On Analytic Equivalence of Glancing Hypersurfaces

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In 1973 Mikio Sato presented the following problem in contact geometry which is concerned with the geometrical structure of boundary value problems describing diffracted rays (cf. [2]).

Let V_1 and V_2 be hypersurfaces in a $(2n+1)$-dimensional contact manifold X which intersect normally at a point P. Assume the Hamilton foliation of V_1 (resp. V_2) is simply tangent to V_2 (resp. V_1) at the point. Let (V_1', V_2') be any other pair of hypersurfaces in X which satisfies the same condition as the pair (V_1, V_2). Then is there a local contact transformation Ψ such that the following condition holds?

$$(1)\qquad \Psi(P)=P \text{ and } \Psi(V_i)=V_i' \text{ for } i=1,2.$$

In 1975 the existence of a counterexample for the problem in the real analytic category was announced at a symposium held at RIMS in Kyoto University (cf. [5]). Here we give it with the whole proof. On the contrary R. Melrose proved in [3] that it is affirmative in the C^∞-category and treated a more degenerate situation in [4]. This suggests us some difficulty in analysing analytic singularities of diffracted rays.

Since our situation is local, all structure will be considered at the germ level at the point P. Thus let (X, ω) be a germ of a real analytic $(2n+1)$-dimensional contact manifold. That is, ω is a germ of Pfaffian form on X such that the $(2n+1)$-from $\omega\wedge(d\omega)^n$ vanishes nowhere on X. Let

$$(2)\qquad V_i=\{x\in X;\ f_i(x)=0\} \text{ for } i=1,2$$

with germs of real-valued real analytic functions f_1 and f_2 which satisfy

$$(3)\qquad \begin{cases} f_1(P)=f_2(P)=[f_1, f_2](P)=0, \\ [f_1,[f_1,f_2]](P)\neq 0,\ [f_2,[f_1,f_2]](P)\neq 0, \\ (\omega\wedge df_1\wedge df_2)(P)\neq 0. \end{cases}$$

This is the assumption of the Sato's problem and the hypersurfaces V_1 and V_2 are said to be glancing at P. A real analytic contact transformation Ψ of (X, ω) is by definition a real analytic diffeomorphism of X satisfying $\Psi^*(\omega)=c\omega$ with a function c and under the canonical coordinate system $(p, x, z)=(p_1, \cdots, p_n, x_1, \cdots, x_n, z)$ of X such that $\omega=dz-\sum_{i=1}^n p_i dx_i$, the Lagrange bracket $[\ ,\ ]$ is expressed as

$$[f, g]=\sum_{i=1}^n\left(\frac{\partial f}{\partial x_i}+p_i\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p_i}-\sum_{i=1}^n\left(\frac{\partial g}{\partial x_i}+p_i\frac{\partial g}{\partial z}\right)\frac{\partial f}{\partial p_i} \tag{4}$$

for functions f and g on X (cf. [1]). Then the following theorem gives the counterexample.

THEOREM *Retain the above notation and assume* $n\geqq 3$. *Use the canonical coordinate system* (p, x, z) *so that* P *corresponds to* $(0, 0, 0)$ *and put*

$$\begin{cases} V_1=V_1'=\{(p, x, z)\in X;\ x_1=0\}, \\ V_2=\left\{(p, x, z)\in X;\ x_1+\dfrac{1}{2}p_1^2+p_2=0\right\}, \\ V_2'=\left\{(p, x, z)\in X;\ x_1+\dfrac{1}{2}p_1^2+p_2+h(p_1, x_2)p_2^l x_3=0\right\} \end{cases} \tag{5}$$

for a non-negative integer l. *Let* $\{C_i\}_{i=0,1,2,\ldots}$ *be a sequence of real numbers satisfying*

$$\overline{\lim}\ \sqrt[i]{|C_i|}<\infty,\quad \overline{\lim}\ \sqrt[i]{i!|C_i|}=\infty. \tag{6}$$

Then if $h(p_1, x_2)=\left(\dfrac{\partial}{\partial x_2}-\dfrac{\partial}{\partial p_1}\right)(p_1\sum_{i=0}^\infty C_i x_2^i)$, *there is no real analytic contact transformation* Ψ *satisfying* (1).

Proof of Theorem. To prove the above statement we prepare the lemma:

LEMMA *Let* $\tilde{X}$ *be a* $2m$*-dimensional symplectic manifold with the canonical coordinate system* $(\xi, x)=(\xi_1, \cdots, \xi_m, x_1, \cdots, x_m)$ *such that the canonical 2-form equals* $\sum_{i=1}^m d\xi_i\wedge dx_i$. *Assume* $m\geqq 2$ *and consider the two hypersurfaces defined near* $\tilde{P}=(0, 0)$:

$$\begin{cases} W_1=\{(\xi, x)\in\tilde{X};\ \xi_1=0\}, \\ W_{2,t}=\left\{(\xi, x)\in\tilde{X};\ \xi_1+\dfrac{1}{2}x_1^2+x_2+\tilde{h}(\xi, x)x_2^l t=0\right\}. \end{cases} \tag{7}$$

If $\tilde{h}=\left(\dfrac{\partial}{\partial\xi_2}-\dfrac{\partial}{\partial x_1}\right)(x_1\tilde{g}(\xi, x))$ *for a real-valued real analytic function* $\tilde{g}(\xi, x)=\sum_{i=0}^\infty\tilde{g}_i(\xi_3, \cdots, \xi_m, x_3, \cdots, x_m)\xi_2^i$ *satisfying* $\overline{\lim}\ \sqrt[i]{i!|\tilde{g}_i(0)|}=\infty$, *there is no local symplectic transformation* Φ_t *with the real analytic parameter* t *near the origin such that*

$$\Phi_0(\tilde{P})=\tilde{P},\ \Phi_t(W_1)=W_1,\ \Phi_t(W_{2,t})=W_{2,0}. \tag{8}$$

Suppose that the required contact transformation Ψ exists. Use the canonical homogeneous coordinate system (ξ, x) of X such that $p_i=-\xi_i/\xi_{n+1}$ for $i=1, \cdots, n$

and $z=x_{n+1}$. Then by the symplectic transformation of the (ξ, x)-space

$$\begin{cases} \xi_1 \longmapsto -x_1(\xi_{n+1}+1)^{2/3}, \ \xi_2 \longmapsto -x_2(\xi_{n+1}+1)^{1/3}, \ \xi_3 \longmapsto -x_3, \\ \xi_j \longmapsto \xi_j \text{ for } j=4, \cdots, n, \ \xi_{n+1} \longmapsto \xi_{n+1}+1, \\ x_1 \longmapsto \xi_1(\xi_{n+1}+1)^{-2/3}, \ x_2 \longmapsto \xi_2(\xi_{n+1}+1)^{-1/3}, \ x_3 \longmapsto \xi_3, \\ x_j \longmapsto x_j \text{ for } j=4, \cdots, n, \ x_{n+1} \longmapsto x_{n+1} - \left(\frac{2}{3} x_1\xi_1 + \frac{1}{3} x_2\xi_2\right)(\xi_{n+1}+1)^{-1} \end{cases}$$

we have

$$V_1 = V_1' = \{(\xi, x) \in X;\ \xi_1 = 0\},$$

$$V_2 = \left\{(\xi, x) \in X;\ \xi_1 + \frac{1}{2} x_1^2 + x_2 = 0\right\},$$

$$V_2' = \left\{(\xi, x) \in X;\ \xi_1 + \frac{1}{2} x_1^2 + x_2 + \tilde{h} x_2^l \xi_3 = 0\right\}$$

where $\tilde{h} = h(x_1(\xi_{n+1}+1)^{-1/3}, \xi_2(\xi_{n+1}+1)^{-1/3})(\xi_{n+1}+1)^{2(1-l)/3}$. The existence of Ψ implies that the existence of an analytic local symplectic transformation $\tilde{\Psi}$: $(\xi, x) \longmapsto (\eta, y)$ of the (ξ, x)-space such that $\tilde{\Psi}((0, 0)) = (0, 0)$ and $\tilde{\Psi}(V_i) = V_i'$ for $i=1, 2$. This shows that the differential $(d\tilde{\Psi})_{(0,0)}$: $(\xi', x') \longmapsto (\eta', y')$ of $\tilde{\Psi}$ at the origin must be of the form

$$\begin{cases} \xi_1' = A_1\eta_1', x_2' = A_2 y_2' + (A_2 - A_1)\eta_1', \\ x_1' = \dfrac{y_1'}{A_1} + \sum_{i=1}^{n+1} A_i'\eta_i' + \sum_{j=2}^{n+1} A_j'' y_j', \cdots \end{cases}$$

with real numbers A_1, A_2, A_i' and A_j'' satisfying $A_1 \neq 0$, $A_2 \neq 0$ and $A_2A_2' = (A_2 - A_1)/A_1$, where (ξ', x') is identified with the tangent vector $\sum_{i=1}^{n+1} \xi_i' \partial/\partial\xi_i + \sum_{i=1}^{n+1} x_i' \partial/\partial x_i$. Since $|\partial(y_1, y_2)/\partial(x_1, x_2)|(0, 0) \neq 0$, by the elementary symplectic transformation with respect to a suitable set of indices $J \subset \{3, 4, \cdots, n+1\}$:

$$x_j \longmapsto \begin{cases} x_j \text{ if } j \notin J \\ \xi_j \text{ if } j \in J \end{cases}, \quad \xi_j \longmapsto \begin{cases} \xi_j \text{ if } j \notin J \\ -x_j \text{ if } j \in J \end{cases}$$

and by the transformation of some indices in $\{3, 4, \cdots, n+1\}$ (remark that these transformations fix V_1 and V_2), we may assume

$$\left|\frac{\partial(y_1, \cdots, y_{n+1})}{\partial(x_1, \cdots, x_{n+1})}\right|(0, 0) \neq 0, \quad \left|\frac{\partial(y_1, y_2, y_4, \cdots, y_{n+1})}{\partial(x_1, x_2, x_4, \cdots, x_{n+1})}\right|(0, 0) \neq 0.$$

Then $\tilde{\Psi}$ can be represented by a generating function $\Omega(\xi, x, y)$ as follows (cf. [1]):

$$\begin{cases} \Omega = \Omega'(\xi, y) - \sum_{j=1}^{n+1} \xi_j x_j, \\ \eta_j = \dfrac{\partial\Omega'}{\partial y_j}(\xi, y), \ x_j = \dfrac{\partial\Omega'}{\partial\xi_j}(\xi, y) \quad \text{for } j=1, \cdots, n+1. \end{cases}$$

Set $\Omega_t'=\Omega'(\xi_1, \xi_2, t, \xi_4, \cdots, \xi_{n+1}, y_1, y_2, 0, y_4, \cdots, y_{n+1})$. Then the equations $\tilde{\eta}_j=\partial\Omega_t'/\partial y_i$, $\tilde{x}_j=\partial\Omega_t'/\partial\xi_j$ for $j=1, 2, 4, \cdots, n+1$ define the symplectic transformation $\Phi_t: (\xi_1, \xi_2, \xi_4, \cdots, \xi_{n+1}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_4, \cdots, \tilde{x}_{n+1})\longmapsto(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_4, \cdots, \tilde{\eta}_{n+1}, y_1, y_2, y_4, \cdots, y_{n+1})$ which satisfies the condition (8) with $m=n$. Thus the proof of the theorem reduces to that of the lemma.

Proof of Lemma. Suppose there exists Φ_t satisfying (8). Replacing Φ_t by $\Phi_0^{-1}\circ\Phi_t$, we may assume $\Phi_0=id$. Then the generating function Ω_t of $\Phi_t: (\xi, x)\longmapsto(\eta, y)$ is of the form

$$(9)\qquad\begin{cases} \Omega_t=\Omega_t'(\xi, y)-\sum_{j=1}^{m}\xi_j x_j, \\ \Omega_t'=\sum_{j=1}^{m}\xi_j y_j+U(\xi, y, t)t,\ U=\sum_{i=0}^{\infty}U_i(\xi, y)t^i, \\ \left.\begin{aligned}\eta_j=\partial\Omega_t'/\partial y_j=\xi_j+t\partial U/\partial y_j \\ x_j=\partial\Omega_t'/\partial\xi_j=y_j+t\partial U/\partial\xi_j\end{aligned}\right\}\ \text{for } j=1, \cdots, m. \end{cases}$$

Since $\Phi_t(W_1)=\{(\eta, y)\in\tilde{X};\ \eta_1=0\}$, we have

$$(10)\qquad \left.\frac{\partial U}{\partial y_1}\right|_{\xi_1=0}=0$$

and since $\Phi_t(W_{2,t})=\{(\eta, y)\in\tilde{X};\ \eta_1+(1/2)y_1^2+y_2=0\}$, we have

$$\xi_1+\frac{1}{2}(y_1+t\partial U/\partial\xi_1)^2+(y_2+t\partial U/\partial\xi_2)+t\tilde{h}(\xi, y+t\partial U/\partial\xi)(y_2+t\partial U/\partial\xi_2)^l$$
$$=(1+t\varphi)\left(\xi_1+t\partial U/\partial y_1+\frac{1}{2}y_1^2+y_2\right)$$

for a function $\varphi=\sum_{i=0}^{\infty}\varphi_i(\xi, y)t^i$, where $y+t\partial U/\partial\xi=(y_1+t\partial U/\partial\xi_1, \cdots, y_m+t\partial U/\partial\xi_m)$. Therefore

$$(11)\qquad -\frac{\partial U}{\partial y_1}+y_1\frac{\partial U}{\partial\xi_1}+\frac{\partial U}{\partial\xi_2}-\left(\xi_1+\frac{1}{2}y_1^2+y_2\right)\varphi$$
$$=-\frac{t}{2}\left(\frac{\partial U}{\partial\xi_1}\right)^2-\left(y_2+t\frac{\partial U}{\partial\xi_2}\right)^l\tilde{h}\left(\xi, y+t\frac{\partial U}{\partial\xi}\right)+t\varphi\frac{\partial U}{\partial y_1}.$$

By the coordinate transformation $\tau_1=\xi_1+(1/2)y_1^2, \tau_2=\xi_2+y_1, z_1=y_1, z_2=y_2$, the left hand side of (11) equals $-\partial U/\partial z_1-(\tau_1+z_2)\varphi$. Hence we have

$$(10)_k\qquad \left.\frac{\partial U_k}{\partial y_1}\right|_{\xi_1=0}=0,$$

$$(11)_k\qquad -\frac{\partial U_k}{\partial z_1}-(\tau_1+z_2)\varphi_k=\text{the function determined only by } (U_i, \varphi_i)_{0\leq i\leq k-1}$$

for $k=0,1,2,\cdots$. Especially when $k=0$,

$$(10)_0 \qquad \left.\frac{\partial U_0}{\partial y_1}\right|_{\xi_1=0}=0\,,$$

$$(11)_0 \qquad -\frac{\partial U_0}{\partial z_1}-(\tau_1+z_2)\varphi_0=-y_2{}^l h(\xi, y)$$

Now we fix a function $f(z_1, z_2, \tau_1, \tau_2)$ such that $\partial f/\partial z_1=y_2{}^l\tilde{h}(\xi, y)$. For simplicity we don't write the variables $(\xi_3, \cdots, \xi_m, y_3, \cdots, y_m)$ hereafter. Then $(10)_0$ and $(11)_0$ imply

$$(12) \qquad \begin{aligned} U_0&=f(z_1, z_2, \tau_1, \tau_2)-(\tau_1+z_2)\phi_0(z_1, z_2, \tau_1, \tau_2)-\phi_1(z_2, \tau_1, \tau_2)\\ &=\phi_2(y_2, \xi_2)+\xi_1\phi_3(y_1, y_2, \xi_1, \xi_2)\end{aligned}$$

for some functions $\phi_0, \cdots, \phi_3$, where $\phi_0=\int_0^{z_1}\varphi_0 dz_1$. Put $\xi_1=0, y_2=-(1/2)y_1{}^2$ and $F(y_1, \xi_2)=f(y_1, -(1/2)y_1{}^2, (1/2)y_1{}^2, \xi_2+y_1)$. Then by (12) we have the representation

$$(13) \qquad F(y_1, \xi_2)=F_1(y_1{}^2, \xi_2+y_1)+F_2(y_1{}^2, \xi_2).$$

On the other hand if F can be written in the above form, it is clear that there exist functions $\phi_0, \cdots, \phi_3$ satisfying (12).

Next we shall study the representation (13) for a given F. Put

$$\begin{aligned}\xi^k y^{2N-k}&=\sum_{j=0}^{N} a_{k,2N-k,j}(\xi+y)^{2j}y^{2(N-j)}\\ &\quad+\sum_{j=0}^{N} b_{k,2N-k,j}\xi^{2j}y^{2(N-j)} \quad \text{for } k=0, \cdots, 2N,\\ \xi^k y^{2N+1-k}&=\sum_{j=0}^{N} a_{k,2N+1-k,j}(\xi+y)^{2j+1}y^{2(N-j)}\\ &\quad+\sum_{j=0}^{N} b_{k,2N+1-k,j}\xi^{2j+1}y^{2(N-j)} \quad \text{for } k=0, \cdots, 2N+1.\end{aligned}$$

Here $a_{k,i,j}$ and $b_{k,i,j}$ are real numbers. Since $2(N+1)-(2N+1)=1$, $a_{k,2N-k,0}+b_{k,2N-k,0}$ and $a_{k,2N-k,j}$ and $b_{k,2N-k,j}$ for $j=1, \cdots, N$ can be uniquely defined and since $2(N+1)-(2N+2)=0$, $a_{k,2N+1-k,j}$ and $b_{k,2N+1-k,j}$ for $j=0, \cdots, N$ can be also uniquely defined. (Their existence will be proved later.) Therefore we can put

$$\begin{cases} a_{k,2i,j}=a_{k,2i-2,j}=\cdots=a_{k,0,j}=0,\\ b_{k,2i,j}=b_{k,2i-2,j}=\cdots=b_{k,0,j}=\begin{cases}1 \text{ if } k=2j \text{ or } 2j+1,\\ 0 \text{ otherwise,}\end{cases}\\ a_{k,2i+1,j}=a_{k,2i-1,j}=\cdots=a_{k,1,j}=a_{k,j},\\ b_{k,2i+1,j}=b_{k,2i-1,j}=\cdots=b_{k,1,j}=b_{k,j}.\end{cases}$$

Here $a_{k,j}$ and $b_{k,j}$ are defined to satisfy

$$\xi^{2k-1}=\sum_{j=0}^{k} a_{2k-1,j}(\xi+1)^{2j}+\sum_{j=0}^{k} b_{2k-1,j}\xi^{2j}, \tag{14}$$

$$\xi^{2k-2}=\sum_{j=0}^{k-1} a_{2k-2,j}(\xi+1)^{2j+1}+\sum_{j=0}^{k-1} b_{2k-2,j}\xi^{2j+1}. \tag{15}$$

Applying $2k\int_0^\xi d\xi$ and $(2k-1)\int_0^\xi d\xi$ to (14) and (15), respectively, we have

$$\begin{aligned}\xi^{2k}=&\sum_{j=0}^{k}\frac{2k}{2j+1}a_{2k-1,j}(\xi+1)^{2j+1}+\sum_{j=0}^{k}\frac{2k}{2j+1}b_{2k-1,j}\xi^{2j+1}\\ &-\sum_{j=0}^{k}\frac{2k}{2j+1}a_{2k-1,j}\end{aligned}$$

$$\begin{aligned}\xi^{2k-1}=&\sum_{j=0}^{k-1}\frac{2k-1}{2j+2}a_{2k-2,j}(\xi+1)^{2j+2}+\sum_{j=0}^{k-1}\frac{2k-1}{2j+2}b_{2k-2,j}\xi^{2j+2}\\ &-\sum_{j=0}^{k-1}\frac{2k-1}{2j+2}a_{2k-2,j}.\end{aligned}$$

Therefore $a_{k,j}$ and $b_{k,j}$ are defined by

$$a_{2k,j}=\frac{2k}{2j+1}a_{2k-1,j},\ a_{0,0}=1,\quad a_{0,j}=0 \text{ for } j\geqq 1,$$

$$b_{2k,j}=\frac{2k}{2j+1}b_{2k-1,j},\ b_{0,0}=-1,\quad b_{0,j}=0 \text{ for } j\geqq 1,$$

$$a_{2k-1,j}=\frac{2k-1}{2j}a_{2k-2,j-1}\quad \text{for } j\geqq 1,$$

$$b_{2k-1,j}=\frac{2k-1}{2j}b_{2k-2,j-1}\quad \text{for } j\geqq 1,$$

$$a_{2k-1,0}=-\sum_{j=1}^{k}\frac{1}{2j+1}a_{2k-1,j}=-\sum_{j=1}^{k}\frac{2k-1}{2j(2j+1)}a_{2k-2,j-1},$$

$$b_{2k-1,0}=-a_{2k-1,0}-\sum_{j=0}^{k-1}\frac{2k-1}{2j+2}a_{2k-2,j}.$$

Hence we have the following

$$\begin{aligned}a_{2k,j}&=\frac{2k}{2j+1}a_{2k-1,j}=\frac{2k(2k-1)}{(2j+1)2j}a_{2k-2,j-1}=\cdots\\ &=\frac{(2k)!}{(2k-2j)!(2j+1)!}a_{2(k-j),0},\end{aligned}$$

$$a_{2k,0}=2ka_{2k-1,0}=-\sum_{j=1}^{k}\frac{(2k)(2k-1)}{(2j)(2j+1)}a_{2k-2,j-1}$$

$$=-\sum_{j=1}^{k}a_{2k,j}=\sum_{j=1}^{k}\frac{-(2k)!}{(2k-2j)!\,(2j+1)!}a_{2(k-j),0}.$$

To estimate the asymptotic behaviour of $a_{2k,0}$ when k tends to infinity, put $c_k=(-1)^k a_{2k,0}/(2k)!$. Then

$$c_k=\frac{1}{3!}c_{k-1}-\frac{1}{5!}c_{k-2}+\frac{1}{7!}c_{k-3}-\cdots-\frac{(-1)^k}{(2k+1)!}c_0,\quad c_0=1 \tag{16}$$

and thus

$$\sum_{k=0}^{\infty}c_k z^{2k}=\left(1+\sum_{j=1}^{\infty}(-1)^j\frac{z^{2j}}{(2j+1)!}\right)^{-1}=\frac{z}{\sin z}\,. \tag{17}$$

Hence $\overline{\lim}\sqrt[k]{|c_k|}=\pi^{-2}$ because the radius of convergence of $z/\sin z$ equals π. Therefore $\overline{\lim}\sqrt[2k]{|a_{2k,0}|/(2k)!}>0$, which proves the lemma. (Remark that $\lim c_{k-1}/c_k=\pi^2$ and $(2k)!c_k=(2^{2k}-2)B_k$ with Bernoulli's numbers B_k.)

Supplement Considering the above argument, we see that the Sato's problem is affirmative in the ring of formal power series.

K. Masuda proved in our seminar that the equation (13) is always solvable in the C^∞-category.

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Bibliography

[1] C. Caratheodory: Calculus of variations and partial differential equations of the first order, Part I, *Holden-Day*, San Francisco, 1965.

[2] M. Kashiwara and T. Kawai: Hidden boundary conditions, *Sûrikaiseki-kenkyûsho Kôkyûroku*, **227** (1975), 39-41 (in Japanese).

[3] R. B. Melrose: Equivalence of glancing hypersurfaces, *Invent. Math.*, **37** (1976), 165-191.

[4] ———: Equivalence of glancing hypersurfaces II, preprint.

[5] T. Oshima: Local equivalence of differential forms and their deformations, *Sûrikaiseki-kenkyûsho Kôkyûroku*, **266** (1976), 108-129 (in Japanese).