschmidt groups. Furthermore, if  $A \subseteq \text{TeSyl}_2(\langle A^g \rangle)$  then  $A = \Omega_1(T)$ .*

Here, we mean by "Goldschmidt groups" the quasisimple groups which Goldschmidt called groups of type I and II.

**GOLDSCHMIDT'S LEMMA.** *Let  $G$  be a finite group,  $S$  be a Sylow 2-subgroup of  $G$ , and  $A$  be an elementary abelian subgroup of  $S$ . Suppose  $A$  is weakly closed in  $S$  with respect to  $G$  and an element  $a \in S - A$  is conjugate to an element of  $A$ . Choose a conjugate  $A_1$  of  $A$  so that*

- (1)  $a \in A_1$ , and
- (2)  $|A \cap A_1|$  is maximal subject to (1),

and set  $X_1 = A \cap A_1$ ,  $X_2 = N_A(\langle X_1, a \rangle)$ . Furthermore, let  $X = C_A(a)$  and  $X_0 = [A, a]$ . Then the following holds:

- (i)  $X_0 X_1 \subseteq X \subseteq X_2$  and if  $A_1^g = A$ ,  $g \in G$ , then  $A \cap X_2^g = X_1^g$  and  $X_2^g \not\subseteq A$ ;
- (ii)  $|A/X| = |X_0| \leq |X_2/X_1|$  and  $|X_0 \cap X_1| = |X_2/X|$ ;
- (iii) we may choose an element  $g \in G$  so that  $A_1^g = A$  and  $N_S(\langle X_1, a \rangle)^g \subseteq S$ .

This result is implicit in the proof of Corollary 4 of Goldschmidt [6]. A proof is given in an author's paper [9, (1G)].

**BURNSIDE'S LEMMA.** *Let  $G$  be a finite group,  $S$  be a Sylow  $p$ -subgroup of  $G$ , and  $W$  be a weakly closed subgroup of  $S$  with respect to  $G$ . If  $A, B$  are subsets of  $S$  that are conjugate in  $G$  and normalized by  $W$ , then  $A, B$  are conjugate in  $N_G(W)$ .*

This is a well-known fact and an easy consequence of Sylow's theorem.

**GLAUBERMAN'S LEMMA.** *Let  $G$  be a finite group,  $S$  be a Sylow  $p$ -subgroup of  $G$ , and  $A$  be an abelian subgroup of  $S$ . If  $A$  is strongly closed in  $S$  with*

respect to  $G$ , then  $N_G(A)$  controls fusion of elements of  $S$ .

This result was first proved by Glauberman [5]. There is an alternative proof based on Alperin's fusion theorem [1, 11].

## 2. Preliminary Lemmas

In this section  $G$  is a finite group satisfying the hypothesis of the theorem and  $S$  is a Sylow 2-subgroup of  $G$ . For any 2-group  $X$ ,  $\mathcal{E}^*(X)$  will denote the set of maximal elementary abelian subgroups of  $X$ . Thus the basic hypothesis of this section may be written as follows.

HYPOTHESIS 1.  $\mathcal{E}^*(S) = \{A, B\}$ ,  $A \neq B$ , and  $S = AB$ .

Under this hypothesis we first prove the following two lemmas.

LEMMA 1.  $A$  and  $B$  are weakly closed in  $S$  with respect to  $G$ .

PROOF.  $N_G(S)$  acts, by conjugation, on  $\mathcal{E}^*(S) = \{A, B\}$ . In particular,  $B$  normalizes  $A$  and, as  $S = AB$ ,  $S$  normalizes  $A$ . Thus  $n = |N_G(S) : N_G(A) \cap N_G(S)|$  is odd, while  $n \leq |\mathcal{E}^*(S)| = 2$ . Therefore,  $N_G(S) \subseteq N_G(A)$  and by symmetry  $N_G(S) \subseteq N_G(B)$ . Now suppose, say,  $A \neq A^g \subseteq S$  for some element  $g \in G$ . Then  $A^g \subseteq B$  by Hypothesis 1. If  $|A| = |B|$ , then  $A^g = B$  and so we may take  $g \in N_G(S)$  by Burnside's lemma. Since this is impossible, it follows that  $|A| < |B|$ . Then  $B$  is weakly closed by Hypothesis 1 and so we may take  $g \in N_G(B)$  again by Burnside's lemma. Since this is impossible,  $A$  is weakly closed and, by symmetry,  $B$  is weakly closed as well.

LEMMA 2. If  $A$  or  $B$  is strongly closed in  $S$  with respect to  $G$ , then respectively (ii) or (iii) of the theorem holds.

Proof. Suppose  $A$  is strongly closed, say, and let  $T = S \cap \langle A^g \rangle$ . Then  $\Omega_1(T) = A$  by the 2-fusion theorem, while  $\mathcal{E}^*(T) = \{A, B \cap T\}$  and  $T = A(B \cap T)$  by Hypothesis 1. Therefore,  $T = A$  and the lemma holds.

Now assume that  $A$  is not strongly closed, and suppose an element  $a \in S - A$  is conjugate to an element of  $A$ . Following Goldschmidt's lemma, we introduce some notation. Let

$$X = C_A(a) \quad \text{and} \quad X_0 = [A, a].$$

Choose a conjugate  $A_1$  of  $A$  so that

$$(1) \quad a \in A_1,$$

and

$$(2) \quad |A \cap A_1| \text{ is maximal subject to (1),}$$

and let

$$X_1 = A \cap A_1 \quad \text{and} \quad X_2 = N_A(\langle X_1, a \rangle).$$

Then  $X_0 X_1 \subseteq X \subseteq X_2$  by Goldschmidt's lemma. Furthermore, let

$$Z = A \cap B.$$

Now Hypothesis 1 implies that  $C_S(x) = B$  for each  $x \in B - A$  and  $C_S(y) = A$  for each  $y \in A - B$ . Therefore,

$$(3) \quad Z(S) = Z.$$

As  $a \in B - A$  by Hypothesis 1, we also have

$$(4) \quad X = Z.$$

Now we may take  $g \in G$  so that  $A_1^g = A$  and  $N_S(\langle X_1, a \rangle)^g \subseteq S$  by Goldschmidt's lemma. As  $\langle X_1, a \rangle \subseteq B$  by Hypothesis 1, we have  $X_2 B \subseteq N_S(\langle X_1, a \rangle)$  and so  $(X_2 B)^g \subseteq S$ . Thus, the weak closure (Lemma 1) of  $B$  yields that

$$(5) \quad g \in N_G(B).$$

As  $X_2^g \subseteq S$  and  $X_2^g \not\subseteq A$  by Goldschmidt's lemma, Hypothesis 1 implies  $X_2^g \subseteq B$ . Thus  $X_2 \subseteq A \cap B = Z$  by (5) and as  $Z \subseteq X_2$  by (3), we have

$$(6) \quad X_2 = Z.$$

Using (4), (6) and Goldschmidt's lemma, we obtain

$$(7) \quad |S/B| = |A/Z| = |X_0| \leq |X_2/X_1| = |X_2^g/X_1^g| \leq |S/A|.$$

We can now prove the following:

LEMMA 3. *If  $|S/A| \neq |S/B|$ , then  $A$  or  $B$  is strongly closed in  $S$  with respect to  $G$ .*

*Proof.* The inequality  $|S/B| \leq |S/A|$  in (7) above was obtained under the hypothesis that  $A$  was not strongly closed. Hence if  $|S/A| < |S/B|$  then  $A$  is strongly closed and by symmetry, if  $|S/B| < |S/A|$ ,  $B$  is strongly closed.

In view of Lemmas 2 and 3, we assume the following from now on.

HYPOTHESIS 2.  $|S/A| = |S/B| = q$ ,  $2 < q$ , and  $A$  is not strongly closed in  $S$  with respect to  $G$ .

As  $|S/A| = |S/B| = q$ , (6) and (7) show

$$(8) \quad |X_0| = |Z/X_1| = q.$$

Also,

$$(9) \quad X_0 \cap X_1 = 1$$

by (4), (6), and Goldschmidt's lemma. Now let

$$R = \langle S, S^{q^{-1}} \rangle \quad \text{and} \quad Q = O^{q'}(N_G(B)),$$

so that  $R \subseteq Q$  by (5). We shall consider the structure of  $Q/B$ .

**LEMMA 4.**  *$N_Q(Z)/B$  is strongly embedded in  $Q/B$  and  $N_R(Z)/B$  is strongly embedded in  $R/B$ .  $Q$  has a normal subgroup  $P$  containing  $B$  such that  $Q/P \cong PSL_2(q)$  and  $|P/B|$  is odd.*

*Proof.* As  $Z(S) = Z \neq Z^{q^{-1}}$  by (3), (6), and Goldschmidt's lemma,  $S$  is not conjugate to  $S^{q^{-1}}$  in  $N_G(Z)$ . Thus  $N_R(Z) \neq R$  and  $N_Q(Z) \neq Q$  by Sylow's theorem. If  $B \subset T \subseteq S$ , then  $Z(T) = Z$  by an analogue of (3) and so  $N_G(T) \subseteq N_G(Z)$ . This implies that  $N_R(Z)/B$  is strongly embedded in  $R/B$  and similarly for  $N_Q(Z)/B$  in  $Q/B$ . As  $S/B$  is elementary abelian of order  $q$  and  $O^{q'}(Q) = Q$ , the second assertion follows from Bender's theorem [3].

We shall next consider the action of  $N_G(B)/B$  on  $B$ . Let

$$A_0 = C_A(O^{2'}(N_G(A))) \quad \text{and} \quad B_0 = C_B(O^{q'}(N_G(B))).$$

As a consequence of Lemma 4 and Bender's theorem [3], we have

$$|Q : N_Q(Z)| = |R : N_R(Z)| = q + 1,$$

and so  $Q = N_Q(Z)R$ . Hence if  $T \in \text{Syl}_2(Q)$ , then  $T = S^{xy}$  with  $x \in N_Q(Z)$  and  $y \in R$  by Sylow's theorem. As  $R \subseteq C_G(X_1)$ , we may deduce as follows:

$$\begin{aligned} [T, X_1] &= [S^{xy}, X_1] = [S^x, X_1]^y \\ &\subseteq [S^x, Z]^y = [S, Z]^{xy} = 1. \end{aligned}$$

Therefore,

$$(10) \quad X_1 = B_0.$$

Henceforth, we assume the following:

**HYPOTHESIS 3.**  $|S/A| = |S/B| = q$ ,  $2 < q$ , and neither  $A$  nor  $B$  is strongly closed in  $S$  with respect to  $G$ .

**LEMMA 5.** *The conjugates of  $(Z/B_0)^*$  under  $Q/B_0$  form a partition of  $(B/B_0)^*$ .  $N_G(Z) \cap N_G(B)$  acts transitively on  $(Z/B_0)^*$  and hence  $N_G(B)$  acts transitively on  $(B/B_0)^*$ .  $N_G(B)$  is 2-constrained.*

*Proof.* Suppose  $B_0 \subset Z \cap Z^x$  for some element  $x \in N_G(B)$ . Then  $|Z/Z \cap Z^x| < |Z/X_1|$  by (10). The equation (8) was obtained under the hypothesis that  $A$  was not strongly closed in  $S$  with respect to  $G$ . Hence  $A$  is strongly closed in  $S$  with respect to  $N_G(Z \cap Z^x)$ , and in particular  $Z = A \cap B$  is normal in  $N_G(Z \cap Z^x) \cap N_G(B)$ . As  $S^x \subseteq N_G(Z \cap Z^x) \cap N_G(B)$ ,  $S^x = S^y$  for some element  $y \in N_G(Z)$  by Sylow's theorem. Thus  $Z^x = Z^y = Z$  by (3). This implies that  $Z/B_0$  is a T.I. set in

$N_G(B)/B_0$ . As  $|Z/B_0|=q$  and  $|B/B_0|=q^2$  by (8) and (9), and as  $|Q:N_G(Z)|=q+1$  by Lemma 4, the first assertion follows.

An analogue for  $A^{g^{-1}}$  of Lemma 4 shows that  $N_G(S^{g^{-1}})$  acts transitively on  $(S^{g^{-1}}/A^{g^{-1}})^*$  and hence on  $(B/Z^{g^{-1}})^*$ . Thus  $N_G(Z^{g^{-1}}) \cap N_G(B)$  acts transitively on  $(B/Z^{g^{-1}})^*$ . It also follows from Lemma 4 and Bender's theorem [3] that

$$N_G(Z^{g^{-1}}) \cap N_G(B) = (N_G(Z) \cap N_G(Z^{g^{-1}}))S^{g^{-1}}.$$

As  $S^{g^{-1}}$  centralizes  $B/Z^{g^{-1}}$ ,  $N_G(Z) \cap N_G(Z^{g^{-1}})$  acts transitively on  $(B/Z^{g^{-1}})^*$  and hence on  $(Z/B_0)^*$ , as  $B/B_0 = Z/B_0 \times Z^{g^{-1}}/B_0$ . This proves the second assertion.

Now the first assertion shows that  $C_G(B) \neq Q$ . The structure of  $Q/B$  (Lemma 4) then forces  $C_G(B) \subseteq P$ , so  $|C_G(B)/B|$  is odd and  $C_G(B)$  is 2-solvable. Therefore,  $N_G(B)$  is 2-constrained.

The following result permits us to use an inductive argument.

LEMMA 6. *If  $W \subseteq B_0$ , then  $S/W \in \text{Syl}_2(C_G(W)/W)$ ,  $\mathcal{C}^*(S/W) = \{A/W, B/W\}$ , and  $S/W = (A/W)(B/W)$ .*

*Proof.* Let  $b \in B - A$ . Then  $b^x \in Z$  for some element  $x \in N_G(B)$  by Lemma 5. As  $X_1 = B_0 \subseteq A \cap A^{x^{-1}}$ , the choices of  $a$  and  $A_1$  show  $A \cap A^{x^{-1}} = B_0$ . Thus  $[[A, b]] = q$  and  $[A, b] \cap B_0 = 1$  by analogues of (8) and (9).

Now let bars denote images in  $C_G(W)/W$ . Then  $\bar{S}$  is a Sylow 2-subgroup of  $\bar{C}_G(\bar{W})$  and  $\bar{S} = \bar{A}\bar{B}$ . Furthermore, if  $b$  is an arbitrary element of  $B - A$  then  $[[\bar{A}, \bar{b}]] = q$  by the above, and so  $C_{\bar{A}}(\bar{b}) = \bar{Z}$ . Thus  $\mathcal{C}^*(\bar{S}) = \{\bar{A}, \bar{B}\}$ .

The following three lemmas deal with the fusion of involutions.

LEMMA 7. *Let  $V \subseteq Z$ . Then  $A$  is not strongly closed in  $S$  with respect to  $C_G(V)$  if and only if  $V \subseteq B_0$ .*

*Proof.* If  $V \subseteq B_0$ , then  $Q \subseteq C_G(V)$  and so  $A$  is not strongly closed in  $S$  with respect to  $C_G(V)$  by Lemma 5. Conversely, if  $A$  is not strongly closed in  $S$  with respect to  $C_G(V)$ , then analogues of (8) and (10) show that there is an element  $h \in C_G(V)$  such that

$$A \cap A^h = C_B(O^2(N_G(B) \cap C_G(V)))$$

and such that

$$|A \cap A^h| = |S|/q^2.$$

As  $B_0 \subseteq C_B(O^2(N_G(B) \cap C_G(V)))$  and  $|B_0| = |S|/q^2$ , it follows that  $A \cap A^h = B_0$ . Thus  $V = V^h \subseteq B_0$ .

LEMMA 8. *Every involution of  $G$  is conjugate to an element of  $Z$ .*

*Proof.* This follows from Lemma 5 and its analogue for  $A$ .

LEMMA 9. *Let  $Z_0 = A_0 \cap B_0$ . Then  $Z_0$  is strongly closed in  $S$  with respect to  $G$ .*

*Proof.* Because of Lemma 1 and Burnside's lemma, it suffices to show that  $\langle N_G(A), N_G(B) \rangle \subseteq N_G(Z_0)$ . Let  $x \in N_G(B)$ . Then  $Z_0^x = A_0^x \cap B_0$  and  $A_0^x = C_{A^x}(O^{2'}(N_G(A^x)))$ . Choose an element  $y \in Q$  so that  $S^x = S^y$ . Then  $A^x = A^y$ , so  $A_0^x = A_0^y$  and  $Z_0^x = A_0^y \cap B_0 = (A_0 \cap B_0)^y = Z_0^y = Z_0$ . Thus  $N_G(B) \subseteq N_G(Z_0)$  and, by symmetry,  $N_G(A) \subseteq N_G(Z_0)$ .

Finally, we prove the following:

LEMMA 10. Assume  $Z_0=1$ . Then either  $A_0=B_0=1$  or  $Z=A_0 \times B_0$ , and in the latter case  $C_G(A_0)$  and  $C_G(B_0)$  are 2-constrained.

*Proof.* As  $Z_0=1$ ,  $Z^*$  is a disjoint union of the sets  $A_0^*$ ,  $B_0^*$ , and  $Z-(A_0 \cup B_0)$ . Moreover, Lemma 1 and Burnside's lemma show that none of them fuses to the others in  $G$ , as  $N_G(B) \subseteq N_G(B_0)$  and  $N_G(A) \subseteq N_G(A_0)$ . Thus  $N_G(Z) \subseteq N_G(A_0) \cap N_G(B_0)$ . Lemma 5 and its analogue for  $A$  now show that  $N_G(Z)$  acts transitively on  $(Z/A_0)^*$  and on  $(Z/B_0)^*$ . Therefore, either  $A_0=B_0=1$  or  $Z=A_0 \times B_0$ .

Assume  $Z=A_0 \times B_0$ . As  $B_0 \not\subseteq A_0$ ,  $B$  is strongly closed in  $S$  with respect to  $C_G(B_0)$  by an analogue of Lemma 7. Let bars denote images in  $C_G(B_0)/B_0 O(C_G(B_0))$  and let  $\bar{K}$  be the normal closure of  $B$  in  $C_G(B_0)$ . Then by the 2-fusion theorem,  $\bar{K}$  is a central product of a 2-group and Goldschmidt groups, and if  $T=S \cap K$  then  $O_2(\bar{K}) \subseteq \bar{B} = \Omega_1(\bar{T})$ . Now Lemma 5 implies that  $N_G(B)$  acts transitively on  $\bar{B}^*$ . This action of  $N_G(B)$  on  $\bar{B}$  forces  $O_2(\bar{K})=1$  or  $\bar{B}$ , as  $N_G(B)$  acts on  $\bar{K}$ . Moreover, if  $O_2(\bar{K})=1$  then  $\bar{K}$  is a simple Goldschmidt group and  $N_G(B)^\infty$  induces a perfect automorphism group of  $\bar{K}$  that normalizes  $\Omega_1(\bar{T})=\bar{B}$ . However, this shows that  $N_G(B)^\infty$  centralizes  $\bar{K}$  [6, Section 3], so  $N_G(B)^\infty \subseteq C_G(B/B_0)$ . Since this is impossible by Lemmas 4 and 5, we must have  $O_2(\bar{K})=\bar{B}$ . This shows that  $BO(C_G(B_0))$  is normal in  $C_G(B_0)$ , so

$$C_G(B_0) = (N_G(B) \cap C_G(B_0)) O(C_G(B_0))$$

by a Frattini argument. Therefore,  $C_G(B_0)$  is 2-constrained by Lemma 5. By symmetry,  $C_G(A_0)$  is 2-constrained as well.

## 2. Proof of the Theorem

In this section we complete the proof of the theorem by induction on  $|G|$ . Let  $G_0 = O^{2'}(G)$ . Then  $S \in \text{Syl}_2(G_0)$  and  $G = N_G(S)G_0$  by a Frattini argument. As  $N_G(S) \subseteq N_G(A) \cap N_G(B)$  by Lemma 1, it follows that  $\langle A^g \rangle = \langle A^{g_0} \rangle$  and  $\langle B^g \rangle = \langle B^{g_0} \rangle$ . Thus if  $G_0 \neq G$ , we can apply the induction hypothesis to  $G_0$ , and obtain the theorem. Therefore, we assume  $G = O^{2'}(G)$ . Also, if  $O(G) \neq 1$  then we can apply the induction hypothesis to  $G/O(G)$ . Therefore, we assume  $O(G)=1$ . Furthermore, in view of Lemmas 2 and 3, we may operate under Hypothesis 3. For a while, however, we shall assume only Hypothesis 3 and prove that if  $Z_0=1$  then  $O^{2'}(G)/O(O^{2'}(G)) \cong PSL_3(q)$  or  $PSp_4(q)$ . It suffices to prove that the centralizer of every non-identity subgroup of  $Z$  is 2-constrained and that  $O_{2',2}(G)=O(G)$ . For Lemma 8 then shows that the centralizer of every involution of  $G$  is 2-constrained. As  $SCN_3(2)$  is non-empty, the "balanced group theorem" [10] shows

that  $G/O(G)$  is of characteristic 2 type. We can then apply previous results [7, 8]. As  $S$  is large enough, the only possibility is that  $O^{2'}(G)/O(O^{2'}(G)) \cong \text{PSL}_3(q)$  or  $\text{PSp}_4(q)$ .

Now let  $1 \neq V \subseteq Z$  and  $H = C_G(V)$ . We show that if  $Z_0 = 1$  then  $H$  is 2-constrained. As  $Z_0 = 1$ , either  $V \not\subseteq A_0$  or  $V \not\subseteq B_0$  and so, by symmetry, we assume  $V \not\subseteq A_0$ . Then  $B$  is strongly closed in  $S$  with respect to  $H$  by an analogue of Lemma 7. If  $A_0 = B_0 = 1$ , then  $V \not\subseteq B_0$  and so  $A$  is also strongly closed in  $S$  with respect to  $H$ . We can then prove that  $H$  is 2-solvable of 2-length 1 and hence 2-constrained [9, the fourth paragraph of the proof of (1H)]. We therefore assume  $Z_0 = A_0 \times B_0$  in view of Lemma 10. As  $N_H(B) \subseteq N_H(B_0)$ ,  $B_0$  is strongly closed in  $S$  with respect to  $H$  by Glauberman's lemma. An analogue for  $H$  of Lemma 2 shows  $S \cap \langle B^H \rangle = B$  and so  $S \cap \langle B_0^H \rangle = B \cap \langle B_0^H \rangle$ . As  $B_0 = \Omega_1(S \cap \langle B_0^H \rangle)$  by the 2-fusion theorem, it follows that  $B_0 \in \text{Syl}_2(\langle B_0^H \rangle)$ . Now we distinguish two cases.

*Case 1.* Assume  $V \not\subseteq B_0$ . Then  $A_0 \in \text{Syl}_2(\langle A_0^H \rangle)$  by symmetry. As  $A_0 \cap B_0 = Z_0 = 1$ , it follows that  $[\langle A_0^H \rangle, \langle B_0^H \rangle] \subseteq O(H)$  and, in particular,  $\langle B_0^H \rangle \subseteq C_H(A_0)O(H)$ . As  $C_H(A_0)$  is 2-constrained by Lemma 10, so also is  $\langle B_0^H \rangle$  and hence  $B_0 O(H)$  is normal in  $H$  by the 2-fusion theorem. Thus  $H = N_H(B_0)O(H)$  by a Frattini argument and, as  $N_H(B_0)$  is 2-constrained by Lemma 10, so also is  $H$ .

*Case 2.* Assume  $V \subseteq B_0$ . Then  $Q \subseteq H$  and  $Q$  centralizes  $B_0 \in \text{Syl}_2(\langle B_0^H \rangle)$ . As  $\langle B_0^H \rangle O(H)/O(H)$  is a central product of a 2-group and Goldschmidt groups, we must have  $[Q^\infty, \langle B_0^H \rangle] \subseteq O(H)$  [6, Section 3]. Now  $Q/B_0$  is perfect by Lemmas 4 and 5. Hence if we set  $W = Z \cap Q^\infty$ , then  $Z = WB_0$  and  $W \not\subseteq B_0$ . Thus  $VW \not\subseteq A_0, B_0$  and so  $C_H(W)$  is 2-constrained by the discussion in Case 1. As  $\langle B_0^H \rangle \subseteq C_H(W)O(H)$ , it follows as in Case 1 that  $H$  is 2-constrained.

It remains to prove  $O_{2',2}(G) = O(G)$ . Let bars denote images in  $G/O(G)$ . The structure of  $\bar{Q}/\bar{B}$  shows  $O_2(\bar{G}) \subseteq \bar{B}$ , and by symmetry  $O_2(\bar{G}) \subseteq \bar{A}$ ; so  $O_2(\bar{G}) \subseteq \bar{Z}$  and then  $O_2(\bar{G}) \subseteq \bar{B}_0$  by Lemma 5. By symmetry  $O_2(\bar{G}) \subseteq \bar{A}_0$  and, as  $\bar{Z}_0 = 1$ ,  $O_2(\bar{G}) = 1$ .

Assume now  $Z_0 \neq 1$  and let  $K = \langle Z_0^G \rangle$ . Assume furthermore that  $O^{2'}(G) = G$  and  $O(G) = 1$ . Then by Lemma 9 and the 2-fusion theorem,  $K$  is a central product of a 2-group and Goldschmidt groups and, if  $T = S \cap K$ , then  $O_2(K) \subseteq \Omega_1(T) = Z_0$ . Since  $[S, Z_0] = 1$  and  $[S, T] \subseteq T \cap Z = Z_0$ , it follows that  $S$  induces inner automorphisms on  $E(K)$  [6, Section 3]. Also,  $[S, O_2(K)] = 1$ . Therefore,  $S \subseteq KC_G(K)$  and, as  $O^{2'}(G) = G$ , we conclude that  $G = KC_G(K)$ .

Now  $C_G(Z_0)/Z_0$  satisfies Hypothesis 3 by Lemmas 6 and 7. Furthermore, the subgroup of  $C_G(Z_0)/Z_0$  corresponding to  $Z_0$  is the identity group. Therefore, the preceding discussion shows that  $C_G(Z_0)/Z_0 O(C_G(Z_0))$  has a normal subgroup of odd index isomorphic to  $\text{PSL}_3(q)$  or  $\text{PSp}_4(q)$ . In particular,  $O_2(C_G(Z_0)/Z_0) = 1$ . As  $Z_0 = \Omega_1(T)$  and  $T \in \text{Syl}_2(K)$ , the structure of  $K$  shows  $O_2(K \cap C_G(Z_0)) = T$  and so  $T/Z_0 \subseteq O_2(C_G(Z_0)/Z_0)$ . Thus  $Z_0 \in \text{Syl}_2(K)$ .

Now let  $L = C_G(Z_0)^\infty$ . Then  $L$  induces a perfect automorphism group on  $K$  centralizing  $Z_0 \in \text{Syl}_2(K)$ . This forces  $[K, L] = 1$  [6, Section 3]. Hence  $L$  is normal in  $KC_G(K) = G$ , as  $C_G(K) \subseteq C_G(Z_0)$ . As  $O(G) = 1$ , the structure of  $C_G(Z_0)/Z_0 O(C_G(Z_0))$  and the definition of  $L$  show that  $L$  is a perfect central extension of  $\text{PSL}_3(q)$  or



$PSp_4(q)$ . Also,  $Z_0L$  has odd index in  $C_G(Z_0)$  and so  $S \subseteq Z_0L \subseteq KL$ . As  $O^{2'}(G)=G$ , it follows that  $G=KL$ . Thus, we have proved that  $G$  is in Case (iv) of the theorem, and the proof of the theorem is complete.

### References

- [1] Alperin, Sylow intersections and fusion, *J. Algebra*, **6** (1967), 222-241.
- [2] M. Aschbacher, A pushing up theorem for characteristic 2 type groups, *Ill. J. Math.* **22** (1978), 108-125.
- [3] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt *J. Algebra*, **17** (1971), 525-554.
- [4] R. Gilman and D. Gorenstein, Finite groups with Sylow 2-subgroups of class two, *Trans. Amer. Math. Soc.*, **207** (1975), 1-126.
- [5] G. Glauberman, A sufficient condition for p-stability, *Proc. London Math. Soc.*, **25** (1972), 253-287.
- [6] D. Goldschmidt, 2-Fusion in finite groups, *Ann. of Math.*, **99** (1974), 70-117.
- [7] K. Gomi, Finite groups all of whose non-2-closed 2-local subgroups have Sylow 2-subgroups of class 2, *J. Algebra*, **35** (1975), 214-223.
- [8] K. Gomi, Sylow 2-intersections and split BN-pairs of rank two, *J. Fac. Sci. Univ. Tokyo*, Sect. IA, **23** (1976), 1-22.
- [9] K. Gomi, Finite groups with a standard subgroup isomorphic to  $Sp(4, 2^n)$ , *Japanese J. Math.*, New Ser. **4** (1978), 1-76.
- [10] D. Gorenstein and J. Walter, Centralizers of involutions in balanced groups, *J. Algebra*, **20** (1972), 284-319.
- [11] T. Kondo, On Alperin-Goldschmidt's fusion theorem, *Sci. Pap. Coll. Gen. Educ. Univ. Tokyo*, **28** (1978), 159-166.