A Formula for Topological Entropy of One-dimensional Dynamics

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0. Introduction

A distinguished property of one-dimensional dynamics is that the existence of periodic orbits (cycles) of various types determines almost all the dynamical structure. (cf. [3–6, 9, 13]) For example, the theorem of Šarkovskii can be refined as follows.

THEOREM 0. *There is a partial order* $\vdash$ *among the types of cycles such that a continuous interval dynamics* (J, f) *has a cycle of type* τ' *if it has a cycle of type* τ *and* $\tau \vdash \tau'$.

Here the type τ *of a cycle of a map* f, $C=\{x_1<\cdots<x_n\}$, *is the cyclic permutation of* $\{1, \cdots, n\}$ *defined by the relation*

$$fx_i = x_{\tau(i)}, \qquad i=1, \cdots, n.$$

In particular, the order $\vdash$ *is linear among the types of cycles under unimodal transformations.*

In the present paper we shall employ a traditional machine in ergodic theory and prove Theorems 1 and 2 on the realization of one-dimensional piecewise continuous maps (J, f) in the first section. The results generalize the theorems for the beta transformations. Using these results and the notion of (topological) tower, we shall give the proof of Theorem 0 (§1) and the proof of the following main theorem.

THEOREM 3. Let (J, f) *be a continuous interval dynamics. Then the topological entropy is given by the formula*

$$\mathrm{ent}\,(J, f) = \sup\,\{\mathrm{ent}\,(J_\tau, f_\tau)\,;\ \tau=\tau(C),\ C\in\Gamma(J, f)\}$$

where $\Gamma(J, f)$ *denotes the totality of cycles under* (J, f), *and the interval dynamics* (J_τ, f_τ), τ *being a cyclic permutation of* $1, \cdots n$, *is defined as follows*: $J_\tau=[1, n]$,

(a) $f_\tau i=\tau(i)$ *for* $i=1, \cdots, n$ and

(b) f_τ *is linear on each interval* $[i, i+1]$, $i=1, \cdots, n-1$.

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As corollaries

(i) $\mathrm{ent}(J, f)=\limsup n^{-1}\log|\Gamma_n(J, f)|$ (the increasing order of the number of cycles of length as n tends to infinity)

(ii) Any continuous interval dynamics (J, f) with positive topological entropy have cycles with periods which are not any powers of 2.

1. Realization of transformations on the interval

In this section we are concerned with a piecewise continuous transformation f on a bounded closed interval J. This class of dynamical systems (J, f) contains all the important one-dimensional dynamical systems: continuous dynamical systems of intervals, continuous dynamical systems of circles and, more generally, of branched one-dimensional manifolds, and number-theoretic transformations.

DEFINITION 1. Closed subintervals $I_1, \cdots, I_l$ are called *lap intervals* of f if the following conditions are verified:

(a) $I_1\cup\cdots\cup I_l=J$ and $\mathrm{int}\, I_i\cap \mathrm{int}\, I_j=\emptyset$ for $i\neq j$.

(b) f is monotone on each interval I_i, $i=1, \cdots, l$.

(c) The number l is minimal under the conditions (a) and (b). The number l is called *lap number* of f and will be denoted by $\mathrm{lap}(f)$.

Remark. The choice of lap intervals is, of course, not unique. It is unique if the inverse image $f^{-1}(x)$ consists of finite points for each x in J.

Let I_a, $a\in A$, be lap intervals of f and $E=\cup\partial I_a$ the end-points of the subintervals I_a. Let us introduce a linear order in the suffix set A by the relation:

$$a<b \quad \textit{if } J_a \textit{ lies on the left side of } I_b.$$

We define a map $\pi=\pi_f: J'\to A^N$ (N being the set of all natural numbers) by the relation:

$$(1) \qquad \pi(x)(n)=a \quad \textit{if } f^n x\in \mathrm{int}\, I_a \quad (n\in \boldsymbol{N},\ a\in A)$$

where $\quad J'=\{x\in J;\ f^n x\in E^c \text{ for all } n\in \boldsymbol{N}\}$

For an element ω of the sequence space A^N or A^Z we shall use the following notations:

$$\omega(L)=(\omega(n))_{n\in L} \qquad \text{for subsets } L \text{ of real line } \boldsymbol{R}$$

$$(\sigma\omega)(n)=\omega(n+1) \qquad \text{(the shift to the left)}$$

LEMMA 1. *Let*

$$W_n(f)=\{\pi(x)[0, n);\ x\in J'\} \qquad (n\in \boldsymbol{N}).$$

Then $u=(a_0,\cdots,a_{n-1})$ *belongs to* $W_n(f)$ *if and only if*

$$I_u{}^0=\bigcap_{i=0}^{n-1} f^{-i}(\mathrm{int}\, I_{a_i})\neq\emptyset.$$

Proof. Obvious.

Let us now introduce a linear order $\prec=\prec_f$ in the sequence space A^N.

Definition 2. Let $\omega, \omega'\in A^N$. Then $\omega\prec\omega'$ iff $\omega=\omega'$ or there is a number n such that

> $\omega(i)=\omega'(i)$ for $0\leqq i<n$ and $\omega(n)<\omega'(n)$ when $\prod_{i=0}^{n-1}\varepsilon(\omega(i))$ is positive and $\omega(n)>\omega'(n)$ when it is negative,

where $\varepsilon(a)$ is $+1$ if f is non-decreasing on I_a and -1 if f is non-increasing on I_a. In a similar way we define the order $\prec=\prec_f$ for words (=finite sequences) $u=(a_0,\cdots,a_{n-1})$.

Remark. (i) The order defined above is "*non-anticipating*: if $\omega\neq\omega'$, then the order relation $\omega\prec\omega'$ is determined by the first n coordinates $\omega(i)$ and $\omega'(i)$, $i\leqq n$, for some n and is independent of the tails of coordinates $\omega(i)$ and $\omega'(i)$, $i>n$. It is easy to prove that the "non-anticipating" property of an order $\prec$ is equivalent to the fact that the upper (closed) segments $\{\omega;\ \omega\succ\omega_0\}$ and the lower segment $\{\omega;\ \omega\prec\omega_0\}$ are closed with respect to the product topology in A^N $(\omega_0\in A^N)$.

(ii) Let x and y be in J'. Then $x\leqq y$ iff $\pi(x)\prec\pi(y)$.

(iii) The following two statements are equivalent for any u and v in the set $W_n(f)$, $n\in \boldsymbol{N}$;

(a) $u\underset{\neq}{\prec}v$, i.e., $u\prec v$ and $u\neq v$.

(b) The interval $I_v{}^0$ lies on the right side of $I_u{}^0$.

LEMMA 2. *Let*

$$\bar{z}_a{}^n=\max\{u;\ u\in W_n(f),\ u(0)=a\}$$

$$\underline{z}_a{}^n=\min\{u;\ u\in W_n(f),\ u(0)=a\}$$

where max *and* min *are taken w.r.t. the order* $\prec$. *Then*

$$W_n(f)=\{u\in A^n;\ \underline{z}_{u(m)}^{n-m}\prec\sigma^m u\prec\bar{z}_{u(m)}^{n-m},\ m=1,\cdots,n-1\}$$

where

$$\sigma^m u=(u(m), u(m+1),\cdots,u(n-1))\qquad \text{if}\quad u=(u(0),\cdots,u(n-1))$$

Proof. The assertion is trivial for $n=1$. Assume that it is true for n and let us prove the relation for $n+1$. Take u from the set $A^{n+1} \cap W_{n+1}(f)^c$ such that $\sigma u \in W_n(f)$. Then $I^0_{\sigma u} \neq \emptyset$ and

$$fI^0_{u(0)} \cap I^0_{\sigma u} = \emptyset.$$

Thus $fJ^0_{u(0)}$ lies either on the left side or on the right side of the interval $fI^0_{\sigma u}$. According to it,

$$\text{either} \qquad u \underset{\neq}{\overset{\prec}{}} \underline{z}^{n+1}_{u(0)} \qquad \text{or} \qquad u \underset{\neq}{\overset{\succ}{}} \bar{z}^{n+1}_{u(0)}.$$

Consequently u does not belong to the set of the right-hand side. The inverse inclusion is obvious by the definition of $\bar{z}^n_a$'s.

Now we can prove the following structure theorem of the realization.

THEOREM 1. *Let*

$$X_f = \{\omega \in A^N;\ \underline{\zeta}_{\omega(n)} \prec \sigma^n \omega \prec \bar{\zeta}_{\omega(n)} \text{ for each } n \in N\},$$

where

$$\bar{\underline{\zeta}} = \lim \bar{\underline{z}}_a{}^n.$$

Then

(i) *The closure $\overline{\pi(J')}$ of $\pi(J')$ is the set X_f.*

(ii) $W_n(f) = \mathcal{W}_n(X_f) \equiv \{\omega[0, n);\ \omega \in X_f\}$

(iii) *The set X_f is a shift invariant closed set.*

Proof. The assertion (ii) follows immediately from Lemma 2 and it implies that the set $\pi(J')$ is dense in X_f. But the set X_f is closed by the definition of the order $\prec$ (See Remark (ii).) Hence we obtain (i). Finally the shift invariantness is obvious by the definition.

Let us construct a "map" $\rho: X_f \to J$. Precisely to say, it is defined as a set-valued function on X_f:

$$\rho(\omega) = \cap f^{-n} I_{\omega(n)}, \qquad \omega \in X_f. \tag{2}$$

First of all we shall show that $\rho(\omega)$ is a non-empty closed interval. If $\omega \in \pi(J')$, then the intersection of the sets $f^{-n} I^0_{\omega(n)}$, $n \in N$, is an interval, which contains, at least, the point x such that $\omega = \pi(x)$. Hence it follows from (ii) of of Theorem 1 that the intersection of the sets $f^{-n} I_{\omega(n)}$, $0 \leqq n < m$, is a non-empty closed interval for each m. Consequently the set $\rho(\omega)$ is also a non-empty closed interval.

THEOREM 2. *Let F_a be the right continuous version of the inverse function of the restriction of f to the interval I_a. Then the following properties are true:*

(i) $\rho(a\omega)=F_a(\rho(\omega))$ *and* $\rho(\sigma\omega)=f(\rho(\omega))$.

(ii) $\rho(\pi(x))\ni x$.

(iii) *The union of all* $\rho(\omega)$ *covers the interval* J.

Furthermore there exists a shift invariant subset X_f^0 *of* X_f *with the following two properties*:

(iv) *The set* $X_f \backslash X_f^0$ *is at most countable and the set* $\rho(\omega)$ *consists of a single point for each* ω *in* X_f^0. (*The point will be denoted by* $\rho(\omega)$, *too.*)

(v) *The map* $\rho: X_f^0 \to J$ *is continuous.*

Remark. The set-valued map $\rho: X_f \to J$ is continuous in the sense that $\rho(\omega)\supset \overline{\lim}\, \rho(\omega_n)$ if $\omega=\lim \omega_n$.

Proof. The assertions (i) and (ii) are obvious by definitions. To show (iii) take x in $J\backslash J'$. Let n be the smallest number for which $c=f^n x$ belongs to the set E, and x belong to I_u^0 for some u in $W_n(f)$. Take the infimum y of the intersection of the set I_u^0 and the connected component of $f^{-n}\{c\}$ containing x. Then,

$$y-\varepsilon \in J' \quad \text{and} \quad \pi(y-\varepsilon)[0, n)=\pi(x)[0, n)$$

for any sufficiently small positive number ε. Put

$$\omega=\sup \pi(y-\varepsilon).$$

Then the sequence ω belongs to X_f and $\rho(\pi(y-\varepsilon))\ni y-\varepsilon$ for small ε. Therefore the closed interval $\bigcap_{n=0}^{m-1} f^{-n} I_{\omega(n)}$ contains the point y for each m. Thus we obtain $y\in\rho(\omega)$ and, hence, $x\in\rho(\omega)$.

Now applying the Baire's theorem to (iii), we obtain at most countably many points ω such that $\rho(\omega)$ has an interior point. Since it is an interval, thus the set $\rho(\omega)$ consists of a single point unless it has an interior point. Consequently we obtain (iv). The continuity (v) follows from the fact that ρ is a monotone (set-valued) function of the ordered space $(X_f, \prec)$ to $(J, \leqq)$.

Corollary. If the lap number lap (f) *is finite, then the topological entropy* ent (J, f) *is given by the following formula*:

$$\text{ent}\,(J, f)=\lim \frac{1}{n} \log |W_n(f)|=\text{ent}\,(X_f, \sigma)$$

Here $|W|$ *is the number of points in the set* W.

Remark. $|W_n(f)|=\text{lap}\,(f^n)$.

Proof. Let μ be the invariant measure of the compact dynamics (X_f, σ) with the maximal entropy, i. e.,

$$h_\mu(X_f, \sigma) = \text{ent}\,(X_f, \sigma).$$

If the entropy is positive, then the probability measure μ is necessarily supported by the set X_f^0 since $X_f \backslash X_f^0$ is countable. In virtue of the injectivity of ρ on X_f^0 we get

$$\text{ent}\,(J, f) \geqq h_{\rho_*\mu}(J, f) = h_\mu(X_f, \sigma) = \text{ent}\,(X_f, \sigma).$$

When $h_\mu(X_f, \sigma)=0$, the above inequality is obvious.

Now let us show

$$\text{ent}\,(X_f, \sigma) \geqq \text{ent}\,(J, f)$$

Take any finite open cover $\mathcal{U}$ of J and let

$$\mathcal{U}_a = \{U \in \mathcal{U}\,;\, U \cap I_a \neq \emptyset\} \qquad (a \in A).$$

Consider an interval $I_u = \bigcap_{m=0}^{n-1} f^{-m} I_{a_m}$ $(n \geqq 1,\ u=(a_0, \cdots, a_{n-1}) \in W_n(f))$. Since f^m, $m=1, \cdots, n$, are monotone on I_u, thus I_u is covered by at most $\sum_{m=0}^{n-1}(|\mathcal{U}_{a_m}|+1)$ members of the cover $\bigvee_{m=0}^{n-1} f^{-m}\mathcal{U}$. Consequently

$$N(\bigvee_{m=0}^{n-1} f^{-m}\mathcal{U}) \leqq kn\, W_n(f),$$

where $k=\max\{|\mathcal{U}_a|+1\,;\, a \in A\}$, and we have

$$h(f, \mathcal{U}) \equiv \limsup \frac{1}{n} \log N(\bigvee_{m=0}^{n-1} f^{-m}\mathcal{U})$$

$$\leqq \limsup \frac{1}{n} \log |W_n(f)| = \text{ent}\,(X_f, \sigma).$$

Hence

$$\text{ent}\,(J, f) = \sup h(f, \mathcal{U}) \leqq \text{ent}\,(X_f, \sigma).$$

Remark. In the case when lap$(f)=2$ the structure of the set X_f is simpler. Let us assume that $fJ=J$. We may assume that $J=[0,1]$ and that $f(1)=0 \leqq f(0) \leqq f(c)=1$ for some c in $(0,1)$. Then $I_1=[0,c]$ and $I_2=[c,1]$ are lap intervals. Under this situation

$$\bar{\zeta}_2 = \max X_f, \qquad \sigma\bar{\zeta}_2 = \underline{\zeta}_1 = \min X_f, \qquad \sigma\underline{\zeta}_2 = \sigma\underline{\zeta}_1 = \bar{\zeta}_2$$

Consequently

$$X_f = \{\omega \in \{1, 2\}^N\,;\, \zeta \prec \sigma^n\omega \quad \text{for any } n \in N\}$$

where $\zeta = \underline{\zeta}_1.$

In particular, the set of realization spaces X_f of lap 2 transformations f is linearly ordered with respect to the inclusion order.

2. A formula for topological entropy

Let us begin with a brief summary of the method developped mainly in [1].

DEFINITION. A subshift is called *p-Markov* if there exists a subset W (*structure set*) of the product space A^{p+1} such that

(1) $$X=\mathcal{M}(W)\equiv\{\omega\in A^T;\ (\omega_n,\cdots,\omega_{n+p})\in W \text{ for each } n\}. \quad (T=\boldsymbol{N} \text{ or } \boldsymbol{Z})$$

Let X be an arbitrary shift invariant closed subset of A^N. Then $X^p=\mathcal{M}(W_{p+1})$ contains X and the intersection of all X^p's is X. In other words, every subshift can be approximated from above by Markov subshifts. Moreover

$$\text{ent}\,(X,\sigma)=\lim \text{ent}\,(X^p,\sigma).$$

LEMMA 1. (i) *If (X,σ) is a Markov subshift, then,*

(2) $$\text{ent}\,(X,\sigma)=\limsup \frac{1}{n}\log|\Gamma_n(X)|,$$

where

$$\Gamma_n(X)=\{\omega\in X;\ \sigma^n\omega=\omega,\ \sigma^m\omega\neq\omega \quad \text{for } 1\leqq m<n\},$$

(ii) *Let (X,σ) be a subshift and assume that there exist Markov subshifts (X_p,σ) such that*

(3) $$X_1\subset X_2\subset\cdots\subset X \quad \text{and} \quad \text{ent}\,(X,\sigma)=\sup \text{ent}\,(X_p,\sigma).$$

Then the equality (2) *holds for (X,σ).*

Proof. The statement (i) is known and found in many literatures but we give a proof for the self-containedness of the proof. It is sufficient to prove (i) for $p=1$ since the general case is reduced to this case by considering A^p in place of A. Define a matrix $M=(M_{ab})_{a,b\in A}$ by

$$M_{ab}=1 \quad \text{if } (a,b)\in W, \quad =0 \quad \text{if } (a,b)\notin W.$$

Then $|W_n(X)|=(M^{n-1}\mathbf{1},\mathbf{1})$ where $\mathbf{1}={}^t(1,\cdots 1)$ and so

$$\text{ent}\,(X,\sigma)=\lim\frac{1}{n}\log|W_n(X)|=\log\lambda,$$

where λ is the Perron-Frobenius eigenvalue of the nonnegative matrix M.

Note that, for any sufficiently small z in $\boldsymbol{C}$,

$$\det(E-zM)=\exp[\text{Tr}\log(E-zM)]$$
$$=\exp\left[-\sum_{n=1}^{\infty}\frac{z^n}{n}\text{Tr}\,M^n\right].$$

Since

$$\mathrm{Tr}\, M^n = |\{u \in W_{n+1}(X)\,;\ u(0)=u(n)\}| = \sum_{m|n} m|\Gamma_m(X)|,$$

thus,

$$\det(E - zM) = \Pi(1-z^m)^{|\Gamma_m(X)|/m}.$$

Consequently we obtain, by comparing the radii of convergence of both sides, that

$$\log \lambda = \limsup_{m\to\infty} \frac{1}{m} \log \frac{|\Gamma_m(X)|}{m} = \limsup \frac{1}{m} \log |\Gamma_m(X)|$$

Using the Markov hull $X^p = \mathcal{M}(W_{p+1}(X))$, we get

$$\limsup \frac{1}{n} \log |\Gamma_n(X)| \leqq \inf_p \limsup_{n\to\infty} \frac{1}{n} \log |\Gamma_n(X^p)|$$

$$= \inf_p \mathrm{ent}\,(X^p, \sigma) = \mathrm{ent}\,(X, \sigma)$$

for an arbitrary subshift (X, σ). Thus it suffices to show the inverse inequality under the assumption of (ii). But it is evident. In fact,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\Gamma_n(X)| \geqq \sup_p \limsup_{n\to\infty} \frac{1}{n} \log |\Gamma_n(X_p)|$$

$$= \sup_p \mathrm{ent}\,(X_p, \sigma) = \mathrm{ent}\,(X, \sigma).$$

DEFINITION 2. Let (X, σ) be a subshift. A subset B of $\mathcal{W}(X)$ is called *orbit basis* if the following map $\varphi: B^\theta \to X$ is bijective:

$$\varphi(i, \eta) = \sigma^i(\cdots, b_0, b_1, \cdots), \qquad \eta = (\cdots, b_0, b_1, \cdots),$$

where

$$B^\theta = \{(i, \eta)\,;\ \eta \in B^{\mathbf{Z}},\ i = 0, 1, \cdots, \theta(\eta) - 1\} \quad \text{and}$$

$$\theta(b_0, b_1, \cdots) = n \quad \text{if} \quad b_0 \in W_n(X).$$

In other words, a subshift (X, σ) admits an orbit basis B if and only if it is conjugate to the tower $(B^\theta, \sigma^\theta)$ over the full shift $(B^{\mathbf{Z}}, \sigma)$ with respect to the ceiling function θ, where

$$\sigma^\theta(i, \eta) = \begin{cases} (i+1, \eta) & \text{if } i+1 < \theta(\eta) \\ (0, \eta) & \text{if } i+1 = \theta(\eta). \end{cases}$$

Remark 1. The orbit basis is not necessarily unique in general.

Remark 2. Let (X, σ) be a *p-Markov* subshift, $u \in \mathcal{W}_p(X)$ and

$$X(u) = \{\omega \in X\,;\ \sigma^i \omega \in [u] \ \textit{for infinitely many } i\text{'s}\}.$$

Then the subshift $(X(u), \sigma)$ admit the orbit basis

$$B(u)=\{uv\in\mathcal{W}(X)\,;\ uvu\in\mathcal{W}_n(X),$$
$$(uvu)[i, i+p)\neq u \quad \text{for } 1\leqq i\leqq n-p-1 \text{ and } n\geqq 2p\}$$

Remark 3. If (X, σ) admits an orbit basis B, then $\exp[-\mathrm{ent}(X, \sigma)]$ is the smallest positive solution of the equation

$$(4) \qquad 1-\sum |B\cap\mathcal{W}_n(X)|t^n=0.$$

In fact, any word $w\in\mathcal{W}_n(X)$ is in one-to-one correspondence to the collection $(j, b_0, \cdots, b_k)$ such that $k\geqq 0$, $b_i\in\mathcal{W}_{n(i)}(X)\cap B$ $(0\leqq i\leqq k)$, $0\leqq j\leqq n(0)-1$, and $1\leqq n+j-n(0)-\cdots-n(k-1)\leqq n(k)$. Thus, for $t\geqq 0$, the series

$$\sum_n |\mathcal{W}_n(X)|t^n$$

converges if and only if the left hand side of (4) is positive.

LEMMA 2. *Let f be a continuous transformation of an interval J into itself. Assume that the realization (X_f, σ) is an irreducible p-Markov subshift for some p. Then there exist piecewise linear transformations (J_n, f_n) with the following three properties*:

(5)
(a) *There are cycles* $C_n=\{x_1{}^n<\cdots<x^n_{p(n)}\}$ *such that*
$f_n x_i{}^n=fx_i{}^n$ *for each* i, $J_n=[x_1{}^n, x^n_{p(n)}]$ *and*
f_n *is linear on each interval* $[x_i{}^n, x^n_{i+1}]$.

(b) $X_{f_n}\subset X_f$ *for each* n.

(c) $\mathrm{ent}(X_f, \sigma)=\sup \mathrm{ent}(X_{f_n})$.

Proof. Let us use the notation in Remark 2. Note that

$$\mathrm{ent}(X, \sigma)=\max\{\mathrm{ent}(X(u), \sigma)\,;\ u\in\mathcal{W}_n(X)\}, \quad X=X_f,$$

for any n. In fact, the nonwandering set of X is contained in the union of $X(u)$ and

$$X'(u)=X\cap\bigcap_n \sigma^n[u]^c,$$

and so in the union of $X(u)$, $X(u)(v)$ and $X'(u)'(v)$ etc. Finally it is contained in the union of $X(u)$, $u\in\mathcal{W}_n(X)$.

Next we may assume that $\mathrm{ent}(X, \sigma)=\mathrm{ent}(X(u), \sigma)$ for some u in $\mathcal{W}_q(X)\cap Z_q{}^c$, where $q\geqq p$ and

$$Z_q=\{\underline{\zeta}_a[n, n+q), \bar{\zeta}_a[n, n+q)\,;\ a\in A, n\geqq 0\}$$

In fact, if it is not true, then,

$$\mathrm{ent}\,(X, \sigma) = \mathrm{ent}\,(X_{\cap} \bigcap_i \sigma^i \bigcup_{Z_q} [v], \sigma)$$

for each q. Since (X, σ) is Markov, thus each ζ_a is a periodic sequence or falls into a periodic sequence under the iteration of σ. Hence Z_q is bounded and

$$\mathrm{ent}\,(X, \sigma) \leqq q^{-1} \log Z_q \to 0 \qquad \text{as } q \to \infty.$$

Let us denote

$$B_n = \{b \in B(u)\,;\ Z_n \cap \mathcal{W}_n(b) = \emptyset \quad \text{or} \quad b \in \mathcal{W}_m(X),\ m \leqq n-1\}$$

Then it follows from $u \in Z_q^c$ that, for $n \geqq q$,

$$\mathcal{W}_n(\varphi(B^0_{n-q})) \cap Z_n = \emptyset$$

Now it follows from the irreducible Markov property of (X_f, σ) that there are periodic sequences $\omega^n \in X_f$, $n \geqq 1$, such that

(6) $$\mathcal{W}_n(\omega^n) \supset Z_n.$$

Furthermore there are cycles C_n of (J, f) which are contained in the orbit of $\rho_f(\omega^n)$. Define the maps f_n by the conditions stated in (a) and denote the natural extension of X by $\bar{X}$. It then follows from (4) and (5)

$$\bar{X}_{f_n} \supset \mathcal{M}(\mathcal{W}_n(X_f)_\cap Z_n^c) \supset \varphi(B^0_{n-q}).$$

Hence, $\exp[-\mathrm{ent}\,(X_{f_{n+q}}, \sigma)]$ is smaller than the smallest positive solution x_n of the equation

$$1 - \sum |B_{n\cap} \mathcal{W}_m(X_f)| t^m = 0.$$

Recall that $x = \exp[-\mathrm{ent}\,(X_f, \sigma)] = \exp[-\mathrm{ent}\,(B^\theta, \sigma^\theta)]$ is the smallest positive zero x of the rational function

$$1 - \sum |B_\cap \mathcal{W}_m(X_f)| t^m.$$

Consequently, x_n converges to x. Hence (c).

LEMMA 3. *If (X_f, σ) is Markov, then,*

$$\mathrm{ent}\,(J, f) = \sup\{\mathrm{ent}\,(J_C, f_C)\,;\ C \in \Gamma(J, f)\}.$$

Proof. Obvious from Lemma 2 in virtue of the irreducible decomposition of X_f.

Next we shall show that the Markov hull of the realization is the realization of a Markov transformation.

LEMMA 4. *Let f be a continuous transformation of an interval J and (X, σ) be the realization of (J, f). Then, for each $p = 0, 1, \cdots$, there exists a continuous Markov transformation f_p of J such that*

$$X_{f_p} = X^p$$

where $X^p = \mathcal{M}(W_{p+1}(X))$ *the p-Markov hull of X.*

Proof. Let $\mathcal{W}_p(X) = \{u_1, \cdots, u_k\}$, $u_1 \prec \cdots \prec u_k$ $(k = |\mathcal{W}_p(X)|)$ and define a piecewise linear continuous transformation f_p by the following two conditions:

(a) f_p is linear on each subinterval I_{au_j} $(j=1, \cdots, k)$ if $au_j \in \mathcal{W}_{p+1}(X)$

(b) $f_p(\min I_{au_j}) = \min I_{u_j}$ or $\max I_{u_j}$,

$f_p(\max I_{au_j}) = \max I_{u_j}$ or $\min I_{u_j}$

according as $\varepsilon(I_{u_j}, f) = +1$ or -1, respectively. Then the relations

$$f_p I_u^0 \cap I_v^0 \neq \emptyset \quad \text{and} \quad f I_u^0 \cap I_v^0 \neq \emptyset$$

are mutually equivalent and so are equivalent to the condition: $u = (a_0, \cdots, a_{p-1})$ and $v = (a_1, \cdots, a_p)$ for some $(a_0, \cdots, a_p) \in \mathcal{W}_{p+1}(x)$. Hence $X_{f_p} = X^p$.

Now we can give the proof of the following.

THEOREM 3. *Let* (J, f) *be a continuous interval dynamics. Then,*

$$\text{(7)} \qquad \operatorname{ent}(J, f) = \sup\{h(C);\ C \in \Gamma(J, f)\}$$

where $h(C)$ *is the "entropy" of cycle C given by the formula* (5). *In particular, if the lap number of* (J, f) *is finite, then,*

$$\text{(8)} \qquad \operatorname{ent}(J, f) = \limsup \frac{1}{n} \log |\Gamma_n(J, f)|$$

Here $\Gamma_n(J, f)$ *is the totality of n-cycles of* (J, f) *and*

$$\Gamma(J, f) = \cup \Gamma_n(J, f).$$

Proof. It follows from Lemmas 3 and 4 that

$$\operatorname{ent}(X^p, \sigma) = \sup\{h(C);\ C \in \Gamma(X^p, \sigma)\}$$

Since the sets $\Gamma(X^p, \sigma)$ are nonincreasing in n and their intersection is $\Gamma(X, \sigma)$, thus,

$$\begin{aligned} \operatorname{ent}(X, \sigma) &= \inf \operatorname{ent}(X^p, \sigma) \\ &= \inf \sup\{h(C);\ C \in \Gamma(X^p, \sigma)\} \\ &= \sup\{h(C);\ C \in \Gamma(X, \sigma)\}. \end{aligned}$$

The latter assertion follows from (6) and Lemma 1 since Theorem 1 guarantees the existence of an increasing sequence of subshifts (X_p, σ) of (X, σ) such that

$$\Gamma(X, \sigma) = \cup \Gamma(X_p, \sigma)$$

if (X, σ) is not Markov, i.e., if the sequences ζ_a and $\bar{\zeta}_a$, $a \in A$, are not periodic.

Remark. The statements of Theorem 3 is also valid for other classes of one-dimensional dynamics. In the case of endomorphisms of the circle S^1 the degree of map f (or the homotopy type of f) must be prescribed. Then the formula (6) is valid if we define the function f_C used in the definition

$$h(C)=\mathrm{ent}\,(S^1, f_C)$$

by the condition that f_C is the piecewise linear continuous transformation which coincides with f on the set C and is homotopic to f. The definition of f_C is similar for the continuous transformations of branched manifolds.

In the case of number-theoretical transformations or, more generally, piecewise continuous transformations of intervals let us call the set

$$\{(x, f(x-0), f(x+0))\,;\ x\in J, f(x-0)\neq f(x+0)\}$$

type of a transformation f. Then the transformation f_C is define by the conditions:

(a) It is piecewise linear and piecewise continuous.
(b) It coincides with f on the set C.
(c) The type of f_C is the same as the type of f.

The proof of these assertions is done by such modification of the lemmas that the type of transformations should be kept unchanged. Since it is so close to the present one, thus we do not repeat it.

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