The Rational Forms on the Weighted Projective Space and the Filtration by the Order of Pole

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§ 0. Introduction.

Let X be a weighted projective space (over $\boldsymbol{C}$) with weight $(b_0, \cdots, b_n)$, where $b_i \in \boldsymbol{Z}_{++} = \{n \in \boldsymbol{Z};\ n > 0\}$ for $i = 0, \cdots, n$. In other words, $X = \operatorname{Proj} \boldsymbol{C}[z_0, \cdots, z_n]$ with grade $(z_i) = b_i$ for $i = 0, \cdots, n$. X is not necessarily smooth but always normal.

Let D be a divisor on X. Then D can be assumed to be defined by a weighted homogeneous polynomial f of weight $\left(\frac{b_0}{d}, \cdots, \frac{b_n}{d}\right)$, $d \in \boldsymbol{Z}_{++}$.

Define sheaves

$$\tilde{\Omega}^q_X(kD) = j_* \Omega^q_{X-\Sigma}(k(X-\Sigma)),$$

where $\Sigma = \operatorname{sing} X$, j is the inclusion map of $X-\Sigma$ to X, and $q, k \in \boldsymbol{Z}_+ = \{n \in \boldsymbol{Z};\ n \geq 0\}$ ([4]). If $k=0$, we simply write $\tilde{\Omega}^q_X$ instead of $\tilde{\Omega}^q_X(0D)$.

Notation. For $q, k \in \boldsymbol{Z}_+$, *put*

$$A^q_k(X) = \Gamma(X, \tilde{\Omega}^q_k(kD)).$$

If either k *or* q *is negative, put*

$$A^q_k(X) = 0.$$

For $q, k \in \boldsymbol{Z}$, *we put*

$$K^{\cdot}_{q-k} = \{\cdots \xrightarrow{d} A^q_k(X) \xrightarrow{d} A^{q+1}_{k+1}(X) \xrightarrow{d} \cdots\},$$

where the d *are induced mappings by the exterior derivatives on* $X-\Sigma$. *Obviously*

$$\cdots K^{\cdot}_i \supset K^{\cdot}_{i+1} \supset \cdots \supset K^{\cdot}_{n+1} = 0.$$

Put $K^{\cdot}=\bigcup_{i=-\infty}^{n} K_i^{\cdot}$, $F^i(K^{\cdot})=K_i^{\cdot}$.

Then we have a filtered complex $(K^{\cdot}, F)$.

J. Steenbrink proved that a complex $(\tilde{\Omega}_X^{\cdot}, d)$ gives a resolution of the constant sheaf $\boldsymbol{C}_X$. Thus we have a filtration F on $H^p(K^{\cdot})=H^p(X-D; \boldsymbol{C})$ for any p. We shall study the spectral sequence $\{E_r^{p,q}(K^{\cdot}, F), E_\infty^n(K^{\cdot}, F)\}$ and abbreviate it to $\{E_r^{p,q}, E_\infty^n\}$.

Denote the affine algebraic set $\{p\in \boldsymbol{C}^{n+1}; f(p)=0\}$ in $\boldsymbol{C}^{n+1}$ by $V(f)$. When $V(f)$ has an isolated singularity at the origin, Steenbrink proved that $E_1^{p,q}=E_\infty^{p,q}$ for any p, q ([4]). His result is a generalization of a theorem by Ph. Griffiths in [1], which claims the same result when X is the ordinary projective space. We can extend the results to

MAIN THEOREM. $E_1^{p,q}=E_\infty^{p,q}=0$ *for any* p, q, *satisfying* $0<p+q<\operatorname{codim}_{\boldsymbol{C}^{n+1}}$ (Sing $V(f))-1$, *and*

$$E_1^{p,-p}=E_\infty^{p,-p}=\begin{cases}\boldsymbol{C}, & p=0,\\ 0, & p\neq 0.\end{cases}$$

This was proved in [5] when $X=\boldsymbol{P}^n$. The proof which we will give here is almost parallel to that of [5].

§1. *G*-invariant forms.

Another definition of the weighted projective space X with grade$(z_i)=b_i$, $i=0, \cdots, n$ is as follows:

Define a subgroup G, whose order is $b_0, \cdots, b_n$, of $PGL(n+1; \boldsymbol{C})$ as

$$G=\left\{\begin{bmatrix}\exp\left(\frac{\beta_0}{b_0}2\pi\sqrt{-1}\right) & & 0\\ & \ddots & \\ 0 & & \exp\left(\frac{\beta_n}{b_n}2\pi\sqrt{-1}\right)\end{bmatrix}\right\},$$

$\beta_i=0, 1, \cdots, b_i-1$.

Then G acts on $\boldsymbol{P}^n$ and the quotient scheme $\boldsymbol{P}^n/G$ is isomorphic to X. Let ρ be the natural projection

$$\rho: \boldsymbol{P}^n\to \boldsymbol{P}^n/G\simeq X.$$

Put $\rho^{-1}(D)=E\subset \boldsymbol{P}^n$. Then G acts on $\rho_*\Omega^q_{P^n}(kE)$ and we have the following

LEMMA 1.1. *$(\rho_*\Omega^q_{P^n}(kE))^G=\tilde{\Omega}^q_X(kD)$, where $q, k\in \boldsymbol{Z}_+$.*

Proof. It suffices to prove

$$(*)\quad (\rho_*\Omega^q_V(kE))^G=\tilde{\Omega}^q_{V/G}(kD)\,,$$

where V is an open set of $\boldsymbol{C}^n$, G is a finite subgroup of $GL(n; \boldsymbol{C})$, and ρ is the natural projection of V onto V/G.

By D. Prill, we have a normal big subgroup G_0 of G and a diagram

$$\begin{tikzcd} V \arrow[r, "\rho_1"] & V/G_0 \arrow[r, "\rho_2"] \arrow[d, "\wr"] & V/G \arrow[d, "\wr"] \\ & V' \arrow[r] \arrow[ur, phantom, "\cap"] & V'/G_s\,, \end{tikzcd}$$

where V' is an open subset of $\boldsymbol{C}^n$, G_s is small, $G/G_0\cong G_s\subset GL(n; \boldsymbol{C})$, and ρ_1, ρ_2 are the natural projections such that $\rho=\rho_2\circ\rho_1$ (see [2]).

Therefore, it suffices to prove $(*)$ when G is small or big.

In the case when G is small, see [4].

Next assume that G is big. Denote the discriminant of ρ by $\mathcal{D}$. Since G is generated by rotations around a hyperplane, $\mathcal{D}$ is the union of the images of these hyperplanes.

Take a smooth point x of $\mathcal{D}$, and $x\in\mathcal{D}_i$, where $\mathcal{D}=\bigcup_i \mathcal{D}_i$ (irreducible decomposition). There exists an open neighborhood U_x of x which doesn't intersect $\bigcup_{j\neq i}\mathcal{D}_j$.

Consider the restriction map

$$\rho'=\rho|_{\rho^{-1}(U_x)}: \rho^{-1}(U_x)\longrightarrow U_x$$

and the subgroup H_i which fixes a hyperplane $\rho^{-1}(\mathcal{D}_i\cap U_x)$.

Then ρ' can be decomposed into $\rho_2'\circ\rho_1'$:

$$\rho^{-1}(U_x)\xrightarrow[\rho_1']{}\rho^{-1}(U_x)/H_i\xrightarrow[\rho_2']{}\rho^{-1}(U_x)/G\,,$$

where ρ_1' and ρ_2' are the natural projections such that $\rho'=\rho_2'\circ\rho_1'$. ρ_2' is etale and ρ_1' can be regarded as the local version of

$$\mu:\ \operatorname{Spec}\boldsymbol{C}[z_1, \cdots, z_n]\longrightarrow \operatorname{Spec}\boldsymbol{C}[z_1^k, z_2, \cdots, z_n]\,,$$

where μ is induced by the inclusion map of

$$\boldsymbol{C}[z_1^k, z_2, \cdots, z_n]\ \text{ into }\ \boldsymbol{C}[z_1, \cdots, z_n]\,,\quad k\in\boldsymbol{Z}_{++}\,.$$

Notice that the holomorphic 1-forms on Spec $\boldsymbol{C}[z_1^k, z_2, \cdots, z_n]$ can be regarded as a free $\boldsymbol{C}[z_1^k, z_2, \cdots, z_n]$-module with a basis $\{z_1^{k-1}dz_1, dz_2, \cdots, dz_n\}$. The holomorphic p-forms are the p-th exterior products of holomorphic 1-forms.

Thus we know by the direct computation that our claim holds true for ρ_1'.

On the other hand the singular points of $\mathcal{D}$ is at least codimension two in $\boldsymbol{C}^n/G$, which proves (1.1) in the light of the extension theorem.

By this lemma, we identify the G-invariant part of $A_k^q=\Gamma(\boldsymbol{P}^n, \Omega^q(kE))$ with $A_k^q(X)$, i.e., $(A_k^q)^G=A_k^q(X)$, for $q, k\in\boldsymbol{Z}_+$.

§ 2. The proof of Main Theorem.

Put $Q(z_0, \cdots, z_n)=f(z_0^{b_0}, \cdots, z_n^{b_n})$. Then Q is a homogeneous polynomial of degree d.

Definition 2.1. *For $i, k\in\boldsymbol{Z}_+$, put*

$$H_k^i=\{\textit{homogeneous rational i-forms } \omega \textit{ on } \boldsymbol{C}^{n+1};\ Q^k\omega \textit{ is holomorphic}\},$$

where by homogeneous forms we mean the invariant forms under the transformations $(z_0, \cdots, z_n)\longmapsto(cz_0, \cdots, cz_n)$ for any $c\in\boldsymbol{C}^*$.

If either i or k is negative, we define $H_k^i=0$.

G acts on H_k^i, because G can be regarded as a subgroup of $GL(n+1; \boldsymbol{C})$. Thus we can define the G-invariant part $(H_k^i)^G$.

Let d_L: $H_k^i\to H_{i+1}^{k+1}$ be the exterior derivative and d'_L: $H_k^i\to H_{k+1}^{i+1}$ be the morphism defined by

$$d'_L\omega=\frac{1}{d}\frac{dQ}{Q}\wedge\omega$$

for any $\omega\in H_k^i$ for $i, k\in\boldsymbol{Z}$.

It is easy to see that d_L and d'_L define

$$d_L:\ (H_k^i)^G\longrightarrow(H_{k+1}^{i+1})^G$$

and

$$d'_L:\ (H_k^i)^G\longrightarrow(H_{k+1}^{i+1})_G\,.$$

Put

$$L_{i-k}^{\cdot}=\{\cdots\xrightarrow{d_L}(H_k^i)^G\xrightarrow{d_L}(H_{k+1}^{i+1})^G\xrightarrow{d_L}\cdots\},$$

$$'L_{i-k}^{\cdot}=\{\cdots\xrightarrow{d'_L}(H_k^i)^G\xrightarrow{d'_L}(H_{k+1}^{i+1})^G\xrightarrow{d'_L}\cdots\},$$

and

$$\cdots\supset L_i^{\cdot}\supset L_{i+1}^{\cdot}\supset\cdots\supset L_{n+2}^{\cdot}=0\,,$$

$$\cdots\supset 'L_i^{\cdot}\supset 'L_{i+1}^{\cdot}\supset\cdots\supset 'L_{n+2}^{\cdot}=0\,.$$

Moreover we put

$$L^{\cdot}=\bigcup_{i=-\infty}^{n+1} L_i^{\cdot}\,, \qquad W^p(L^{\cdot})=L_p^{\cdot}\,,$$

$$'L^{\cdot}=\bigcup_{i=-\infty}^{n+1} {}'L_i^{\cdot}\,, \qquad 'W^p('L^{\cdot})={}'L_p^{\cdot}\,.$$

Thus we have two filtered complexes $(L^{\cdot}, W)$ and $('L^{\cdot}, 'W)$.

Define *the Euler vector field*

$$\theta=\sum_{j=1}^{n+1} z_j \frac{\partial}{\partial z_j}$$

on $\boldsymbol{C}^{n+1}$ and a homomorphism

$$\langle\theta, \quad\rangle: \ H_k^i \longrightarrow H_k^{i-1}$$

for $i, k\in \boldsymbol{Z}$ by the contraction. Then $\langle\theta, \quad\rangle$ induces the restriction

$$\langle\theta, \quad\rangle: \ (H_k^i)^G \longrightarrow (H_k^{i-1})^G\,.$$

We have another homomorphism

$$\pi: \ (A_k^i)_G \longrightarrow (H_k^i)^G$$

for $i, k\in \boldsymbol{Z}$, where π is the pull back by the natural projection: $\boldsymbol{C}^{n+1}-\{0\}\to \boldsymbol{P}^n$.

The following lemma is due to K. Saito:

LEMMA 2.2. *$H^q('L_i^{\cdot})=0$ for $q<s$, where*

$$s=ht\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right)=\operatorname{codim}_{\boldsymbol{C}^{n+1}}(\operatorname{Sing} V(f))\,.$$

Proof. For any $\omega\in(H_k^i)^G$, $Q^k\omega$ is of degree dk and is an element of the free $\boldsymbol{C}[z_0^{b_0}, \cdots, z_n^{b_n}]$-module with a basis $\{z_0^{b_0-1}dz_0, \cdots, z_n^{b_n-1}dz_n\}$. Thus apply Saito's generalized de Rham lemma ([3]).

This key lemma proves Main Theorem by almost the same methods as in [5].

COROLLARY 2.3. *Let q be an integer satisfying $0<q<s-1$. If $\varphi\in A_k^q(X)$ is closed, then there exists $\psi\in A_{k-1}^{q-1}(X)$ such that $d\psi=\varphi$.*

Remark. Let g be a weighted homogeneous polynomial of weight $\left(\frac{b_0}{d}, \cdots, \frac{b_{n-1}}{d}\right)$. Put $f(z_0, \cdots, z_n)=g(z_0, \cdots, z_{n-1})-z_n^d$. Let X be a weighted projective space of weight $(b_0, \cdots, b_{n-1}, 1)$ and D a divisor defined by $\{f=0\}$. Let U be a Zariski open set of X defined by $\{z_n\neq 0\}$.

Put $X_\infty = X - U$, $D_\infty = D \cap X_\infty$, $D_1 = D - D_\infty$. We have an exact sequence

$$\cdots \longrightarrow H^q(X_\infty - D_\infty) \longrightarrow H^q(X-D) \longrightarrow H^q(U-D_1) \longrightarrow H^{q-1}(X_\infty - D_\infty) \longrightarrow \cdots,$$

where the coefficients of the cohomology groups are all $\boldsymbol{C}$. So far we have filtrations on $H^q(X-D)$ and $H^{q-1}(X_\infty - D_\infty)$. These filtrations can introduce two filtrations on $H^q(U-D_1) \simeq H^{q-1}(D_1)$ and these give a mixed Hodge structure on $H^{q-1}(D_1)$ (see [4]).

By applying Main Theorem we have

$$H^q(X-D)=0$$

for $0<q<ht\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right) = ht\left(\frac{\partial g}{\partial z_0}, \cdots, \frac{\partial g}{\partial z_{n-1}}\right) - 1 = \operatorname{codim}_{C^n}(\operatorname{Sing} D_0) - 1$.

Moreover,

$$H^{q-1}(X-D)=0$$

for $0<q-1<ht\left(\frac{\partial g}{\partial z_0}, \cdots, \frac{\partial g}{\partial z_{n-1}}\right) = \operatorname{codim}_{C^n}(\operatorname{Sing} D_0)$.

This shows $H^q(D_1)=0$ for $0<q<\operatorname{codim}_{C^n}(\operatorname{Sing} D_0)$, which is known in topology.

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