The Traces of Hecke Operators in the Space of the 'Hilbert Modular' Type Cusp Forms of Weight Two

By Hirofumi ISHIKAWA

College of Liberal Arts and Science, Okayama University
Tsushimanaka, Okayama 700, Japan

(Received January 19, 1979)
(Introduced by H. Shimizu)

Introduction

The purpose of the present note is to calculate the trace of Hecke operators acting on the space of cusp forms of weight two belonging to a Hilbert modular group over a totally real algebraic number field. Arthur [1] has given the proof of 'Selberg trace formula' for a reductive algebraic group over a number field F whose semi-simple component is of F-rank one. We shall apply his results to the group $SL(2)$ and $GL(2)$, and carry out the trace formula for Hecke operators.

§1 is concerned with preliminary statements. In §2, an explicit formula for the trace of $T(\Gamma\alpha\Gamma)$ in the space of cusp forms of weight two belonging to $SL_2(\mathfrak{o})$ will be given (Theorem 1). In §3, we shall give the trace of $T(UyU)$ in $S'=S_1\times\cdots\times S_h$, where S_ν is the cusp form space of weight two belonging to the discrete subgroup Γ_ν of $GL_2(\boldsymbol{R})\times\cdots\times GL_2(\boldsymbol{R})$ (Theorem 2). We shall apply Theorem 2 to the operator $T(\mathfrak{q})$ defined in Shimura [17] and give the trace of $T(\mathfrak{q})$ (Theorem 2′). In §§3.4–3.5, we shall give some application of Theorem 2′ and the numerical examples.

About the calculation of the trace of Hecke operators in the space of various cusp forms of the several variables, Shimizu [16] has given its explicit formula for the case of weight greater than two. Also for the case of automorphic forms of weight greater than or equal to two for a discontinuous group with a compact fundamental domain, it has been calculated in our previous paper [10]. Recently, the dimension formula for the space of 'Hilbert modular' type cusp forms of weight two belonging to a quadratic real number field is given by Hirzebruch [8]. The author would like to express his sincere thanks to Prof. Y. Ihara, Prof. H. Shimizu and to Prof. T. Shintani who encouraged him with many suggestions during the preparation of the paper.

Notation

If H, v stand for a F-subgroup of G and any place of F, H_F, H_v and H_A denote the groups of F-rational points, F_v-rational points and the adèlized group, respectively. H_∞, H_f denote the infinite part and the finite part of H_A $(H_A = H_\infty H_f)$.

$$n(x)(x \in \boldsymbol{R}),\ a(y),\ (y \in \boldsymbol{R}_+),\ k(\theta)(0 \leq \theta < 2\pi),\ h_t(t \in \boldsymbol{R}),\ w$$

denote the elements $\begin{bmatrix} 1, & x \\ 0, & 1 \end{bmatrix}$, $\begin{bmatrix} y^{1/2}, & 0 \\ 0, & y^{-1/2} \end{bmatrix}$, $\begin{bmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{bmatrix}$, $\begin{bmatrix} e^t, & 0 \\ 0, & e^{-t} \end{bmatrix}$ and $\begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix}$ in $SL_2(\boldsymbol{R})$. If, for $g \in SL_2(\boldsymbol{R})$, g decomposes into $n(x)a(y)k(\theta)$, we shall denote x, y, θ by $x(g), y(g), \theta(g)$, respectively. Denote by σ_m an irreducible representation of $SO_2(\boldsymbol{R})$ satisfying $\sigma_m(k(\theta)) = e^{-im\theta}$, $(m \in \boldsymbol{Z})$.

§ 1. Preliminaries

1.1. Let F be a totally real algebraic number field of finite degree n over $\boldsymbol{Q}$. Let F_v be the completion of F with respect to a valuation v in F and F_A the adèle ring of F. Denote by $\mathfrak{o}, E_0$ the ring of all integers in F and the group of units in $\mathfrak{o}$. For a prime ideal $\mathfrak{p}$ in $\mathfrak{o}$, let us denote by $\mathfrak{o}_\mathfrak{p}$ the valuation ring in $F_\mathfrak{p}$.

Let G be $SL(2)$ or $GL(2)$ which are considered algebraic groups defined over F. Now we shall define some F-subgroups of G. Let N, A and Z be the subgroup consisting of matrices $\begin{bmatrix} 1, & n \\ 0, & 1 \end{bmatrix}$ in G, the group of diagonal matrices in G and the center of G, respectively. Put $P = NA$, so that P is a minimal parabolic subgroup of G. Define a subgroup A_∞^+ of the identity component of A_∞ by $\left\{ \begin{bmatrix} a, & 0 \\ 0, & a^{-1} \end{bmatrix} \in A_\infty;\ a_1 = \cdots = a_n \in \boldsymbol{R}_+^\times \right\}$ $(G = SL(2))$ or by $\left\{ \begin{bmatrix} a, & 0 \\ 0, & d \end{bmatrix} \in A_\infty;\ a_1 = \cdots = a_n,\ d_1 = \cdots = d_n \in \boldsymbol{R}_+^\times \right\}$ $(G = GL(2))$, (a_i, d_i denoting the infinite components of a, d with respect to the infinite place v_i in F). Put $K_\mathfrak{p} = U_\mathfrak{p} = SL_2(\mathfrak{o}_\mathfrak{p})$ or $GL_2(\mathfrak{o}_\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ in $\mathfrak{o}$ and $K_{v_i} = SO_2(\boldsymbol{R})$ or $O_2(\boldsymbol{R})$ for an infinite place v_i, according to $G = SL(2)$ or $GL(2)$. Here K_v is a maximal compact subgroup of G_v and we say*) that $G_v = P_v K_v$. Set $K = \prod_v K_v$, $U = \{x \in G_A;\ x_\mathfrak{p} \in U_\mathfrak{p}, (\mathfrak{p} < \infty)\}$ and $U_0 = \{x \in U;\ x_{v_i} = 1, (1 \leq i \leq n)\}$. We can also say that $G_A = P_A K$.

We shall take as measures on G_A, A_A the Tamagawa measures, and on Z_∞, A_∞^+, the measures which correspond to the Euclidian measures on $(\boldsymbol{R}^\times)^n$, and on $(\boldsymbol{R}_+^\times)^\varepsilon$ ($\varepsilon = 1$ or 2, according to $SL(2)$ or $GL(2)$). Let dn be the Haar measure on N_A which makes the measure of $N_F \backslash N_A$ equal to one. As measure on K, we shall choose the normalized Haar measure dk. Define the right and the left Haar measures on P_A by

*) Bruhat, F., $\mathfrak{p}$-adic groups, in Algebraic Groups and Discontinuous groups, A.M.S., 1966.

$$\int_{P_A} h(p)d_r p = \int_{N_A}\int_{A_A} h(na)dnda$$

$$(h \in C_c^\infty(P_A)) .$$

$$\int_{P_A} h(p)d_l p = \int_{A_A}\int_{N_A} h(an)dadn$$

There is a homomorphism $\delta_P(p)$ of P_A to $\boldsymbol{R}_+^\times$ such that $d_r p = \delta_P(p)d_l p$, which is given by $|a/d|_A$ $\left(p=\begin{bmatrix} a, & b \\ 0, & d\end{bmatrix}\right)$. Put $H(p) = \frac{1}{2}\log \delta_P(p)$. Besides we can extend H to the mapping of G_A to $\boldsymbol{R}$ such that $H(x)=H(p)$ if $x=pk$.

We denote by $L^2(Z_\infty G_F\backslash G_A)$ the space of the square integrable functions over $Z_\infty G_F\backslash G_A$. Let us define the subspace $L_0^2(Z_\infty G_F\backslash G_A)$ of $L^2(Z_\infty G_F\backslash G_A)$, which consists of functions h satisfying

$$\int_{N_F\backslash N_A} h(nx)dn = 0 , \quad \text{for almost all } x \text{ in } G_A .$$

Also we denote by $L_0^2(\{P\})$, $L_1^2(\{P\})$ the space of the residues of the Eisenstein series and the continuous part of $L^2(Z_\infty G_F\backslash G_A)$, respectively. According to ([1], p. 344), $L^2(Z_\infty G_F\backslash G_A)$ decomposes into the direct sum of the above three subspaces.

Let λ be the right regular representation of $Z_\infty\backslash G_A$ on $L^2(Z_\infty G_F\backslash G_A)$. Suppose f is a complex valued function on $Z_\infty\backslash G_A$ satisfying the following assumption.

ASSUMPTION 1. ([1], Assumption 3.5) *f is the convolution of a left K-finite function f_1 and a right K-finite function f_2 such that the function f_i is of form $f_\infty f_f$, $f_f = \prod_{\mathfrak{p}<\infty} f_\mathfrak{p}$, where f_∞ is an infinitely differentiable function with compact support on $Z_\infty\backslash G_\infty$, where $f_\mathfrak{p}$ is a locally constant function with compact support on G and where, for almost all $\mathfrak{p}$, $f_\mathfrak{p}$ is equal to the characteristic function of $U_\mathfrak{p}$.*

Now we define an operation $\lambda(f)$ of $L^2(Z_\infty G_F\backslash G_A)$ by

$$\lambda(f)h(y) = \int_{Z_\infty\backslash G_A} f(x)(\lambda(x)h)(y)dx, \ (h \in L^2(Z_\infty G_F\backslash G_A)) .$$

Also according to ([1], Corollary 2.10), the subspaces $L_0^2(Z_\infty G_F\backslash G_A)$, $L_0^2(\{P\})$ and $L_1^2(\{P\})$ are $\lambda(f)$-invariant. Let $\lambda_0(f)$ be the restriction of $\lambda(f)$ to $L_0^2(Z_\infty G_F\backslash G_A)\oplus L_0^2(\{P\})$. $\lambda(f)$ is written as an integral operator with a kernel $K(f; x, y)$, namely,

$$(1.1) \quad \begin{aligned} (\lambda(f)h)(x) &= \int_{Z_\infty G_F\backslash G_A} K(f; x, y)h(y)dy \\ K(f; x, y) &= \sum_{\gamma \in Z_F\backslash G_F} f(x^{-1}\gamma y) . \end{aligned}$$

Here, we see that the summation of K is finite if x, y lie in fixed compact subsets of $Z_\infty\backslash G_A$.

Next we shall give expression of $\lambda_0(f)$. Let Φ denote the space of square integrable functions ϕ on $Z_\infty N_A A_F A_\infty^+ \backslash G_A$. For a character χ of $Z_\infty A_F A_\infty^+ \backslash A_A$, $\Phi(\chi)$ denotes the subspace of Φ which consists of the functions ϕ satisfying the equation $\phi(ax)=\chi(a)\phi(x)$ for $a \in A_A$. If X is the group of characters of $Z_\infty A_F A_\infty^+ \backslash A_A$, Φ decomposes to the direct sum of $\Phi(\chi)(\chi \in X)$. For an irreducible representation τ of K, we denote by $\Phi(\chi, \tau)$ the subspace of $\phi(\chi)$ on which the right regular representation of K is equivalent to τ. Then the dimension of $\Phi(\chi, \tau)$ is finite and $\Phi(\chi)$ is the direct sum of $\Phi(\chi, \tau)$. According to the above decomposition to the direct sum, we fix an orthonormal basis $\{\phi_\alpha\}_{\alpha \in A_\chi}$ of $\Phi(\chi)$. For any $z \in \boldsymbol{C}$, there is a representation $\pi(z)$ of $Z_\infty \backslash G_A$ on Φ defined by

$$(\pi(z, y)\phi)(x)=\phi(xy)e^{(z+1)H(xy)}e^{-(z+1)H(x)} .$$

Also define an operation $\pi(z, f)$ on Φ by

$$\pi(z, f)\phi=\int_{Z_\infty \backslash G_A} f(y)\pi(z, y)\phi dy .$$

For $\phi \in \Phi$, $z \in \boldsymbol{C}$ (Im $(z)>1$), $x \in G_A$, we define the Eisenstein series $E(\phi, z, x)$ associated to ϕ. Put

$$E(\phi, z, x)=\sum_{\delta \in P_F \backslash G_F} \phi(\delta x)e^{(z+1)H(\delta x)} . \tag{1.2}$$

Then $E(\phi, z, x)$ converges uniformly for x in a compact set of $Z_\infty \backslash G_A$ and z in a compact set of $\{z \in \boldsymbol{C}; \text{Im}(z)>1\}$. If $M(z)$ is an analytic function of $\{z \in \boldsymbol{C}; \text{Im}(z)>1\}$ to the space of linear operators on Φ defined by the Fourier expansion of $E(\phi, z, x)$ (see Appendix), it is known that $E(\phi, z, x)$ and $M(z)\phi$ can be continued to meromorphic functions on $\boldsymbol{C}$. Put

$$\begin{aligned} K_1(f; x, y)&=\frac{1}{4\pi}\sum_{\chi \in X}\int_{-i\infty}^{i\infty}\sum_{\alpha, \beta \in A_\chi}(\pi(z, f)\phi_\beta, \phi_\alpha)E(\phi_\alpha, z, x)\overline{E(\phi_\beta, z, y)}d|z| \\ K_0(f; x, y)&=K(f; x, y)-K_1(f; x, y) \end{aligned} \tag{1.3}$$

Because of Assumption 1, the summation over A_χ is finite.

THEOREM A. ([1], Theorem 3.6 & 3.9). *$\lambda_0(f)$ is of trace class and the kernel $K_0(f; x, y)$ is integrable over the diagonal of $Z_\infty G_F \backslash G_A \times Z_\infty G_F \backslash G_A$ and its integral equals the trace of $\lambda_0(f)$.*

Combining Theorem A with ([6], § 1.2.3 Lemma), we have

THEOREM B. *$L_0^2(Z_\infty G_F \backslash G_A)$ decomposes into the direct sum of countably many invariant subspaces of irreducible unitary representations. Each irreducible representation enters into $L_0^2(Z_\infty G_F \backslash G_A)$ with a finite multiplicity.*

THEOREM C. ([1], § 9). *The trace of $\lambda_0(f)$ is the sum of the following terms;*

(i) measure $(Z_\infty G_F\backslash G_A)f(1)$,

(ii) $\sum_{\gamma\in\{G_e\}}\int_{Z_\infty G(\gamma)_F\backslash G_A} f(x^{-1}\gamma x)dx$,

(iii) $-\frac{c}{2}\sum_{\substack{\mu\in Z_F\backslash A_F\\ \mu\notin Z_F}}\int_K\int_{N_A} f(k^{-1}n^{-1}\mu nk)H(wn)dndk$,

(iv) $\lim_{z\to 0}\frac{d}{dz}\{z\theta(z,f)\}$,

(v) $\frac{1}{4\pi}\sum_{\chi\in X}\int_{-i\infty}^{i\infty}\mathrm{trace}_{\Phi(\chi)}\, M(-z)\left(\frac{d}{dz}M(z)\right)\pi(z,f)d|z|$.

(vi) $-\frac{1}{4}\mathrm{trace}\, M(0)\pi(0,f)$,

where

$$\theta(z,f)=\int_{Z_\infty G_F\backslash G_A}\sum_{\delta\in P_F\backslash G_F}\sum_{v\in N_F,v\neq 1} f(g^{-1}\delta^{-1}v\delta g)e^{-2H(\delta g)z}dg\,.$$

The notations are defined as follows. Let G_e be a subset of G_F consisting of elements which are not G_F-conjugate to any element in P_F. For x', x in G_F, x, x' will be called equivalent if $x'=zgxg^{-1}$ for some $z\in Z_F$, $g\in G_F$. $\{G_e\}$ denotes a fixed set of representatives of equivalence classes in G_e. The positive constant c is defined by the equation

$$\int_{Z_\infty\backslash G_A}h(x)dx=c\int_K\int_{Z_\infty\backslash P_A}h(pk)d_l p dk,\quad (h\in C_c^\infty(Z_\infty\backslash G_A))\,.$$

1.2. Let H be a space of continuous functions h on $Z_\infty G_F\backslash G_A$ satisfying the following conditions

(H.1) $h(xk)=\prod_{i=1}^n\sigma_2(k_{v_i})h(x)$, for all $k\in\prod_{i<\infty}K_i\prod_{i=1}^n SO_2(F_{v_i})$,

(H.2) for any compact set C of $Z_\infty G_F\backslash G_A$ and for any constant $c>0$, there exist constants c_1, c_2 such that $\left|h\left(\begin{bmatrix}a, 0\\ 0, d\end{bmatrix}x\right)\right|\leq c_1|a/d|_A^{c_2}$ for all $x\in C$ and for all $\begin{bmatrix}a, 0\\ 0, d\end{bmatrix}\in A_A$, $|a/d|_A\geq c$.

(H.3) $\lambda(D_{v_i})h=0$, where $D_{v_i}=(0,\cdots,0,D,0,\cdots,0)$ is an element of the universal enveloping algebra of $\mathfrak{G}_\infty\otimes \boldsymbol{C}$ $\Big(\mathfrak{G}_\infty$ being Lie algebra of $GL_2(F_\infty)$, $D=X_1^2+X_2^2-X_3^2$, $X_1=\begin{bmatrix}1, & 0\\ 0, & -1\end{bmatrix}$, $X_2=\begin{bmatrix}0, 1\\ 1, 0\end{bmatrix}$, $X_3=\begin{bmatrix}0, 1\\ -1, 0\end{bmatrix}\Big)$, (H.4) $\int_{N_F\backslash N_A} h(nx)dn=0$, for almost all x.

By Godement**), H coincides with the space of all K-finite functions in $L_0^2(Z_\infty G_F\backslash G_A)$ satisfying the conditions (H.1, 3). For every y in G_A, yUy^{-1} is commensurable with U; so we can define the Hecke ring $R(U, G_A)$ which is a free $\boldsymbol{Z}$-module generated by all UyU $(y\in G_A)$ with a structure of ring. For UyU in $R(U, G_A)$, define a linear operator $T(UyU)$ in H by

$$T(UyU)h(x)=\sum_i h(xz_i)\,, \tag{1.4}$$

where $\{z_i\}$ is a system of representatives for the right cosets of U in UyU. We identify $R(U, G_A)$ with the tensor product of the rings $R(U_\mathfrak{p}, G_\mathfrak{p})$ by the correspondence:

$$UyU\longleftrightarrow\otimes U_\mathfrak{p} y_\mathfrak{p} U_\mathfrak{p}\,.$$

For UyU in $R(U, G_A)$, we fix $f_\mathfrak{p}$ once for all, the characteristic function of $U_\mathfrak{p} y_\mathfrak{p} U_\mathfrak{p}$, and f_f for $\prod_{\mathfrak{p}<\infty} f_\mathfrak{p}$.

An irreducible unitary representation r of $Z_\infty\backslash G_A$ is regarded as a tensor product representation of irreducible representations r_∞ of $Z_\infty\backslash G_\infty$ and r_f of G_f. Moreover r_∞ is given by the tensor product representation of each irreducible representation of $Z_{v_i}\backslash G_{v_i}$ $(1\leq i\leq n)$, whose classification has been given by Bargmann (c.f. ([9], Proposition 1)). For a representation r, χ_r will denote its character. If f satisfies Assumption 1, then $\chi_r(f)=\chi_{r_\infty}(f_\infty)\chi_{r_f}(f_f)$. Set $I=\{1, 2, \cdots, n\}$. For a subset J in I, $R_\infty(J)$ will be denoted the subset of equivalent classes of irreducible representations $r_\infty=\otimes r_j$ of $Z_\infty\backslash G_\infty$ such that r_j is contained in the discrete series or in the principal or supplemental series, according as j is contained in J or not. For $J\subseteq I$, we shall call that a function $f_\infty\in C_c^\infty(Z_\infty\backslash G_\infty)$ is of type J if f_∞ satisfies the condition: for any $k=(k_{v_i})$, $k'=(k'_{v_i})$ in $\prod_{i=1}^n SO_2(F_v)$,

$$f_\infty(k_\infty x_\infty k'_\infty)=\prod_{j\in J}\sigma_2(k_{v_j}k'_{v_j})^{-1}f_\infty(x_\infty)\,.$$

From ([9] Proposition 4) it follows that unless J contains J', $\chi_{r_\infty}(f_\infty)$ vanishes for any r_∞ in $R_\infty(J')$ and for any function f_∞ of type J, and that, for $r_\infty=(r_j)\in R_\infty(I)$, when $\chi_{r_\infty}(f_{\infty,I})$ does not vanish, a function φ which is contained in the representation space satisfies $\lambda(D)\varphi=0$. Up to equivalence, there exists uniquely such r_∞ (in the case $G=SL(2)$) or the restriction of such r_∞ into $Z_\infty\backslash(G_\infty)_+$ (in the case $G=GL(2)$). We denote by s_∞ such an irreducible representation in $R_\infty(I)$.

Now we shall consider a system of functions $\{f_J\}$ for all $J\subseteq I$.

Assumption 2. *$\{f_J\}$ satisfies the following conditions,*

**) Godement, R., Notes on Jaquet-Langland's theorey (§ 3.1). Lecture note, Institute for Advanced Study, Princeton, 1970.

(i) $f_J(x)=f_{\infty,J}(x_\infty)f_f(x_f)$ *satisfies Assumption* 1; *moreover* $f_{\infty,J}$ *is of type* J *and* f_f *is determined by* UyU *as previously stated.*

(ii) $\chi_{r_\infty}(f_{\infty,J_1})=\chi_{r_\infty}(f_{\infty,J_2})$, *for any* J_1, J_2 *and* $J'\subseteq J_1\cap J_2$ *and for any* $r_\infty\in R_\infty(J')$.

(iii) *In the case* $G=GL(2)$, $f_{\infty,J}(x_\infty)=f_{\infty,J}((\det x_\infty)^{-1/2}x_\infty)$ *for* $x_\infty\in(G_\infty)_+$, *and* $f_{\infty,J}=0$ *outside* $(G_\infty)_+$.

For $J\subseteq J$, set sign $(J)=(-1)^n(-1)^{|J|}$, where $|J|$ denotes the number of elements in J.

Now we consider the trace of the alternating sum of operator sign $(J)\lambda_0(f_J)$ over all subsets J in I.

PROPOSITION 1. *If* $\{f_J\}$ *satisfies Assumption* 2, *we have*

$$\sum_{J\subseteq I}\text{sign}\,(J)\,\text{trace}\,\lambda_0(f_J)=\chi_{s_\infty}(f_{\infty,I})\,\text{trace}_H\,T(UyU) \\ +(-1)^n\,\text{trace}\,(\lambda_0(f_\phi)|L_0^2(\{P\})\,, \tag{1.5}$$

where J *runs over all subsets of* I.

Proof. Let $R(J)$ be the subset of equivalent classes of irreducible representations $r=r_\infty\otimes r_f$ of $Z_\infty\backslash G_A$ such that r_∞ is contained in $R_\infty(J)$. For $r\in R(J)$, we denote by m_r the multiplicity with which r enters in $L_0^2(Z_\infty G_F\backslash G_A)$. By Theorem B, we have

$$\text{trace}\,(\lambda_0(f_J)|L_0^2(Z_\infty G_F\backslash G_A))=\sum_{J'\subseteq J}\ \sum_{r=r_\infty\otimes r_f\in R(J')} m_r\chi_{r_\infty}(f_{\infty,J})\chi_{r_f}(f_f)\,.$$

Because of Assumption 2 (ii), we get

$$\sum_{J\subseteq I}\text{sign}\,(J)\,\text{trace}\,(\lambda_0(f_J)|L_0^2(Z_\infty G_F\backslash G_A))=\sum_{s\in R(I)} m_s\chi_s(f_I)\,.$$

Suppose $\chi_s(f_I)\neq 0$. If V_s stands for the representation space of s in $L_0^2(Z_\infty G_F\backslash G_A)$, then $\lambda(D_{v_i})\varphi=0$ for $\varphi\in V_s$ ([10], Proposition 1). As $f_I(kxk')=\prod_j \sigma_2(k_{v_j}k'_{v_j})^{-1}f_I(x)$ for $k, k'\in\prod_{\mathfrak{p}} K_{\mathfrak{p}}\prod_j SO_2(F_{v_j})$, we only consider the trace $\lambda_0(F_I)$ in the subspaces V_s' consisting of φ in V_s such that $\lambda(k)\varphi=\prod_j \sigma_2(k_{v_j})\varphi$ for all k in $\prod_{\mathfrak{p}} K_{\mathfrak{p}}\prod_j SO_2(F_{v_j})$. But, in the case $G=GL(2)$, it follows from Assumption 2 (iii) that the value of $\chi_{s_\infty}(f_{\infty,I})$ does not depend on the choice of s_∞. Thus we have

$$\sum_{s\in R(I)} m_s\chi_s(f_I)=\text{trace}_H\,\lambda_0(f_I)=\chi_{s_\infty}(f_{\infty,I})\,\text{trace}_H\,T(UyU)\,.$$

On the other hand, we can see that the trace of $\lambda_0(f_J)$ in $L_0^2(\{P\})$ does not vanish if and only if J is the empty set.

Q.E.D.

§ 2. The trace of Hecke operators for $\Gamma = SL_2(\mathfrak{o})$

2.1. Throughout this section, we shall be dealing with $G=SL(2)$. We shall only consider the case $F \neq \boldsymbol{Q}$. We denote by h the class number of F. We can regard an element in G_F as an element in G_∞ through the projections of G_F into each components of G_∞ with respect to infinite places. Then we can see that the projection of Γ into G_∞ is a discrete subgroup of G_∞; we also denote by Γ its projection. The next proposition follows from the strong approximation theorem (c.f. [4]).

PROPOSITION 2. *There is a canonical bijection between* $G_F \backslash G_A / U_0$ *and* $\Gamma \backslash G_\infty$.

Let $\mathfrak{H}^n$ be the direct product of n upper half planes. G_∞ will act on $\mathfrak{H}^n$ as the linear fractional transformation on each component. dz denotes an invariant measure on $\mathfrak{H}^n$ defined by $\prod_j dx_j dy_j / y_j^2$ $(z_j = x_j + iy_j)$. By a classical cusp form of weight two belonging to Γ, we understand a holomorphic function h on $\mathfrak{H}^n$ satisfying the following conditions;

(S.1) $h(\gamma z) = j(\gamma, z)^{-1} h(z)$ for $\gamma \in \Gamma$,

(S.2) $h(z)$ is regular at every parabolic point κ of Γ, and a constant term in the Fourier expansion of h at κ vanishes.

Here $j(x, z)$ denotes $\prod_{j=1}^{n} (c_j z_j + d_j)^{-2}$ for $x = (x_{v_j}) \in G_\infty$ $\left(x_{v_j} = \begin{bmatrix} a_j, & b_j \\ c_j, & d_j \end{bmatrix} \right)$. The linear space consisting of all h is denoted by S. For an element α in G_∞ such that $\alpha \Gamma \alpha^{-1}$ is commensurable with Γ, we define a linear operator $T(\Gamma \alpha \Gamma)$ in S, say Hecke operator, by

$$(2.1) \qquad T(\Gamma \alpha \Gamma) h(z) = \sum_{\mu=1}^{d} j(\alpha_\mu^{-1}, z) h(\alpha_\mu^{-1} z), \quad \left(\Gamma \alpha \Gamma = \bigcup_{\mu=1}^{d} \alpha_\mu \Gamma \right).$$

Using Proposition 2, for $h \in S$, we can define a function h' of H by $h'(x) = h(x_\infty(i)) j(x_\infty, i)$ if $x = g x_\infty u$ for $g \in G_F$, $x_\infty \in G_\infty$, $u \in U_0$. Thus we can establish the following proposition.

PROPOSITION 3. *The map* ι *from* h *to* h' *describes an isomorphism between* S *and* H *as linear spaces over* $\boldsymbol{C}$.

By ([17] Proposition 1.4), we have

LEMMA 1. *Let* α *be an element of* G_F. *Then we have* $\Gamma \alpha U = U \alpha U$.

For UyU (§ 1.2), we take an element α in G_F and any z in UyU satisfying $z \in \alpha U$. It follows from Lemma 1 that $\Gamma \alpha \Gamma$ is uniquely determined by UyU independent of the choice of z and α. As $\alpha \Gamma \alpha^{-1}$ is commensurable with Γ for $\alpha \in G_F$, an operator $T(UyU)$ in H induces an operator $T(\Gamma\alpha\Gamma)$ in S; and the action on S induced by $T(UyU)$ through ι coincides with the action of $T(\Gamma\alpha\Gamma)$. Therefore the calculation of the trace of $T(\Gamma\alpha\Gamma)$ in S is referable to that of the trace of $T(UyU)$ in H. On the other hand, because of Proposition A-1, the trace of $\lambda_0(f_\phi)$ in $L_0^2(\{P\})$ equals the trace in the constant function field.

Next, we construct a "convenient" system $\{f_{\infty,J}\}$ of functions on $Z_\infty\backslash G_\infty$ satisfying Assumption 2. Let ϕ_j $(1\leq j\leq n)$ be an entire function of exponential type on $\boldsymbol{C}$ satisfying the following condition:

(i) $\phi_j(z)$ is rapidly decreasing uniformly in every strip of finite width which is parallel to the imaginary axis when the imaginary part of z tends to infinity,

(ii) $\phi_j(-z)=\phi_j(z)$, $\phi_j(1/2)\neq 0$.

(From ([13], V-§3), it follows that an entire function satisfying the above conditions is obtained by the Mellin transform of an infinite differentiable function h with compact support such that $h(a)=h(a^{-1})$ on $\boldsymbol{R}^+$). Put

$$g_j(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_j(i\rho)e^{-i\rho u}d\rho\,,\quad g_j[\cosh(u)]=2g_j(u)\,,\quad (u\geq 0)\,.$$

For $e=0$ or 2, put

$$f_{e,j}[x]=(-1/\sqrt{2}\,\pi)\int_x^{\infty}g_j'[y]T_e(x,y)\sqrt{y-x}^{\,-1}dy\,,\quad (x\geq 1)\,,$$

$$T_e(x,y)=\begin{cases}1, & \text{if } e=0\\ 2\dfrac{y+1}{x+1}-1, & \text{if } e=2\,.\end{cases}$$

For $x=k(\theta)h_tk(\phi)\in SL_2(\boldsymbol{R})$ $(t\geq 0)$, put

$$f_{e,j}(x)=e^{ie(\theta+\phi)}f_{e,j}[\cosh(2t)]\,.$$

Note that $f_{e,j}$ does not depend on the decomposition of x. For $x=(x_{vj})\in G_\infty$ and for $J\subseteq I$, put

$$f_{\infty,J}(x_\infty)=\prod_{j\in J}f_{2,j}(x_{vj})\prod_{j'\notin J}f_{0,j'}(x_{vj'})\,. \tag{2.2}$$

The integral operator with the kernel $f_{e,j}$ has an eigenfunction $e^{-ie\theta(x)}y(x)^{1/2+z}$ if z is a complex number. Thus we regard its eigenvalue as a function of z, say Selberg's transformation of $f_{e,j}$.

LEMMA 2. *Selberg's transformation of $f_{e,j}$ is given by ϕ_j.*

Proof. We consider the following formula:

$$\int_{SL_2(\boldsymbol{R})} f_{e,j}(g^{-1}g')e^{ie(\theta(g)-\theta(g'))}y(g')^{1/2+z}y(g)^{-1/2-z}dg'$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{e,j}(n(x)h_t)e^{2tz-t}dxdt\,.$$

Choose elements $k(\theta), h_\tau, k(\psi)$ satisfying $n(x)h_t=k(\theta)h_\tau k(\psi)$. Then we have

$$\cosh(2\tau)=2\left|\cosh(t)+i\frac{x}{2}e^{-t}\right|^2-1\,,\quad e^{i(\theta+\psi)}=\frac{\cosh(t)-i\frac{x}{2}e^{-t}}{\left|\cosh(t)-i\frac{x}{2}e^{-t}\right|}\,.$$

By the definition of $f_{e,j}[x]$, its integral equals

$$\int_{-\infty}^{\infty}\left\{\sqrt{2}\int_{\cosh(2\tau)}^{\infty} f_{e,j}[x]T_e(x,\cosh(2\tau))\sqrt{x-\cosh(2\tau)}^{-1}dx\right\}e^{2\tau z}d\tau\,.$$

But the inverse formula of $g_j[y]$ from $f_{e,j}[x]$ is given by

$$g_j[y]=\sqrt{2}\int_y^{\infty} f_{e,j}[x]T_e(x,y)\sqrt{x-y}^{-1}dx\,.$$

Therefore its integral equals $\int_{-\infty}^{\infty} g_j[\cosh(2t)]e^{2tz}dt$. By the inverse formula of the Fourier transform, it becomes $\psi_j(z)$. Q.E.D.

Because of ([3] Theorem 2.1), $f_{e,j}$ is an infinitely differentiable function on $SL_2(\boldsymbol{R})$ with compact support. Thus $\{f_J=f_{\infty,J}f_f\}$ satisfies Assumption 2. Further we have

$$\chi_{s_\infty}(f_{\infty,J})=\chi_{id.}(f_{\infty,\phi})=\prod_j \psi_j\left(\frac{1}{2}\right).$$

2.2. The purpose of this subsection is to calculate the left hand side of (1.5), using Theorem C. Firstly we consider the non-parabolic part (Theorem C (i), (ii)). Define an equivalence relation of elements of $\Gamma\alpha\Gamma$ by

$$x\sim x' \longleftrightarrow x'=\pm\gamma x\gamma^{-1}\quad\text{for } \gamma\in\Gamma\,.$$

Let $[x]$ denote an equivalent class in $\Gamma\alpha\Gamma$ containing x. For $g\in G_\infty$, put $\Gamma(g)=\{\gamma\in\Gamma\,;\,g=\pm\gamma g\gamma^{-1}\}$.

Case i). The contribution from the singular part to (1.5) is

$$\text{measure}\,(G_F\backslash G_A)\sum_{J\subseteq I}\text{sign}\,(J)f_J(1)$$

By the definition of f_f, it does not vanish only if $UyU=U$ Combining ([6], § 5.5 (2)) with Lemma A-3, we get

LEMMA 3.

$$f_{e,j}(1)=\frac{1}{4\pi}\left\{\int_{-\infty}^{\infty}\phi_j(i\rho)\tanh(\pi\rho)d\rho+(e/2)\phi_j\left(\frac{1}{2}\right)\right\}.$$

Therefore

$$\text{measure}\,(G_F\backslash G_A)\sum_{J\subset I}\text{sign}\,(J)f_{\infty,J}(1)=\text{measure}\,(G_F\backslash G_A)(4\pi)^{-n}\prod_j\phi_j\left(\frac{1}{2}\right).$$

On the other hand, measure $(G_F\backslash G_A)$ equals the volume of the fundamental domain of Γ in $\mathfrak{H}^n$ with respect to dz. By ([15], No. 29 (53)), making use of the zeta function ζ_F and of the discriminant D_F of F, we express the volume in the form

$$(2/\pi^n)|D_F|^{3/2}\zeta_F(2).$$

Case ii). In view of the definition of f_f, the contribution from the elliptic part to (1.5) is written in the form

$$\sum_{J\subset I}\text{sign}\,(J)\sum_{[g]_{g\in B'}}\int_{\Gamma(g)\backslash G_\infty}f_{\infty,J}(x^{-1}gx)dx,$$

where B' denotes a subset of $\Gamma\alpha\Gamma$ consisting of elements g such that no fixed point of g is a parabolic point of Γ. An element g in B' can be classified to one of the following types: (ii-1) g is elliptic, (ii-2) g is hyperbolic and no fixed point of g is a parabolic point of Γ, (ii-3) g is mixed. (We say the an element g in G_∞ is elliptic, hyperbolic or parabolic as all g_j are of the corresponding types in the usual sense. If g belongs to none of the above types and $g\neq\pm1$, we say that g is mixed).

Case ii-1). Combining ([6], § 5.4 (4)), with Lemma A-1, we get

LEMMA 4. *Let g be an elliptic element of $G_v=SL_2(\boldsymbol{R})$ in the usual sense. Then*

$$\int_{G_v(g)\backslash G_v}f_{e,j}(x^{-1}gx)dx$$
$$=(2\sin\theta_g)^{-1}\left\{\int_{-\infty}^{\infty}\phi_j(i\rho)\frac{\cosh((\pi-2\theta_g)\rho)}{2\cosh(\pi\rho)}d\rho+i(e/2)e^{i\theta_g}\phi_j\left(\frac{1}{2}\right)\right\}.$$

where θ_g satisfies $\dfrac{gz-z_0}{gz-\bar{z}_0}=e^{2i\theta_g}\dfrac{z-z_0}{z-\bar{z}_0}$, ($z_0$ being the fixed point of g in the upper half plane).

Let $\{B_e\}$ be a complete system of inequivalent elliptic elements in B'. The contribution from this part is referred to

$$\sum_{J\subset I}\operatorname{sign}(J)\sum_{[g]\in\{B_e\}}(\Gamma(g):\{\pm1\})^{-1}\int_{G_\infty(g)\backslash G_\infty} f_{\infty,J}(x^{-1}gx)dx$$
$$=\sum_{[g]\in\{B_e\}}(\Gamma(g):\{\pm1\})^{-1}\prod_{j=1}^{n}\frac{ie^{i\theta(g_j)}}{2\sin\theta(g_j)}\prod_{j=1}^{n}\phi_j\Big(\frac{1}{2}\Big)$$
$$=\sum_{[g]\in\{B_e\}}(\Gamma(g):\{\pm1\})^{-1}\prod_{j=1}^{n}\frac{\eta_j}{\zeta_j-\eta_j}\prod_{j=1}^{n}\phi_j\Big(\frac{1}{2}\Big),$$

where η_j, ζ_j are the eigenvalues of g_j satisfying $e^{2i\theta(g_j)}=\eta_j\zeta_j^{-1}$.

Case ii-2).

Lemma 5. *Let g be a hyperbolic element of $G_v=SL_2(\boldsymbol{R})$ in the usual sense. Then we get*

$$\int_{G_v(g)\backslash G_v} f_{e,j}(x^{-1}gx)dx=\frac{1}{2\pi}\frac{1}{\lambda-\lambda^{-1}}\int_{-\infty}^{\infty}\phi_j(i\rho)\lambda^{2i\rho}d\rho, \tag{2.4}$$

where $\lambda>1$ is the eigenvalue of g.

Proof. We write $I(g)$ for the left hand side of (2.4). Then $I(g)$ is transformed into

$$\frac{1}{2(\lambda-\lambda^{-1})}\int_{-\infty}^{\infty} f_{e,j}\left(\begin{bmatrix}\lambda, n\\ 0, \lambda^{-1}\end{bmatrix}\right)dn.$$

As we choose elements $k(\theta), h_{t'}$ and $k(\phi)$ satisfying the equation $\begin{bmatrix}\lambda, n\\ 0, \lambda^{-1}\end{bmatrix}=k(\theta)h_{t'}k(\phi)$, we get

$$\cosh(2t')=\frac{\lambda^2+\lambda^{-2}}{2}+\frac{n^2}{2},\quad e^{i(\theta+\phi)}=\frac{\lambda+\lambda^{-1}+i(n/2)}{|\lambda+\lambda^{-1}+i(n/2)|}.$$

It follows from the definition of $f_{e,j}[q]$ that

$$I(g)=\frac{1}{2(\lambda-\lambda^{-1})}\int_{-\infty}^{\infty}\left\{\frac{\lambda+\lambda^{-1}+i(n/2)}{|\lambda+\lambda^{-1}+i(n/2)|}\right\}^{e} f_{e,j}\left[\frac{\lambda^2+\lambda^{-2}}{2}+\frac{n^2}{2}\right]dn.$$

Put $p=\dfrac{\lambda^2+\lambda^{-2}}{2}$, $q=p+\dfrac{n^2}{2}$. Changing the variable n into p, we have

$$\frac{1}{\lambda-\lambda^{-1}}\int_p^{\infty} T_e(q,p)f_{e,j}[q]\frac{dq}{\sqrt{2}\sqrt{q-p}}.$$

By the inverse formula of $g_j[p]$ from $f_{e,j}[q]$, $I(g)$ becomes $\dfrac{1}{2(\lambda-\lambda^{-1})}g_j[p]$. In view of the definition of ϕ_j, $I(g)$ coincides with the right hand side of (2.4).

Q.E.D.

Thus, since $\int_{G_\infty(g)\backslash G_\infty} f_{\infty,J}(x^{-1}gx)dx$ does not depend on J for any hyperbolic element g, their alternating sum over J vanishes. We conclude that the contribution from hyperbolic elements to (1.5) vanishes.

Case ii-3). g is mixed. At least one of g_{v_j}'s is hyperbolic in the usual sense ([15], No. 22). It follows from Lemma 5 that the contribution from this part to (1.5) also vanishes.

Secondly, we treat the parabolic part (Theorem C (iii-vi)). G_F decomposes to double cosets of P_F and Γ in the form

$$G_F=\bigcup_{\nu=1}^{h} P_F\gamma_\nu\Gamma\,,\quad (\gamma_\nu\in G_F)\,. \tag{2.5}$$

From now on, we fix a set of Γ-inequivalent cusps $\kappa_\nu=\gamma_\nu^{-1}\infty$. Put

$$\Gamma_{\kappa_\nu}^{(1)}=\{\gamma\in\Gamma\,;\,\gamma\kappa_\nu=\kappa_\nu\}\,,\quad \Gamma_{\kappa_\nu}=\{\gamma\in\Gamma_{\kappa_\nu}^{(1)}\,;\,\gamma_\nu\gamma\gamma_\nu^{-1}\in N_F\}\,,\quad B_{\kappa_\nu}=\{g\in\Gamma\alpha\Gamma\,;\,\gamma_\nu g\gamma_\nu^{-1}\in N_F\}\,,$$

$$A_\nu=\left\{(a_jd_j)\,;\,\gamma_\nu\gamma\gamma_\nu^{-1}=\left(\begin{bmatrix}a_j, & b_j\\ 0, & d_j\end{bmatrix}\right),\ \gamma\in\Gamma_{\kappa_\nu}^{(1)}\right\},\quad M_\nu=\left\{(b_j)\,;\,\gamma_\nu\gamma\gamma_\nu^{-1}=\left(\begin{bmatrix}1, & b_j\\ 0, & 1\end{bmatrix}\right),\ \gamma\in\Gamma_{\kappa_\nu}\right\},$$

$$N_\nu=\left\{(b_j)\,;\,\gamma_\nu\gamma\gamma_\nu^{-1}=\left(\begin{bmatrix}1, & b_j\\ 0, & 1\end{bmatrix}\right),\ \gamma\in B_{\kappa_\nu}\right\},\quad \mathfrak{A}_\nu=(c_\nu, d_\nu)\left(\gamma_\nu=\begin{bmatrix}a_\nu, & b_\nu\\ c_\nu, & d_\nu\end{bmatrix}\right).$$

(By [18], it is known that $A_\nu=E_0^2$, $M_\nu=\mathfrak{A}_\nu^{-2}$). As M_ν is a discrete subgroup of $\boldsymbol{R}^n$ of rank n ([15], Theorem 3.1), for a set of generators $\{\mu_i^{(j)}\,;\,1\le j\le n\}$ of M_ν, put $d(M_\nu)=|\det(\mu_i^{(j)})|$. Hereafter, we often use the following fact. If u_j, v_j, v_j' are any complex numbers $(1\le j\le n)$ and $n\ge 2$, then

$$\sum_{J\subset I}\operatorname{sign}(J)\Big(\sum_{j\in J} v_j\prod_{k\ne j}u_k+\sum_{j'\notin J}v_{j'}\prod_{k\ne j'}u_k\Big)=0\,. \tag{2.6}$$

Case iii). Let μ be an element in A_F-Z_F. Using the fact that $y(wn(x))=(1+x^2)^{-1}$ in Iwasawa's decomposition of $wn(x)$ in $SL_2(\boldsymbol{R})$, μ's term in (iii) of Theorem C is transformed into

$$\begin{aligned} &c\sum_{j=1}^{n}\int_{\boldsymbol{R}} f_{e,j}(n(x)^{-1}\mu_{v_j}n(x))\log(1+x^2)dx\\ &\qquad\times\prod_{k\ne j}\int_{\boldsymbol{R}} f_{e,k}(n(x)^{-1}\mu_{v_k}n(x))dx\,, \end{aligned} \tag{2.7}$$

where the constant c is independent of the choice of e. If λ_k is an eigenvalue of μ_{v_k} such that $|\lambda_k|>1$, it follows from Lemma 5 that the second integral of (2.7) equals $|\lambda-\lambda^{-1}|^{-1}g_k(2\log|\lambda|)$. Therefore the alternating sum of (2.7) over J is equal to

$$c\sum_{J\subset I}\operatorname{sign}(J)\sum_j\left\{\prod_{k\ne j}\frac{g_k(2\log|\lambda_k|)}{|\lambda_k-\lambda_k^{-1}|}\int_{\boldsymbol{R}} f_{e,j}(n(x)^{-1}\mu_{v_j}n(x))\log(1+x^2)dx\right\}. \tag{2.8}$$

Since $n\ge 2$, it follows from (2.6) that (2.8) vanishes.

Case iv). By virtue of (2.5), we can transform $\theta(z,f_J)$ into

$$\sum_{\nu=1}^{h}\int_{G_F\backslash G_A}\sum_{\delta\in\Gamma_{\kappa_\nu}^{(1)}\backslash\Gamma}\sum_{v\in\gamma_\nu^{-1}N_F\gamma_\nu, v\neq 1} f_J(g^{-1}\delta^{-1}v\delta g)e^{-2H(\gamma_\nu\delta g)z}dg$$

$$=\sum_{\nu=1}^{h}\int_{\Gamma\backslash G_\infty}\sum_{\delta\in\Gamma_{\kappa_\nu}^{(1)}\backslash\Gamma}\sum_{v\in B_{\kappa_\nu}, v\neq 1} f_{\infty,J}(g^{-1}\delta^{-1}v\delta g)\prod_j (y_j(\gamma_\nu\delta g))^{-z}dg\,.$$

Lemma 6. ([16], Lemma 3.1). *N_ν is the union of a finite number of cosets of M_ν. If $\mu\in N_\nu$, $\lambda\in\Lambda_\nu$, then $\lambda\mu$ is contained in N_ν.*

Set $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)$ ($\varepsilon_j=1$ or -1). Define a Dirichlet series by

$$L_{N_\nu/\Lambda_\nu}(z,\varepsilon)=\sum_{\mu\neq 0}(\prod_j|\mu_j|)^{-(1+z)}\,,$$

where μ runs over full representatives of N_ν/Λ_ν except $\mu=0$ such that $\varepsilon_j\mu_j>0$ for all j. Put

$$L_{N_\nu/\Lambda_\nu}(z)=\sum_{\mu\in N_\nu/\Lambda_\nu,\,\mu\neq 0}\prod_j \operatorname{sign}(\mu_j)(\prod_j|\mu_j|)^{-(1+z)}\,.$$

$L_{N_\nu/\Lambda_\nu}(z)$ is equal to $\sum_\varepsilon(\prod_j\varepsilon_j)L_{N_\nu/\Lambda_\nu}(z,\varepsilon)$. (Note that $L_{M_\nu/\Lambda_\nu}(z)$ is introduced in ([15], No. 21)).

Now we consider the expression of $\theta(z,f_J)$ making use of $L_{N_\nu/\Lambda_\nu}(z,\varepsilon)$. For simplicity, fix one of κ_ν's, say κ_1, and assume $\kappa_1=\infty$, $\gamma_1=1$. Then

$$\int_{\Gamma\backslash G_\infty}\sum_{\delta\in\Gamma_{\kappa_1}^{(1)}\backslash\Gamma}\sum_{v\in B_{\kappa_1}, v\neq 1} f_{\infty,J}(g^{-1}\delta^{-1}v\delta g)\prod_j (y_j(\delta g))^{-z}dg$$

$$=\int_{\Gamma_{\kappa_1}\backslash G_\infty}\sum_{v\in B_{\kappa_1}/\Lambda_1, v\notin\Lambda_1} f_{\infty,J}(g^{-1}vg)\prod_j (y_j(g))^{-z}dg$$

$$=d(M_1)L_{N_1/\Lambda_1}(z,\varepsilon)\prod_j\int_{\boldsymbol{R}^+} f_{e,j}(a(y)^{-1}n(\varepsilon_j)a(y))y^{-(2+z)}dy\,.$$

It can be shown that $L_{N_1/\Lambda_1}(z,\varepsilon)$ has a simple pole at $z=0$ and that its residue is $m_1\int_{\boldsymbol{R}^n/\Lambda_1,\,\prod_j|x_j|\leq 1,\,x_j\varepsilon_j>0} dx$ (m_1 being the number of cosets of M_1 in N_1). As its residue does not depend on ε, we denote it by λ_{-1}. We also denote by $\lambda_0^{(\varepsilon)}$ the constant term of the Laurent expansion of $L_{N_1/\Lambda_1}(z,\varepsilon)$ at $z=0$. Therefore, we get

$$\lim_{z\to 0}\frac{d}{dz}\left(zL_{N_1/\Lambda_1}(z,\varepsilon)\prod_j\int_{\boldsymbol{R}^+} f_{e,j}(a(y)^{-1}n(\varepsilon_j)a(y))y^{-(2+z)}dy\right)$$

$$=\lambda_0^{(\varepsilon)}\prod_j\int_{\boldsymbol{R}^+} f_{e,j}(a(y)^{-1}n(\varepsilon_j)a(y))y^{-2}dy$$

$$+\lambda_{-1}\sum_j\Big[\Big[\int_{R^+} f_{e,j}(a(y)^{-1}n(\varepsilon_j)a(y))(\log y)y^{-2}dy$$

$$\times\prod_{k\neq j}\Big\{\int_{R^+} f_{e,k}(a(y)^{-1}n(\varepsilon_k)a(y))y^{-2}dy\Big\}\Big]\Big].$$

Concerning the first integral of the above formula, we need the following lemma.

LEMMA 7. *For $\varepsilon=1$ or -1, we get*

$$\int_{R^+} f_{e,j}(a(y)^{-1}n(\varepsilon)a(y))y^{-2}dy=\frac{1}{2}g_j(0)+(e/2)(i\varepsilon/2\pi)\phi_j\Big(\frac{1}{2}\Big). \tag{2.10}$$

Proof. We only show this lemma in the case $\varepsilon=1$. The left hand side of (2.10) can be expressed by

$$\lim_{\theta\to 0}\frac{\sin\theta}{\pi}\int_{SO_2(R)\backslash SL_2(R)} f_{e,j}(gk(\theta)g^{-1})dg\,, \tag{2.11}$$

since

$$\lim_{\theta\to 0}\frac{\sin\theta}{\pi}\int_0^\infty f_{e,j}\Big(\begin{bmatrix}\cos\theta, & t^2\sin\theta\\ -t^2\sin\theta, & \cos\theta\end{bmatrix}\Big)\frac{t^2-t^{-2}}{t}dt$$

$$=\int_0^\infty f_{e,j}(n(x))dx=\int_0^\infty f_{e,j}(a(y)^{-1}n(1)a(y))y^{-2}dy\,.$$

On the other hand, because of Lemma 3, (2.11) equals

$$\frac{1}{2\pi}\Big\{\int_{-\infty}^\infty \phi_j(i\rho)\frac{d\rho}{2}+i(e/2)\phi_j\Big(\frac{1}{2}\Big)\Big\}=\frac{1}{2}g_j(0)+(i/2\pi)(e/2)\phi_j\Big(\frac{1}{2}\Big). \qquad \text{Q.E.D.}$$

Thus, making use of (2.6), λ_{-1}-term vanishes. Consequently

$$\sum_{J\subseteq I}\operatorname{sign}(J)\lim_{z\to 0}\Big\{\frac{d}{dz}z\theta(z,f_J)\Big\}=\Big(\frac{1}{2\pi}\Big)^n\sum_{\nu=1}^h d(M_\nu)L_{N_\nu/A_\nu}(0)\prod_{j=1}^n\phi_j\Big(\frac{1}{2}\Big).$$

Next we shall see the contribution from the part of Eisenstein series.

Case v). In view of Assumption 2, note that unless $\Phi(\chi)$ contains a U_0-invariant element, the trace of $\pi(z,f_J)$ in $\Phi(\chi)$ vanishes. For $J\subseteq I$ and for $\phi_{m,\chi}$ defined in (A.1), $\pi(z,f_J)\phi_{m,\chi}$ does not vanish only if $m_j=2$ for $j\in J$, $m_{j'}=0$ for $j'\notin J$. Put $m_J=(m_1,\cdots,m_n)$ such that $m_j=2$ for $j\in J$, $m_{j'}=0$ for $j'\notin J$. Then

$$\pi(z,f_J)\phi_{m_J,\chi}(x)=\prod_j\phi_j\Big(\frac{z}{2}+2\pi i\xi_j\Big)\sum_l\phi_{m_J,\chi}(xz_l)\,,$$

where ξ_j is a real number attached to χ (Appendix 1). It follows from Proposition A-1 & Corollary A-2 that

$$M(-z)\frac{d}{dz}M(z)\pi(z,f_J)\phi_{m_J,\chi}(x)=\Big\{m(-z,\bar\chi)\frac{d}{dz}m(z,\chi)$$

$$+\sum_{j\in J}A(-z,2,-\xi_j)\frac{d}{dz}A(z,2,\xi_j)\Big\}\prod_j\phi_j\Big(\frac{z}{2}+2\pi i\xi_j\Big)\sum_i\phi_{m_J,\chi}(xz_i)\,.$$

Therefore

$$\sum_{J\subseteq I}\mathrm{sign}\,(J)\,\mathrm{trace}_{\Phi(\chi)}M(-z)\frac{d}{dz}M(z)\pi(z,f_J)$$

$$=\sum_{J\subseteq I}\mathrm{sign}\,(J)m(-z,\bar\chi)\frac{d}{dz}m(z,\chi)\,\mathrm{trace}_{\Phi(\chi)}\,\pi(z,f_J)$$

$$+\sum_{J\subseteq I}\mathrm{sign}\,(J)\sum_{j\in J}A(-z,2,-\xi_j)\frac{d}{dz}A(z,2,\xi_j)\,\mathrm{trace}_{\Phi(\chi)}\,\pi(z,f_J)\,.$$

As each term of the first part of above formula does not depend on J, its first part vanishes. Since $n\geq 2$, its second part also vanishes in view of (2.6). Thus, there are no contributions from (v) to (1.5).

Case vi). As $M(z)$ maps $\phi_{m,\chi}$ to $\phi_{m,\bar\chi}$, the trace over $\Phi(\chi_0)$ only contributes to (1.5) (χ_0^2 being the identity character). Since $A(0,e,0)=1$ for $e=0,2$ because of Proposition A-1, we have

$$\sum_{J\subseteq I}\mathrm{sign}\,(J)\,\mathrm{trace}_{\Phi(\chi_0)}\,M(0)\pi(0,f_J)$$

$$=m(0,\chi_0)\sum_{J\subseteq I}\mathrm{sign}\,(J)\,\mathrm{trace}_{\Phi(\chi_0)}\,\pi(0,f_J)=0\,.$$

Therefore, there is no contribution from (vi) to (1.5).

2.3. Summing up the above results, we obtain

THEOREM 1. *The trace of $T(\Gamma\alpha\Gamma)$ in S is given by*

$$\mathrm{Trace}\;T(\Gamma\alpha\Gamma)=\delta\frac{2}{(2\pi)^{2n}}|D_F|^{3/2}\zeta_F(2)$$

$$+\sum_{[g]\in\{B_e\}}(\Gamma(g):\{\pm 1\})^{-1}\prod_{j=1}^{n}\frac{\eta_j}{\zeta_j-\eta_j}$$

$$+(i/2\pi)^n\sum_{\nu=1}^{h}d(M_\nu)L_{N_\nu/A_\nu}(0)-(-1)^n\cdot d\,.$$

The notations used in this formula are defined as follows.

$$\delta=\begin{cases}1 & \text{if}\quad \Gamma\alpha\Gamma=\Gamma\\ 0 & \text{if}\quad \Gamma\alpha\Gamma\neq\Gamma\end{cases}$$

$\{B_e\}$; *a complete system of inequivalent elliptic elements in $\Gamma\alpha\Gamma$.*

ζ_j,η_j; *the eigenvalues of an elliptic element g_j satisfying the condition in Lemma* 4.

d; the number of right Γ-cosets in $\Gamma\alpha\Gamma$.
The others are referred to the previous definitions.

Remark. By [15], the elliptic term is written by

$$\frac{1}{h}\sum_{\mathfrak{O}\in\Omega}\frac{h(\mathfrak{O})}{w(\mathfrak{O})}\sum_{g\in B_e\cap\Omega g \bmod \{\pm 1\}}\Psi(g)\,.$$

Here Ω is the set of all orders $\mathfrak{O}$ (taken up to isomorphism) in the quadratic extensions of F. $h(\mathfrak{O}), w(\mathfrak{O})$ is the class number of $\mathfrak{O}$ or is the index of E_0 in the group of units in $\mathfrak{O}$. Put $K=F(\mathfrak{O})$. Denote by $C''(\mathfrak{o}), C(\mathfrak{O})$ the ideal class group of $\mathfrak{o}$ modulo $\prod_{i=1}^{n} v_j$, the ideal class group of $\mathfrak{O}$ and by N the mapping of $C(\mathfrak{O})$ into $C''(\mathfrak{o})$ induced by $N_{K/F}$. If $(C''(\mathfrak{o}):N(C(\mathfrak{O})))=1$, put $\Psi(g)=1/2(-1)^n$. In the other case, i.e., $(C''(\mathfrak{o}):N(C(\mathfrak{O})))=2$, call σ_j either one of the two extensions of K of the isomorphism of F into F_{v_j}. Take g^* a generator of $\Gamma(g)$ and assume ζ_j^* is the eigenvalue of g_j^* satisfying the condition in Lemma 4. We define $\varepsilon(\sigma_j)$ to be 1 or -1 according as $\sigma_j g_j^*=\zeta_j^*$ or ζ_j^{*-1}. Put

$$\Psi(g)=\sum_{(\sigma_1,\cdots,\sigma_n)}\prod_j\frac{(\sigma_j g)^2}{1-(\sigma_j g)^2}$$

where $(\sigma_1,\cdots,\sigma_n)$ is taken over all the combinations such that $\prod_{j=1}^{n}\varepsilon(\sigma_j)=1$.

§ 3. The trace of Hecke operators for GL(2).

3.1. In this section, let G be $GL(2)$. Also we shall only consider the case $F\neq \boldsymbol{Q}$. By Eichler's approximation theorem, we have

$$G_A=\bigcup_{\nu=1}^{h} G_F x_\nu U,\quad (x\in G_A, h \text{ being the class number of } F). \tag{3.1}$$

Put $U_\nu=x_\nu U x_\nu^{-1}$, $\Gamma_\nu=U_\nu\cap G_F$ $(1\leq\nu\leq h)$. Let $\mathfrak{H}^n$ be the set of all $z=(z_1,\cdots,z_n)$ with $z_j\in\boldsymbol{C}$, $\operatorname{Im}(z_j)\neq 0$. Then G_∞ acts on $\mathfrak{H}^n$ as the linear fractional transformation on each component. As the same way as § 2.1, we regard G_F as subgroup of G_∞. dz also denotes the invariant measure on $\mathfrak{H}^n$ defined by $\prod_j dx_j dx_j/y_j^2$ $(z_j=x_j+iy_j)$. By S_ν, we understand the space of cusp forms of weight two on $\mathfrak{H}^n$ with respect to Γ_ν consisting of a holomorphic function on each connected component of $\mathfrak{H}^n$ which satisfies the conditions (S.1), (S.2) in § 2.1 replacing $\Gamma, j(\gamma,z)$ by $\Gamma_\nu, j'(\gamma,z)=\prod_j(\det\gamma_j(c_jz_j+d_j)^{-2})$. Let S' be the direct product of $S_1,\cdots,S_h$. By (3.1), we can see the following proposition.

PROPOSITION 4.***) *For $h_\nu\in S_\nu(1\leq\nu\leq h)$, define a function h' on $Z_\infty G_F\backslash G_A$ by*

***) Shimizu, H., Theta series and automorphic forms on GL_2, (§ 6, 8), J. of Math. Soc. Japan, 24, (1972).

$h'(x)=h_\nu(x_\infty(i))j'(x_\infty, i)$, if $x=gx_\nu x_\infty u$ for $g\in G_F$, $x_\infty\in G_\infty$, $u\in U_0$. Then the map ι' from (h_ν) to h' describes an isomorphism between S' and H as linear spaces over $\boldsymbol{C}$.

Next we define the Hecke operators on S'. We need the following lemma.

LEMMA 7. ([17], Proposition 2.3). *Let α be an element of G_F. Then $U_\mu\alpha U_\nu = \Gamma_\mu\alpha U_\nu$.*

For UyU (§ 1.2), suppose there exists z in UyU such that $x_\mu z\in\beta x_\nu U$ with $\beta\in G_F$. Then μ is uniquely determined by ν, because of (3.1). We take an element β in $U_\mu x_\mu y x_\nu^{-1}U_\nu\cap G_F$. It follows from Lemma 7 that $\Gamma_\mu\beta\Gamma_\nu$ is uniquely determined by UyU. Thus we define a linear operator $T(UyU)$ on S' by

$$(3.2)\qquad \begin{aligned} &T(\Gamma_\mu\beta\Gamma_\nu)h_\mu(z)=\sum_j j'(\beta_j^{-1}, z)h_\mu(\beta_j^{-1}z)=g_\nu(z) \quad (\Gamma_\mu\beta\Gamma_\nu=\bigcup_j \beta_j\Gamma_\nu)\\ &T(UyU)(h_1, \cdots, h_h)=(g_1, \cdots, g_h)\,. \end{aligned}$$

Also the action of $T(UyU)$ on S' coincides with the action of $T(UyU)$ on H through the isomorphism ι'.

By ([1], Lemma 2.6), $L_0^2(\{P\})$ owes the residues of the Eisenstein series. The poles of $E(\phi, z, x)$ occur at $z=1$ only when $\tau(k)=\xi(\det(k))$ with a Grössen-character ξ of $F_A^\times/F^\times$ such that ξ^2 is trivial on $F_\infty^\times$. Put $D_\xi(x)=\xi(\det(x))$. Then $L_0^2(\{P\})$ is equivalent to the direct sum of the one-dimensional spaces $\boldsymbol{C}D_\xi$. In order to calculate the trace of $\lambda_0(f_\phi)$ we limit ourselves to the right U-invariant subspaces. Such spaces occur in the case ξ is trivial on $\prod_{\infty}\mathfrak{o}_p^\times$. As is well known that $(F_A: F^\times(F_\infty^\times)_+\prod_{p<\infty}\mathfrak{o}_p^\times)=2^n h/(E_0:E_0^+)$, we have

$$\operatorname{trace}(\lambda_0(f_\phi)|L_0^2(\{P\}))=\delta'\frac{2^n h}{(E_0:E_0^+)}d\prod_{j=1}^n \psi_j\left(\frac{1}{2}\right),$$

(d being the number of right U_0-cosets in UyU, and $\delta'=1$ if $\det y_f\in F^\times(F_\infty^\times)_+\Pi\mathfrak{o}_p^\times$, otherwise $\delta'=0$).

Next, we can extend $f_{\infty,J}$ which is defined in § 2.1 to the function of G_∞ such that it satisfies Assumption 2 (iii). For the present case, we shall define a system $\{f_J\}$ satisfying Assumption 2. Combining above arguments with Proposition 1, we obtain

$$(3.3)\qquad \begin{aligned} &\sum_{J\subset I}\operatorname{sign}(J)\operatorname{trace}\lambda_0(f_J)\\ &=\left\{\operatorname{trace}_{S'} T(UyU)+(-1)^n\delta'\frac{2^n}{(E_0:E_0^+)}h\cdot d\right\}\prod_{j=1}^n\psi_j\left(\frac{1}{2}\right). \end{aligned}$$

3.2. In this subsection, we shall calculate the left hand side of (1.5) with the aid of Theorem C. Put $Z_\nu=\Gamma_\nu\cap Z_\infty$, $B_\nu=U_\nu x_\nu y x_\nu^{-1}U_\nu\cap G_F$. We also define the equivalence relation in B_ν replacing $(\Gamma\alpha\Gamma, \Gamma, \{\pm1\})$ by $(B_\nu, \Gamma_\nu, Z_\nu)$, and denote by

$[x]$ an equivalence class containing x. For $g \in G_\infty$, put $\Gamma_\nu(g)=\{g \in \Gamma_\nu;\ g=\varepsilon\gamma g\gamma^{-1}$ for some $\varepsilon \in Z_\nu\}$.

Case i). Unless y is contained in U, the contribution from the singular part vanishes. By Lemma 3, for the case $UyU=U$, its contribution is

$$\text{measure}\,(Z_\infty G_F \backslash G_A)(4\pi)^{-n} \prod_{j=1}^{n} \psi_j\left(\frac{1}{2}\right).$$

On the other hand, by (3.1) measure of $Z_\infty G_F\backslash G_A$ equals the sum of the volumes of the fundamental domain of Γ_ν in $\mathfrak{H}^n$ with respect to dz. It is known****) that its sum is written by $\dfrac{2h}{\pi^n}|D_F|^{3/2}\zeta_F(2)$.

Case ii). Let B_ν' be a subset of B_ν consisting of elements g such that no fixed point of g is a parabolic point of Γ_ν. Then the contribution from the elliptic part to (1.5) is given

$$\sum_{J\subset I} \operatorname{sign}(J) \sum_{\nu=1}^{h} \sum_{[g]\ g\in B_\nu'} \int_{Z_\infty \Gamma_\nu(g)\backslash G_\infty} f_{\infty,J}(x^{-1}gx)dx\,.$$

According to § 2.2, we classify an element in B_ν'.

Case ii-1). Let $\{B_{\nu,e}\}$ be a complete system of inequivalent elliptic elements in B_ν'. Note that if $B_{\nu,e}\neq\{\phi\}$, $\det y_f \in F^\times(F_\infty^\times)_+\Pi\mathfrak{o}_v^\times$. The contribution from elliptic elements in B_ν' is written by

$$\sum_{J\subset I} \operatorname{sign}(J) \sum_{[g]} (\Gamma_\nu(g):Z_\nu)^{-1} \sum_{\varepsilon} \int_{G_\infty(g)\backslash(G_\infty)_+} f_{\infty,J}(x^{-1}g^\varepsilon x)dx\,,$$

where $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)$ runs over all combinations of $\varepsilon_j=1,\ -1\ (1\leq j\leq n)$. Let η_j, ζ_j be the eigenvalues of g_j satisfying $e^{2i\theta(g_j)}=\eta_j\zeta_j^{-1}$. By Lemma 4, we have

$$\sum_{J\subset I} \operatorname{sign}(J) \sum_{\varepsilon} \int_{G_\infty(g)\backslash G_\infty} f_{\infty,J}(x^{-1}g^\varepsilon x)dx$$
$$=\sum_{\varepsilon}\left\{\prod_{\varepsilon_j=1}\frac{\eta_j}{\zeta_j-\eta_j}\prod_{\varepsilon_k=-1}\frac{\zeta_k}{\eta_k-\zeta_k}\right\}\prod_j \psi_j\left(\frac{1}{2}\right)=(-1)^n\prod_j \psi_j\left(\frac{1}{2}\right).$$

Case ii-2, 3). Applying arguments of § 2.2 to the present case, we see that there are no contributions from hyperbolic and mixed elements to (1.5).

Case iv). Let $\kappa_\nu=\gamma_\nu^{-1}\infty\ (1\leq\nu\leq h)$ be inequivalent cusps of Γ_μ. We define $(\Gamma_\mu^{(1)})_{\kappa_\nu}, (\Gamma_\mu)_{\kappa_\nu}, (B_\mu)_{\kappa_\nu}, (A_\mu)_\nu, (M_\mu)_\nu, (N_\mu)_\nu$ and the Dirichlet series $L_{(N_\mu)_\nu/(A_\mu)_\nu}(z)$ by the same way as in § 2.2 replacing Γ by Γ_μ. Applying calculation of § 2.2 to the present case together with (3.1), $\theta(z,f_J)$ can be written in the form

$$\theta(z,f_J)=\sum_\mu\sum_\nu \theta_{\mu\nu}(z,f_J)$$

$$\theta_{\mu\nu}(z,f_J)=d((M_\mu)_\nu)L_{(N_\mu)_\nu/(A_\mu)_\nu}(z)$$
$$\times\prod_j\int_{R^+}\{f_{e,j}(a(y)^{-1}n(1)a(y))+f_{e,j}(a(y)^{-1}n(-1)a(y))\}y^{-(2+z)}dy\,.$$

****) Shimizu, H., On zeta functions of quaternion algebras, Ann. of Math. 81 (1965) pp. 166–193.

But, by Lemma 7, we have

$$\int_{R^+}\{f_{e,j}(a(y)^{-1}n(1)a(y))+f_{e,j}(a(y)^{-1}n(-1)a(y))\}y^{-2}dy=g_j(0)\,.$$

Therefore

$$\sum_{J\subseteq I}\operatorname{sign}(J)\lim_{z\to 0}\left\{\frac{d}{dz}(z\theta_{\mu\nu}(z,f_J))\right\}$$
$$=\sum_{J\subseteq I}\operatorname{sign}(J)L_{(N_\mu)_\nu/(A_\mu)_\nu}(0)\prod_j g_j(0)=0\,.$$

Thus, in the present case, this part has no contribution to (1.5).

Case iii, v, vi). By the same reason as § 2.2, there are no contributions from the parts (iii), (v), (vi) to (1.5).

3.3. Summing up the above things, we obtain

THEOREM 2. *The trace of $T(UyU)$ in S' is given by*

$$\text{Trace } T(UyU)=\delta\frac{2h}{(2\pi)^{2n}}|D_F|^{3/2}\zeta_F(2)+(-1)^n\sum_{\nu=1}^{h}\sum_{[g]}(\Gamma_\nu(g):Z_\nu)^{-1}-(-1)^n\frac{2^n h}{(E_0:E_0^+)}\cdot d\,, \tag{3.4}$$

if $\det y_f\in F^\times(F_\infty^\times)_+\Pi\mathfrak{o}_\mathfrak{p}^\times$. *Otherwise, it vanishes. The notations are defined as follows.* $\delta=1$ *if* $y\in U$ *and otherwise* $\delta=0$. $[g]$ *runs over a full system of inequivalent elliptic elements in* B_ν'. d *is a number of right U-cosets in* UyU.

Let $\mathfrak{a}$ be an integral ideal in F. Following from ([17], § 3.4), we define an operator $T(\mathfrak{a})$ in S' by the sum of $T(UyU)$ over all double cosets UyU such that the right $M_2(\mathfrak{o})$ ideal $\bigcap_\mathfrak{p} y\,M_2(\mathfrak{o})$ is integral and of norm $\mathfrak{a}$. Unless $\mathfrak{a}$ is a principal ideal $a\mathfrak{o}$ with a totally positive element a, trace $T(\mathfrak{a})=0$. Combining Theorem 2 with ([9], § 5.1) & ([16], § 4), we get

THEOREM 2$'$. *Let* $\mathfrak{a}=a\mathfrak{o}$ *be a principal ideal with a totally positive element* a *in* $\mathfrak{o}$. *The* trace $T(\mathfrak{a})$ *in* S' *is given by the following formula.*

$$\text{Trace } T(\mathfrak{a})=\delta(\mathfrak{a})\frac{2h}{(2\pi)^{2n}}|D_F|^{3/2}\zeta_F(2)+\frac{(-1)^n}{2}\sum_{\mathfrak{O}\in\Omega}\frac{h(\mathfrak{O})}{w(\mathfrak{O})}|I(\mathfrak{a},\mathfrak{O})\,mod\,E_0|-(-1)^n\frac{2^n h}{(E_0:E_0^+)}\sum_{\mathfrak{n}|\mathfrak{a}}N_{F/Q}(\mathfrak{n})\,. \tag{3.5}$$

The notations are as follows. $\delta(\mathfrak{a})=1$ *if* $\mathfrak{a}=a_0^2\mathfrak{o}$ *for some* $a_0\in\mathfrak{o}$ *and otherwise* $\delta(\mathfrak{a})=0$. $\mathfrak{n}$ *runs over all divisors of* $\mathfrak{a}$. Ω *is the set of all orders* $\mathfrak{O}$ *(taken up to isomor-*

phism) in totally imaginary quadratic extensions of F. $h(\mathfrak{O})$ is the class number of $\mathfrak{O}$, and $w(\mathfrak{O})$ is the index of E_0 in the group of units in $\mathfrak{O}$. Let K be the quadratic extension of F containing $\mathfrak{O}$; ϕ is an embedding of the adèle of K into $M_2(K)_A$ such that $\phi(\mathfrak{O}_{\mathfrak{p}})=\phi(K_{\mathfrak{p}})\cap M_2(\mathfrak{o}_{\mathfrak{p}})$ for all $\mathfrak{p}$; $I(\mathfrak{a},\mathfrak{O})$ is the set of all $x\in K-F$ such that $\phi(x)$ is contained in the union of double cosets UyU appearing in $T(\mathfrak{a})$.

3.4. Here, we shall give some relations between the various relative class numbers of the totally imaginary quadratic extensions over some fixed real quadratic field. When $F=\boldsymbol{Q}(\sqrt{5})$, $\boldsymbol{Q}(\sqrt{2})$, $\boldsymbol{Q}(\sqrt{13})$ or $\boldsymbol{Q}(\sqrt{17})$, the space of cusp forms of weight two with respect to F vanishes (Theorem 2). It follows from Theorem 2′

PROPOSITION 5. *Let q be a totally positive element in $\mathfrak{o}$. For above F, we have*

$$\delta(\mathfrak{q})\frac{1}{8\pi^4}D_F^{3/2}\zeta_F(2)\sum_{\mathfrak{n}\mid\mathfrak{q}}N_{F/Q}(\mathfrak{n})+\frac{1}{2}\sum_{s,f}\frac{h(\mathfrak{O}(s,f))}{w(\mathfrak{O}(s,f))}=0\,.$$

Here, $\mathfrak{q}$ is the principal ideal in $\mathfrak{o}$ generated by q; s,f denote integers in F, and $\mathfrak{O}(s,f)$ denotes the order in $F(\sqrt{s^2-4q})$ with the discriminant $(s^2-4q)f^{-2}$. s runs over all integers in F such that $s^2-4q<0$, $(s^{\epsilon})^2-4q^{\epsilon}<0$, $((s\pm\sqrt{s^2-4q})/2 \bmod E_0)$ and (f) runs over the divisors of (f_0) where $\mathfrak{O}(s,f_0)$ is the principal order in $F(\sqrt{s^2-4q})$, (ϵ being a generator of $Gal(F/\boldsymbol{Q})$).

Remark. $\frac{1}{8\pi^4}D_F^{3/2}\zeta_F(2)$ is $\frac{1}{60}$, $\frac{1}{24}$, $\frac{1}{12}$ or $\frac{1}{6}$, according to $F=\boldsymbol{Q}(\sqrt{5})$, $\boldsymbol{Q}(\sqrt{2})$, $\boldsymbol{Q}(\sqrt{13})$ or $\boldsymbol{Q}(\sqrt{17})$, respectively.

3.5. Numerical Examples

i) Set $F=\boldsymbol{Q}(\sqrt{N})$ ($N>5$ being a square-free positive number). We assume that the class number of F is one and that F contains a unit of norm -1. It follows from Theorem 2′ that

$$\dim S'=\frac{1}{8\pi^4}D_F^{3/2}\zeta_F(2)-1+\frac{1}{2}\sum\frac{h(\mathfrak{O}(s,\mathfrak{f}))}{w(\mathfrak{O}(s,\mathfrak{f}))}(w(\mathfrak{O}(s,\mathfrak{f}))-1)\,. \tag{3.6}$$

Here,

s; integers in F, satisfying $4-s^2$ is totally positive,
$\mathfrak{f}$; integral ideals in F, satisfying $(s^2-4)\mathfrak{f}^{-2}$ is an integral ideal in F,
$\mathfrak{O}(s,\mathfrak{f})$; the order in $F(\sqrt{s^2-4})$ with discriminant $(s^2-4)\mathfrak{f}^{-2}$.

The summation runs over all $\mathfrak{O}(s,\mathfrak{f})$ (taken up to isomorphism). As $N>5$, in the summation of right hand side of (3.6) s which satisfies the conditions is only 0 or ± 1, and for any cases of $s, \mathfrak{f}=\mathfrak{o}$. The consideration of the zeta function of $F(\sqrt{s^2-4})$ gives

LEMMA 8. *Suppose $K=\boldsymbol{Q}(\sqrt{N}, \sqrt{-D})$ with a square-free positive number D. Then $h(K)=(1/2)h(-D)h(-ND)$, where $h(-L)$ denotes the class number of $\boldsymbol{Q}(\sqrt{-L})$.*

Thus

$$\dim S' = \frac{1}{8\pi^4} D_F{}^{3/2}\zeta_F(2) - 1 + \frac{h(-3N)}{6} + \frac{h(-N)}{8}.$$

Moreover we assume that N is not divided by 2 or 3 and $N \geq 37$. It follows from Theorem 2′ that the trace of $T((2))$ and of $T((3))$ in S' are also given by

$$\text{(3.7)} \quad \begin{aligned} \operatorname{Trace} T((2)) &= \frac{h(-2N)}{4} + \frac{h(-7N)}{2} + \frac{h(-N)}{4} - \delta_2(N), \\ \operatorname{Trace} T((3)) &= \xi_2(N)\frac{h(-3N)}{2} + \frac{h(-11N)}{2} + \frac{h(-2N)}{2} - \delta_3(N). \end{aligned}$$

Here,

$$\delta_2(N) = \begin{cases} 9 & \text{if } \left[\frac{2}{N}\right]=1 \\ 5 & \text{if } \left[\frac{2}{N}\right]=-1 \end{cases} \qquad \delta_3(N) = \begin{cases} 16 & \text{if } \left[\frac{3}{N}\right]=1 \\ 10 & \text{if } \left[\frac{3}{N}\right]=-1 \end{cases}$$

$$\xi_2(N) = \begin{cases} 3 & \text{if } \left[\frac{2}{N}\right]=1 \\ 1 & \text{if } \left[\frac{2}{N}\right]=-1 \end{cases}.$$

where $\left[\frac{p}{N}\right]$ denotes the Legendre symbol. Using the electric computer at Okayama University, for a prime number N satisfying the above conditions, we calculate the dimension of S', the trace of $T((2))$, and of $T((3))$ which are given in Appendix 3.

ii) By application of Proposition 5, we get some class numbers of biquadratic fields. Let $N=13$ or 17 and consider the trace of $T((2))$. In these cases, s which appears in Proposition 5, is 0, ± 1, ± 2, $\pm(1\pm\sqrt{13})/2$ (if $N=13$) or $\pm(1\pm\sqrt{17})/2$ (if $N=17$). Then the value of trace of $T((2))$ is the sum of the terms appearing in the right hand side of (3.7) and of

$$2h(F(\sqrt{(-9+\sqrt{13})/2})), \quad \text{or} \quad 2h(F(\sqrt{(-7+\sqrt{17})/2})),$$

according to $N=13$ or 17. Thus

$$h\left(F\sqrt{\frac{-9+\sqrt{13}}{2}}\right)=1, \quad (F=\boldsymbol{Q}(\sqrt{13}))$$

$$h\left(F\sqrt{\frac{-7+\sqrt{17}}{2}}\right)=1, \quad (F=\boldsymbol{Q}(\sqrt{17})).$$

Appendix 1.

The purpose of this appendix is to give the definition of the Eisenstein series and some examinations of its Fourier expansion. Through this section, we take $G=SL(2)$.

We define the Eisenstein series $E(\phi, z, x)$ on $G_F\backslash G_A$ which is right U_0-invariant. Let χ be a Grössen-character of $F_A^{\times}/F^{\times}$ with a conducter $\mathfrak{o}$ such that its restriction $\chi(x_\infty)$ to $(F_A^{\times})_\infty$ is $\prod_{j=1}^{n}|x_j|^{4\pi i\xi_j}$, where $\xi=(\xi_1, \cdots, \xi_n)$ in $\boldsymbol{R}^n$ satisfies the following conditions

(i) $\xi_1 \log|\varepsilon_1^{(i)}|+\cdots+\xi_n \log|\varepsilon_n^{(i)}|\in \boldsymbol{Z}, \ (1\leq i\leq n-1)$

(ii) $\xi_1+\cdots+\xi_n=0,$

($\{\varepsilon^{(1)}, \cdots, \varepsilon^{(n-1)}\}$ being a complete set of generators in the group of E_0^2). For an element $\begin{bmatrix} a, 0 \\ 0, a^{-1}\end{bmatrix}$ in A_A, we put $\chi\left(\begin{bmatrix} a, 0 \\ 0, a^{-1}\end{bmatrix}\right)=\chi(a)$; thus we can regard χ as a character of $A_F A_{\infty}^{+}\backslash A_A$. For $m=(m_1, \cdots, m_n)$ (m_j being an even integer) and for $x=nak$ ($n\in N_A$, $a\in A_A$, $k\in K$), we define a function $\phi_{m,\chi}$ in $\Phi(\chi)$ and the Eisenstein series $E(\phi_{m,\chi}, z, x)$ by

$$\text{(A.1)} \qquad \begin{aligned} \phi_{m,\chi}(x) &= \prod_{j=1}^{n} \sigma_{m_j}(k_{v_j})\chi(a) \\ E(\phi_{m,\chi}, z, x) &= \sum_{\delta\in P_F\backslash G_F} \phi_{m,\chi}(\delta x)e^{(z+1)H(\delta x)}. \end{aligned}$$

For a constant term of the Fourier expansion of the Eisenstein series, we have

PROPOSITION A-1. *The constant term of* $E(\phi_{m,\chi}, z, x)$, *i.e.*,

$$E^0(\phi_{m,\chi}, z, x)=\int_{N_F\backslash N_A} E(\phi_{m,\chi}, z, nx)dn$$

is given by the expression

$$E^0(\phi_{m,\chi}, z, x)=\phi_{m,\chi}(x)e^{(z+1)H(x)} + m(z, \chi)\prod_{j=1}^{n} A(z, m_j, \xi_j)\phi_{m,\bar{\chi}}(x)e^{(1-z)H(x)},$$

(A.2)
$$m(z,\chi)=\frac{\xi(z,\chi)}{\xi(1+z,\chi)},$$
$$\xi(z,\chi)=\left\{\frac{|D_F|^{1/2}}{\Gamma\left(\frac{1}{2}\right)}\right\}^z\prod_{j=1}^{n}\Gamma\left(\frac{z}{2}+2\pi i\xi_j\right)L(z,\chi),$$
$$A(z,m_j,\xi_j)=\begin{cases}1 & (m_j=0),\\ \prod_{k=1}^{m_j/2}\dfrac{2k-1-z-4\pi i\xi_j}{2k+1+z+4\pi i\xi_j} & (m_j\geq 2),\end{cases}$$

where $L(z,\chi)$ denotes Hecke's L-function with a Grössen-character χ.

Proof. $E^0(\phi_{m,\chi},z,x)$ becomes

$$\sum_{\{\delta\}\in P_F\backslash G_F/N_F}\int_{\delta^{-1}P_F\delta\cap N_F\backslash N_A}\phi_{m,\chi}(\delta nx)e^{(z+1)H(\delta nx)}dn\,.$$

Because of Bruhat's decomposition of $G_F=P_F\cup P_FwN_F$, it equals

$$\phi_{m,\chi}(x)e^{(z+1)H(x)}+\int_{N_F}\phi_{m,\chi}(wnx)e^{(z+1)H(wnx)}dn\,.$$

By Iwasawa's decomposition of $x=n'a'k'$ ($n'\in N_A$, $a'\in A_A$, $k'\in K$), we have

$$H(wnx)=-H(x)+H(w(a'^{-1}nn'a'))\,,$$
$$\chi(wnx)=\chi(x)^{-1}\chi(w(a'^{-1}nn'a'))\,.$$

Thus the later integral of above formula is

(A.3)
$$\begin{aligned}&\phi_{m,\bar\chi}(x)e^{(1-z)H(x)}\int_{N_A}e^{i\sum_j m_j\theta_j(wn)}\chi(wn)e^{(z+1)H(wn)}dn\\&=\phi_{m,\bar\chi}(x)e^{(1-z)H(x)}\int_{N_\infty}e^{i\sum_j m_j\theta_j(wn_\infty)}\chi(wn_\infty)\prod_j y_j(wn_\infty)^{(z+1)/2}dn\\&\quad\times\int_{N_f}\chi(wn_f)e^{(z+1)H(wn_f)}dn_f\,.\end{aligned}$$

By the same way as ([12], (6.1.3)), the first integral equals

$$|D_F|^{-1/2}\prod_j A(z,m_j,\xi_j)B\left(\frac{1}{2},\frac{z}{2}+2\pi i\xi_j\right).$$

For a finite prime $\mathfrak{p}$, we define a mapping $H_\mathfrak{p}$ of $G_\mathfrak{p}$ to $\boldsymbol{R}$ by the same way as the definition of H. It follows from Iwasawa's decomposition of $wn_\mathfrak{p}$ that

$$\int_{N_\mathfrak{p}}\chi(wn_\mathfrak{p})e^{(z+1)H_\mathfrak{p}(wn_\mathfrak{p})}dn_\mathfrak{p}=\sum_{m=0}^{\infty}\chi(\mathfrak{p}^m)\varphi(\mathfrak{p}^m)N_{F/Q}(\mathfrak{p}^m)^{-(1+z)}=\frac{1-\chi(\mathfrak{p})N_{F/Q}(\mathfrak{p})^{-(1+z)}}{1-\chi(\mathfrak{p})N_{F/Q}(\mathfrak{p})^{-z}},$$

where φ denotes Euler's function. If S_k is a set consisting of the first k finite primes, the second integral in (A.3) becomes

$$\lim_{k\to\infty}\prod_{\mathfrak{p}\in S_k}\int_{N_\mathfrak{p}}\chi(wn)e^{(z+1)H(wn)}dn=\prod_{\mathfrak{p}\neq\infty}\frac{1-\chi(\mathfrak{p})N_{F/Q}(\mathfrak{p})^{-(1+z)}}{1-\chi(\mathfrak{p})N_{F/Q}(\mathfrak{p})^{-z}}=\frac{L(z,\chi)}{L(1+z,\chi)}.$$

Combining the above things, we get the expression (A.2). Q.E.D.

By the functional equation of Hecke's L-function, we have

COROLLARY A-2. $m(z,\chi)m(-z,\bar{\chi})=1$.

Appendix 2.

We shall examine the correspondence between the situations of ([6], § 5) and ours. Here $\varphi(g), h^+(\rho), h_n^\pm$ denote the functions defined in ([6], §§ 5.1-5.2).

LEMMA A-3. Let take $f_{e,j}(g)$ for $\varphi(g)$. Then

(1) $h^+(\rho)=\phi_j\left(i\frac{\rho}{2}\right)$.

(2) $h_n^+=0$ if $n\neq 1$ or if $n=1$, $e=0$; $h_n^-=0$ for all n; $h_1^+=\phi_j\left(\frac{1}{2}\right)$.

Proof. (1) Let g be a hyperbolic element in $G=SL_2(\boldsymbol{R})$ in the usual sense. By Lemma 5, we have the following equation

$$\text{(A.4)}\qquad \begin{aligned}&\int_{-\infty}^{\infty}|\lambda_g-\lambda_g^{-1}|\int_{G(g)\backslash G}f_{e,j}(x^{-1}gx)dx|\lambda_g|^{i\rho}\frac{d\rho_g}{\rho_g}\\&=\int_{-\infty}^{\infty}g_j(2\log|\lambda_g|)|\lambda_g|^{i\rho}\frac{d\lambda_g}{\lambda_g},\end{aligned}$$

where λ_g is the eigenvalue of g satisfying $|\lambda_g|>|\lambda_g^{-1}|$ By ([6], § 5.3 (5)), its right hand side becomes $h^+(\rho)$. On the other hand, because of the definition of g_j, its left hand side equals $\phi_j\left(i\frac{\rho}{2}\right)$.

(2) Applying the argument of ([9], § 2.2), it follows first two assertions. Now let us take $n=1$, $e=2$. It follows from the definition of h_1^+ that

$$h_1^+=\int_G f_{2,j}(g)dg+\int_{(G)_h}f_{2,j}(g)\frac{3\lambda_g^{-1}-\lambda_g}{\lambda_g-\lambda_g^{-1}}dg,$$

where $(G)_h$ denotes the set of all hyperbolic elements g in G. By the definition of $f_{2,j}$, the first integral vanishes. By ([6], § 5.3 (5)), we have

$$\int_{(G)_h}f_{2,j}(g)\frac{\lambda_g^{-1}}{\lambda_g-\lambda_g^{-1}}dg=\int_{|\lambda_h|>1}(\lambda_h-\lambda_h^{-1})\lambda_h^{-1}\int_{(G)_h\backslash G}f_{2,j}(x^{-1}hx)dxdh.$$

By Lemma 5, its integral equals

$$\frac{1}{2\pi}\int_{|\lambda|=1}\left\{\int_{-\infty}^{\infty}\psi_j(i\rho)|\lambda|^{2i\rho-1}d\rho\right\}\frac{d\lambda}{\lambda}=-\frac{1}{2\pi i}\int_{\mathrm{Re}(z)=0}\frac{\psi_j(z)}{z-\frac{1}{2}}\,dz\,.$$

By the same way, we have

$$\int_{(G)_h}f_{2,j}(g)\frac{\lambda_g}{\lambda_g-\lambda_g^{-1}}\,dg=\frac{1}{2\pi i}\int_{\mathrm{Re}(z)=1}\frac{\psi_j(z)}{z-\frac{1}{2}}\,dz\,.$$

It follows from Cauchy's integral formula that the second integral becomes $\psi_j\left(\frac{1}{2}\right)$.

Appendix 3.

N	DIM S'	TR $T((2))$	TR $T((3))$	N	DIM S'	TR $T((2))$	TR $T((3))$
29	1	-1	1	337	21	13	21
37	1	0	1	349	16	8	46
41	1	1	-2	353	18	32	15
53	2	2	2	373	19	8	34
61	2	0	8	389	18	10	22
73	2	3	3	397	18	14	51
89	3	7	1	409	28	13	21
97	3	6	6	421	21	8	45
101	4	3	9	433	29	22	28
109	4	1	14	449	28	27	13
113	4	11	5	457	32	18	27
137	5	14	7	461	21	14	33
149	6	4	11	509	23	18	34
157	6	6	23	521	35	29	15
173	7	8	15	541	30	7	54
181	7	5	27	557	27	17	31
193	9	9	13	569	41	24	16
197	8	8	15	593	38	40	24
233	11	18	12	601	50	15	29
241	13	8	13	613	35	14	52
269	11	10	22	617	41	42	29
277	13	6	28	641	48	31	19
281	15	16	8	653	33	17	36
293	12	13	25	661	37	15	71
313	18	17	22	673	57	23	33
317	13	13	24	677	34	21	41

N	DIM S'	TR $T((2))$	TR $T((3))$
701	39	13	31
709	43	12	63
757	47	16	57
769	71	18	39
773	38	27	46
797	41	26	42
809	67	30	24
821	46	18	41
829	52	16	77
853	51	19	84
857	65	49	28
877	56	17	70
881	74	40	27
929	78	46	23
937	90	29	49
941	53	24	51
953	79	48	32
977	80	57	31
997	65	19	86
1013	56	29	52
1021	68	20	91
1033	106	30	53
1049	95	45	27
1061	64	20	52
1069	76	18	86
1097	91	61	30
1109	68	25	48
1117	80	15	76
1153	121	44	52
1181	71	28	58
1193	105	67	35
1201	141	23	49
1213	90	21	77
1217	106	62	42
1237	85	29	105
1249	147	22	49
1277	77	32	52
1289	129	44	30

N	DIM S'	TR $T((2))$	TR $T((2))$
1301	84	27	52
1321	158	37	54
1361	138	44	30
1381	113	14	84
1409	148	53	29
1433	137	73	47
1453	116	25	85
1481	161	44	30
1493	98	28	67
1549	130	21	101
1553	150	78	40
1597	130	25	100
1609	214	33	62
1613	107	30	67
1621	143	16	101
1637	108	39	68
1657	204	47	73
1669	139	23	126
1693	136	26	118
1697	172	86	49
1709	125	27	64
1721	193	59	39
1733	124	28	59
1741	150	22	121
1753	224	57	72
1777	229	46	75
1789	160	19	104
1801	258	34	59
1861	163	25	130
1873	259	49	78
1877	134	36	75
1889	222	58	32
1913	213	82	48
1933	172	23	106
1949	149	34	73
1973	142	41	70
1993	272	43	83

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