On the Structure of Integrals of Power Product of Linear Functions

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Introduction

1.

Let $f_0, f_1, \cdots, f_m, m \geqq n+1$ be linear functions in the complex affine space $\boldsymbol{C}^n$ and S be the union of the hyperplanes S_j: $f_j=0, 1 \leqq j \leqq m$. We are going to compute the *global monodromy* and the *Gauss-Manin connection* of the integral:

$$J=\begin{cases} \int f_1^{\lambda_1}\cdot f_2^{\lambda_2}\cdots f_m^{\lambda_m} dx_1\wedge\cdots\wedge dx_n \quad \text{or} \\ \int f_1^{\lambda_1}\cdot f_2^{\lambda_2}\cdots f_m^{\lambda_m}\exp(f_0) dx_1\wedge\cdots\wedge dx_n \end{cases} \tag{1.1}$$

for $\lambda=(\lambda_1, \cdots, \lambda_m)\in\boldsymbol{C}^m$ and to express them *in terms of the simplices* $\langle i_1, \cdots, i_n\rangle$ or $\langle i_0, i_1, \cdots, i_n\rangle$ attached to the configuration S (See §4. Theorem and the Appendix). To apply for *J Pham's generalized Picard-Lefschetz formula* ([10] p. 128) we shall define certain real twisted cycles called "*visible cycles*" and derive a connection formula among them (See [1] p. 253 and also [4]). As is shown in [2], *by Deligne-Grothendieck comparison theorem*, the above integral can be regarded as the duality between de Rham rational cohomology $H^n(M, \nabla_\omega)$ and the homology $H_n(M, \mathcal{S}_{-\omega})$ in the space $M=\boldsymbol{C}^n-S$, where ∇_ω denotes the Gauss-Manin connection corresponding to the 1-form $\omega=\sum_{j=1}^m \lambda_j df_j/f_j$ or $\sum_{j=1}^m \lambda_j df_j/f_j+\sum_{j=1}^m \mu_j dx_j$ with $f_0=\sum_{j=1}^m \mu_j x_j$, and $\mathcal{S}_{-\omega}$ the *local system* or the *relative local system modulo at infinity* defined by the holonomy group of $\nabla_{-\omega}$ respectively, according as $\boldsymbol{\mu}=(\mu_1, \cdots, \mu_n)$ equal to zero or not.[1] In the sequel we shall assume

(H, 1) *$S_1, \cdots, S_m, S_{m+1}$ are all real and normally crossing each other where S_{m+1} denotes the hyperplane at infinity.*

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1) The integral (1, 1) can also be formulated by means of Ext of certain modules of differential operators with coefficients of hyperfunctions done by M. Kashiwara and T. Kawai. For this see [5].

(H, 2) *$\lambda_1, \lambda_2, \cdots, \lambda_m$ are all real and positive and μ is real. These are all generic, namely $\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_p}\notin \boldsymbol{Z}$ for any $1\leqq i_1<i_2<\cdots\leqq m+1, 1\leqq p\leqq n$, where λ_{m+1} denotes $-\sum_{j=1}^{m}\lambda_j$.*

The integral (1, 1) contains in its limit cases various classes of hypergeometric functions, for example, of P. Appell or L. Pochhammer (See [6] and [11]). $H_n(M, \mathcal{S}_{-\omega})$ has also been determined in [4]. In the case where $m=4, n=2, \mu_i=0 (1\leqq i\leqq 2)$ and λ_i are all equal to 1/2 a cell structure of algebraic surfaces related to (1, 1) has been investigated by T. Nakamura in an unpublished note [8].

2. Numbering of visible cycles.

The following lemma is proved in [1] and [2] (See [2] pp. 292–294).

LEMMA 1. *$H^n(M, \mathcal{F}_\omega)\cong \Omega^n(M, \log S)/\omega\wedge\Omega^{n-1}(M, \log S)$ if $\mu=0$ and $\cong\Omega^n(M, \log S)$ if $\mu\neq 0$, where $\Omega^q(M, \log S)$ denotes the space of logarithmic p-forms on M along S, spanned by the forms $d\log f_{i_1}\wedge\cdots\wedge d\log f_{i_p}$ with $1\leqq i_1<i_2<\cdots<i_p\leqq m$. Therefore its rank is equal to $\binom{m-1}{n}$ if $\mu=0$ or $\binom{m}{n}$ if $\mu\neq 0$. If $\mu=0$ this is also equal to the number of relatively compact connected components of $\boldsymbol{R}^n-\boldsymbol{R}^n\cap S$. If $\mu\neq 0$, this is equal to the number of relatively compact components and components contained in the halfspace $\sum_{j=1}^{n}\mu_j x_j<0$ at the infinity. Actually the set of the closures in $\boldsymbol{R}^n$ of these components can be regarded as "twisted cycles" which constitute a basis (denoted by δ) of $H_n(M, \mathcal{S}_{-\omega})$.*

The set of the closures in $\boldsymbol{R}^n$ of components of $\boldsymbol{R}^n-\boldsymbol{R}^n\cap S$ will be denoted by $\hat{\delta}$. Each element of δ will be called "*visible cycles*". We denote by $[I]=[i_1, i_2, \cdots, i_n], 1\leqq i_1<i_2<\cdots<i_n\leqq m+1$ the point in $\boldsymbol{R}^n$ defined by $f_{i_1}=f_{i_2}=\cdots=f_{i_n}=0$ (Remark that $[I]$ is to be at the infinity if $i_n=m+1$). We shall fix the lexicographic order for $[I]$ as follows: $[I]<[J]$ if and only if $i_1=j_1, \cdots, i_{p-1}=j_{p-1}$ and $i_p<j_p$ for some p. Let f_i have an expression $a_{i0}+\sum_{\nu=1}^{n}a_{i\nu}\cdot x_\nu$ and $\bar{f}_i$ be the principal part $\sum_{\nu=1}^{n}a_{i\nu}\cdot x_\nu$. We choose a special configuration $\mathfrak{K}_0$ of S such that if $[I]<[J]$, then

$$\bar{f}_0([I])=\bar{f}_0([J]), \cdots, \bar{f}_{p-1}([I])=\bar{f}_{p-1}([J]) \quad \text{and} \quad (-1)^n\bar{f}_p([I])<(-1)^n\bar{f}_p([J])$$

for some p (See the figures 1, 2). We denote by

$$\mathcal{Z}_-[i_1, \cdots, i_n; \mathfrak{K}_0] \ (\text{or } \mathcal{Z}_+[i_1, \cdots, i_n; \mathfrak{K}_0]), 1\leqq i_1<i_2<\cdots<i_n\leqq m+1,$$

the relative chain in $(\boldsymbol{R}^n, S\cap\boldsymbol{R}^n)$ defined by $(-1)^{n-\nu}f_{i_\nu}\geqq 0$ (or $(-1)^{n-\nu}f_{i_\nu}\leqq 0$ respectively) for $1\leqq\nu\leqq n$ with the orientation $(-1)^{\frac{n(n+1)}{2}}df_{i_1}\wedge df_{i_2}\wedge\cdots\wedge df_{i_n}>0$, and by $\mathcal{Z}_-^*[i_1, \cdots, i_n; \mathfrak{K}_0]$ the intersection of $\mathcal{Z}_-[I; \mathfrak{K}_0]$ with the half-space $(-1)^n f_1\geqq 0$. We put $\mathcal{Z}_-^*[1, i_2, \cdots, i_n; \mathfrak{K}_0]$ to be equal to 0 for $1<i_2<\cdots<i_n\leqq m+1$. Let $\Delta(I; \mathfrak{K}_0)$ be be the closure of the unique connected component of $\boldsymbol{R}^n-S\cap\boldsymbol{R}^n$ contained in

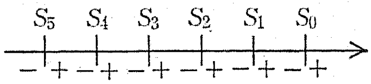

Fig. 1 $(n=1, m=5)$

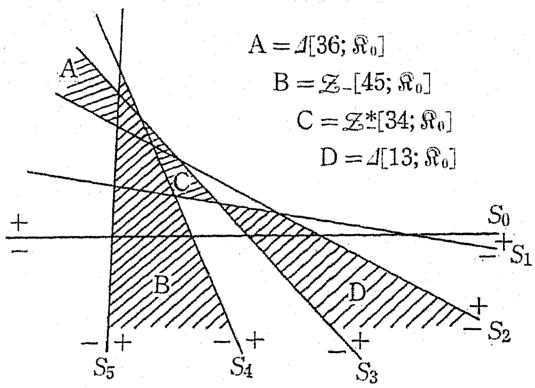

Fig. 2 $(n=2, m=5)$

$\mathfrak{Z}_-[I; \mathfrak{R}_0]$, which contains the point $[I]$ but any point $[J]$ for $[J]>[I]$. We put $\Delta(i_1, \cdots, i_{n-1}, m+2; \mathfrak{R}_0)$ to be $\Delta(1, i_1, \cdots, i_{n-1}; \mathfrak{R}_0)(-1)^{n-1}, 1<i_1$, by introducing an accessary hyperplane S_{m+2}. In view of Lemma 1, the following Proposition can be proved by induction with respect to m and n.

PROPOSITION 1. *In the configuration $\mathfrak{R}_0$ of S $\mathfrak{Z}_-^*[i_1, \cdots, i_n; \mathfrak{R}_0] 2\leqq i_1 < \cdots < i_n \leqq m$, $\mu=0$ or $\mathfrak{Z}_-[i_1, \cdots, i_n; \mathfrak{R}_0], 1\leqq i_1<\cdots<i_n\leqq m, \mu\neq 0$ form a basis of $H_n(M, \mathcal{S}_{-\omega})$. In fact $\Delta(I; \mathfrak{R}_0)$ can be expressed as*

$$\Delta(I; \mathfrak{R}_0) = \begin{cases} \tilde{T}_1 \cdot \tilde{T}_2 \cdots \tilde{T}_n \mathfrak{Z}_-[I; \mathfrak{R}_0], \ \mu \neq 0 \quad \text{and} \\ \tilde{T}_1 \cdot \tilde{T}_2 \cdots \tilde{T}_n \mathfrak{Z}_-^*[I; \mathfrak{R}_0], \ \mu = 0, \end{cases} \tag{2, 1}$$

where $\tilde{T}_\nu$ denotes the ν-th difference operator:

$$\tilde{T}_\nu \mathfrak{Z}_-[i_1, \cdots, i_\nu, \cdots, i_n; \mathfrak{R}_0] = \mathfrak{Z}_-[i_1, \cdots, i_\nu, \cdots, i_n; \mathfrak{R}_0] - \mathfrak{Z}_-[i_1, \cdots, i_\nu - 1, \cdots, i_n; \mathfrak{R}_0]. \tag{2, 2}$$

By replacing them by the contour integrals if necessary, the non-compact components of $\boldsymbol{R}^n - S \cap \boldsymbol{R}^n$ can also be regarded as cycles in $H_n(M, \mathcal{S}_{-\omega})$. $\Delta(J; \mathfrak{R}_0)$, $1\leqq j_1 < \cdots < j_n \leqq m+1$, is contained in $\mathfrak{Z}_-[I; \mathfrak{R}_0]$ if and only if $j_1 \leqq i_1 < j_2 \leqq i_2 < \cdots < j_n \leqq i_n$ and in $\mathfrak{Z}_+[I; \mathfrak{R}_0]$ if and only if $i_1 < j_1 \leqq i_2 < \cdots \leqq i_n < j_n$. Let L be a real line in $\mathfrak{Z}_-[I; \mathfrak{R}] \cup \mathfrak{Z}_+[I; \mathfrak{R}_0]$, going through $[I]$ and choose an affine coordinate v on L such that $\mathfrak{Z}_-[I; \mathfrak{R}_0] \cap L$ corresponds to the half-line $v\leqq 0$. We denote by $v_1, \cdots, v_r, 0, v_{r+1}, \cdots, v_{r+s}$ the intersection points of $L \cap S_j$ in increasing order. Then $v_{r+s} = +\infty$ and v_1 corresponds to $L \cap S_1$, and each interval corresponds to $L \cap \Delta(J; \mathfrak{R}_0)$ for a certain $\Delta(J; \mathfrak{R}_0)$. *We first assume* $\mu=0$. Consider a loop γ_1 homotopic to zero in $L^C - L^C \cap (S \cup S_{m+1})$, L^C being the complexification of L, and turning round once counter-clockwise all the points $v_1, v_2, \cdots, v_r, 0, v_{r+1}, \cdots, v_{r+s}$ and retract it into the upper and lower sides of the real interval $[v_1, +\infty]$ (See the figure 3).

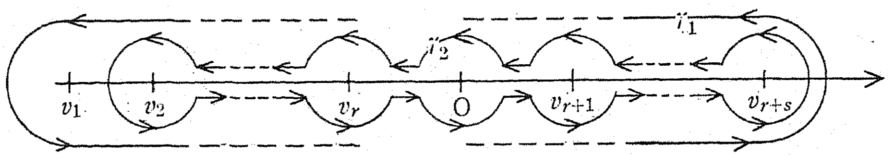

Fig. 3

The branch of the function $U=\exp(f_0)\cdot f_1^{\lambda_1}\cdots f_m^{\lambda_m}$ is different on the upper and lower sides but does not depend on the variation of L provided $L\subset\mathfrak{Z}_-[I;\mathfrak{R}_0]\cup\mathfrak{Z}_+[I;\mathfrak{R}_0]$. In fact the branch of U on the upper side of $L\cap\Delta[J;\mathfrak{R}_0]$ is

$$\exp 2\pi i(\lambda_{m+1}+\sum_{\substack{j_1\leq\sigma\leq i_2-1\\ \cdots\\ j_n\leq\sigma\leq m}}\lambda_\sigma)\quad\text{times}$$

that of U on the lower side of $L\cap\Delta[J;\mathfrak{R}_0]$, because $\Delta[J;\mathfrak{R}_0]$ is divided from the point $[I]$ by the hyperplanes S_k where $i_1<k<j_1,\cdots$, or $i_n<k<j_n$ or $j_1<k<i_1,\cdots$, $j_n<k<i_n$ according as $\Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_+[I;\mathfrak{R}_0]$ or $\Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_-[I;\mathfrak{R}_0]$. Therefore we have the identity in $H_n(M,\mathcal{S}_{-\omega})$

$$(2,3)\qquad \sum_{\substack{\Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_+\cup\mathfrak{Z}_-\\ j_n=m+1}}(1-\exp(2\pi i\lambda_{m+1}))\Delta(j_1,\cdots,j_{n-1},m+1;\bar{\mathfrak{R}}_0)$$
$$+\sum_{\substack{\Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_+\cup\mathfrak{Z}_-\\ j_n\leq m}}\{1-\exp(2\pi i(\lambda_{m+1}+\sum_{\substack{j_1\leq\sigma\leq i_2-1\\ \vdots\\ j_{n-1}\leq\sigma\leq i_{n-1}\\ j_n\leq\sigma\leq m}}\lambda_\sigma))\}\cdot\Delta(J;\mathfrak{R}_0)=0$$

Now consider the path γ_2 starting from $-\infty$ to v_1, turning round v_1 counter-clockwise and coming back from v_1 to $-\infty$, and retract it into the upper and lower side of the real interval $[v_2,+\infty]$ (See the figure 3). The same argument as in the preceding shows that

$$(2,4)\qquad \sum_{\substack{\Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_+\cup\mathfrak{Z}_-\\ j_n=m+1}}\{\exp(2\pi i\lambda_{m+1})-1\}\Delta(j_1,\cdots,j_{n-1},m+1;\mathfrak{R}_0)$$
$$+\sum_{\substack{1\leq j_1<\cdots<j_n\leq m\\ \Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_+\cup\mathfrak{Z}_-}}\{\exp 2\pi i(-\sum_{\substack{1\leq\sigma\leq j_1\\ i_2\leq\sigma\leq j_2\\ \vdots\\ i_n\leq\sigma\leq j_n}}\lambda_\sigma)-1\}\Delta(j_1,\cdots,j_n;\mathfrak{R}_0)=0$$

in $H_n(M,\mathcal{S}_{-\omega})$.

In case of $\mu\neq 0$, by eliminating from (2, 3) and (2, 4) the term $\Delta(j_1,\cdots,j_{n-1},m+1;\mathfrak{R}_0)$ we have

$$(2,5)\qquad \sum_{\substack{\Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_+\cup\mathfrak{Z}_-\\ j_n\leq m}}\{1-\exp 2\pi i(\sum_{\substack{j_1\leq\sigma\leq i_2-1\\ \vdots\\ j_{n-1}\leq\sigma\leq i_{n-1}\\ j_n\leq\sigma\leq m}}\lambda_\sigma+\lambda_{m+1})\}\Delta(J;\mathfrak{R}_0)$$
$$+\sum_{\substack{\Delta(J;\mathfrak{R}_0)\subset\mathfrak{Z}_+\cup\mathfrak{Z}_-\\ j_n\leq m}}\{\exp 2\pi i(-\sum_{\substack{1\leq\sigma\leq j_1\\ i_2\leq\sigma\leq j_2\\ \vdots\\ i_n\leq\sigma\leq j_n}}\lambda_\sigma)-1\}\Delta(J;\mathfrak{R}_0)=0,$$

seeing that the chain $\Delta(j_1, \cdots, j_{n-1}, m+1; \mathfrak{K}_0)$ cannot be defined as an element in $H_n(M, \mathcal{S}_{-\omega})$ if $\mu \neq 0$.

It seems probable that any chain $\Delta(I; \mathfrak{K}_0)$ can be expressed as a linear combination of the basis in Proposition 1 *by means of the identities* (2, 3), (2, 4) *and* (2, 5).[2)]

3. Choice of special paths in the configuration space of S.

From now on we shall fix all $a_{i\nu}, n \geqq \nu \geqq 1, m \geqq i \geqq 1$. We denote by $\mathcal{X}_{m-\nu+1}$ the affine spaces $C^{m-\nu+1}$ of the points $(t_1, \cdots, t_m)$ where $t_i = -a_{i0}$. The configuration $\mathfrak{K}$ of S becomes *degenerated* if and only if the determinant

$$(3,1) \qquad D[i_1, \cdots, i_n, i_{n+1}] = \begin{vmatrix} a_{i_1 0} & \cdots a_{i_1 n-1} a_{i_1 n} \\ a_{i_2 0} & \cdots a_{i_2 n-1} a_{i_2 n} \\ \vdots & \\ a_{i_{n+1} 0} & \cdots a_{i_{n+1} n-1} a_{i_{n+1} n} \end{vmatrix}$$

vanishes for a certain sequence $[i_1, i_2, \cdots i_{n+1}]$, which defines the hyperplane $\mathfrak{H}(i_1, \cdots, i_n, i_{n+1})$ in $\mathcal{X}_m$. We denote by $\mathcal{Y}_m$ the union of such hyperplanes. We shall call $\mathcal{X}_m - \mathcal{Y}_m$ to be the "*restricted configuration space*". Let $b = (b_1, \cdots, b_m) \in \mathcal{X}_m - \mathcal{Y}_m$ denote the base point corresponding to $\mathfrak{K}_0$.

For an arbitrary i_0 we fix $t_1, \cdots, t_{i_0-1}, t_{i_0+1}, \cdots, t_m$ and varies t_{i_0} in C. We denote by $g_{i_0}[i_1, \cdots, i_n]$ the unique solution of the equation $D[i_0, i_1, \cdots, i_n] = 0, i_0 < i_1 < \cdots < i_n$, with respect to t_{i_0}. Then by (H, 1) $g_{i_0}[i_1, \cdots, i_n]$ are all real and we have $g_{i_0}[I] < g_{i_0}[J]$ provided $[I] < [J]$. Now consider the path $\alpha[i_0, I]$ in $C - \cup_{[J]<[I]} g_{i_0}[J]$ as follows. Let ε be a small positive number such that $g_{i_0}[J_1] + \varepsilon < g_{i_0}[J_2] - \varepsilon$ if $[J_1] < [J_2]$. Let $u_1, u_2, \cdots, u_l$ be all the values $g_{i_0}[J]$ for $[J] < [I]$ in increasing order, so that $b_{i_0} < u_1 < u_2 < \cdots < u_l < g_{i_0}[I] - \varepsilon$. Then $\alpha[i_0, I]$ is defined as:

$$(3,2) \quad \begin{cases} t_{i_0} = \tau(g_{i_0}[I] - \varepsilon), 0 \leqq \tau \leqq (u_1 - \varepsilon)/(g_{i_0}[I] - \varepsilon), \\ \cdots\cdots \\ t_{i_0} = u_\nu - \varepsilon \exp\left(\pi i \dfrac{(g_{i_0}[I] - \varepsilon)\tau - (u_\nu - \varepsilon)}{2\varepsilon}\right), \dfrac{u_\nu - \varepsilon}{g_{i_0}[I] - \varepsilon} \leqq \tau \leqq \dfrac{u_\nu + \varepsilon}{g_{i_0}[I] - \varepsilon} \\ t_{i_0} = \tau(g_{i_0}[I] - \varepsilon), (u_\nu + \varepsilon)/(g_{i_0}[I] - \varepsilon) \leqq \tau \leqq (u_{\nu+1} - \varepsilon)/g_{i_0}[I], \\ \cdots\cdots \\ t_{i_0} = \tau(g_{i_0}[I] - \varepsilon, (u_l + \varepsilon)/(g_{i_0}[I] - \varepsilon) \leqq \tau \leqq 1, \end{cases}$$

(See the figure 4). We denote by $\sigma_-[i_0, I]$ (or $\sigma_+[i_0, I]$) the displacement of t_{i_0} in $C - \cup_{[J]<[I]} g_{i_0}[J]$ from $g_{i_0}[I] - \varepsilon$ to $g_{i_0}[I] + \varepsilon$ (or from $g_{i_0}[I] + \varepsilon$ to $g_{i_0}[I] - \varepsilon$) along the lower half-circle (or the upper half-circle respectively): and by $\beta[i_0, I]$ the composite of $\sigma_-[i_0, I]$ and $\sigma_+[i_0, I]$:

2) This problem was stated by A. Hattori.

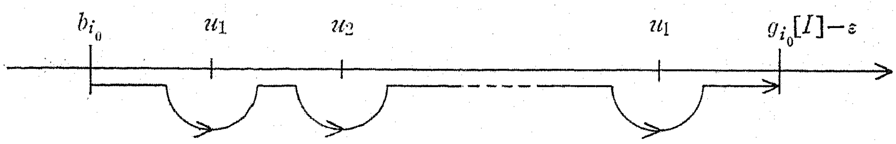

Fig. 4 $\alpha[i_0, I]$

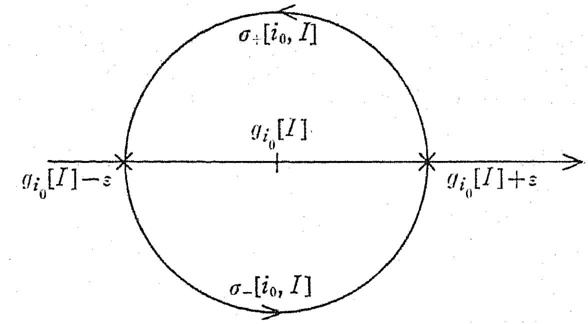

Fig. 5 $\sigma_{\pm}[i_0, I]$

$$\beta[i_0, I]=\sigma_+[i_0, J], \sigma_-[i_0, I]. \tag{3, 3}$$

Then the path $\gamma[i_0, I]=\alpha[i_0, I]^{-1}\cdot\beta[i_0, I]\cdot\alpha[i_0, I]$ defines an element of the fundamental group $\pi_1(\mathcal{X}_m-\mathcal{Y}_m, \mathfrak{K}_0)$ with base point $\mathfrak{K}_0$. The following Proposition immediately follows from the definition.

PROPOSITION 2. *The variation of* $\mathfrak{Z}_-[J; \mathfrak{K}_0]$ *in* $H_n(M, S_{-\omega})$ *from* $\mathfrak{K}_0$ *to* $\alpha[i_0, I]$ $\mathfrak{K}_0=\mathfrak{K}$ *along the path* $\alpha[i_0, I]$ *is given by the formula*;

$$\mathfrak{Z}[J; \mathfrak{K}]=V(\alpha[i_0, I])\cdot\mathfrak{Z}_-[J; \mathfrak{K}]=\overleftarrow{\prod_{K<I}} V(\sigma_-[i_0, K])\cdot\mathfrak{Z}_-[J; \mathfrak{K}_0] \tag{3, 4}$$

where $\overleftarrow{\prod}_{K\in\mathfrak{Z}}\sigma_-$ *denotes the ordered product* $\sigma_-[i_0, K_s].\sigma_-[i_0, K_{s-1}]\cdot\cdots\cdot\sigma_-[i_0, K_1]$ *for* $K_1<K_2<\cdots<K_s$ *and* $\mathfrak{Z}=\{K_1, K_2, \cdots, K_s\}$.

Let $\mathcal{Y}'_{m-\nu+1}$ be the set of points in $\mathcal{X}_{m-\nu+1}$ such that for at least one μ with $\mu<\nu$ and $I, J\subset\{\nu, \nu+1, \cdots, m\}$ we have $f_\mu([I])=f_\mu([J])$. Then $\mathcal{Y}'_{m-\nu+1}\supset\mathcal{Y}_{m-\nu+1}$ and $\mathcal{X}_m-\mathcal{Y}'_m$ is the topological fibre bundle over $\mathcal{X}_{m-1}-\mathcal{Y}'_{m-1}$ with the fibre F_m homeomorphic to $\boldsymbol{C}-\{f_1([I]), I=\{i_1, \cdots, i_n\}, 2\leqq i_1\}$. The group $\pi_1(F_m)$ is isomorphic to the free group generated by $\gamma[1, i_1, \cdots, i_n], 2\leqq i_1<\cdots<i_n$ and we have the exact sequences

$$\{1\}\longrightarrow\pi_1(F_m)\longrightarrow\pi_1(\mathcal{X}_m-\mathcal{Y}'_m)\longrightarrow\pi_1(\mathcal{X}_{m-1}-\mathcal{Y}'_{m-1})\longrightarrow\{1\}$$

and

$$p_{m-1}: \pi_1(\mathcal{X}_{m-1}-\mathcal{Y}'_{m-1})\longrightarrow\pi_1(\mathcal{X}_{m-1}-\mathcal{Y}_{m-1})\longrightarrow\{1\}$$

where $\mathrm{Ker}\, p_{m-1}$ is generated by the loops $g_1[J,K]$ (and their conjugates) turning round the hyperplanes defined by $f_1([J])=f_1([K])$ for $J, K\subset\{2,\cdots,m\}$ such that $|J\cap K|$, the size of $J\cap K$, $\leqq n-2$.

According to Zariski-Van Kampen theorem (see [3]), in a similar way to Artin formula (see Theorem N 8, p. 173 in [7]) we have

PROPOSITION 3. *The group* $\pi_1(\mathcal{X}_m-\mathcal{Y}_m)$ *is generated by* $\gamma[i_1, i_2,\cdots,i_{n+1}], 1\leqq i_1<i_2<\cdots<i_{n+1}\leqq m$ *with the defining relations* i)

$$(3,4)\qquad \overleftarrow{\prod_{\nu=1}^{n+2}}\gamma[i_1,\cdots,\overset{\nu}{\vee},\cdots,i_{n+2}]\cdot\gamma[i_1,\cdots,\overset{\mu}{\vee},\cdots,i_{n+2}]=\gamma[i_1,\cdots,\overset{\mu}{\vee},\cdots,i_{n+2}]\cdot\overleftarrow{\prod_{\nu=1}^{n+2}}[i_1,\cdots,\overset{\nu}{\vee},\cdots,i_{n+2}],\ 1\leqq\mu\leqq n+2,$$

ii) *For* $i_1<j_1$,

$$(3,5)\qquad \gamma[i_1,\cdots,i_{n+1}]\cdot\gamma[j_1,\cdots,j_{n+1}]=\gamma[j_1,\cdots,j_{n+1}]\cdot\gamma[i_1,\cdots,i_{n+1}]$$

if $|\{i_1,\cdots,i_{n+1}\}\cap\{j_1,\cdots,j_{n+1}\}|<n$, *and* $\{j_1,\cdots,j_n\}>\{i_2,\cdots,i_{n+1}\}$ *or* $\{i_2,\cdots,i_{n+1}\}>\{j_2,\cdots,j_{n+1}\}$, *and* iii) *For* $i_1<j_1$,

$$(3,6)\qquad \begin{aligned}&\gamma[j_1,\cdots,j_n,j_{n+1}]\cdot\gamma[i_1,\cdots,i_n,i_{n+1}]\cdot\gamma[j_1,\cdots,j_n,j_{n+1}]^{-1}\\&=\omega_\mu(i_1,j_1,\cdots,j_n,j_{n+1})^{-1}\cdot\gamma[i_1,\cdots,i_n,i_{n+1}]\cdot\omega_\mu(i_1,j_1,\cdots,j_n,j_{n+1})\end{aligned}$$

if $|\{i_1,\cdots,i_{n+1}\}\cap\{j_1,\cdots,j_{n+1}\}|<n$ *and* $\{j_2,\cdots,j_{n+1}\}>\{i_2,\cdots,i_{n+1}\}>\{j_1,j_2,\cdots,\overset{\mu}{\vee},\cdots,j_{n+1}\}$, $\{i_2,\cdots,i_{n+1}\}<\{j_1,j_2,\cdots,\overset{\mu-1}{\vee},\cdots,j_{n+1}\}$, *where* $\omega_\mu(i_1,j_1,\cdots,j_{n+1})$ *denotes the words*, $1\leqq\mu\leqq n+1$,

$$\left\{\overleftarrow{\prod_{\nu=\mu}^{n+1}}\gamma[i_1,j_1,\cdots,\overset{\nu}{\vee},\cdots,j_{n+1}]\right\}^{-1}\cdot\gamma[i_1,j_2,\cdots,j_{n+1}]^{-1}\cdot\overleftarrow{\prod_{\nu=\mu}^{n+1}}\gamma[i_1,j_1,\cdots,\overset{\nu}{\vee},\cdots,j_{n+1}]\cdot\gamma[i_1,j_2,\cdots,j_{n+1}].$$

We shall now define the space of full configurations of S $\hat{\mathcal{X}}_m-\hat{\mathcal{Y}}_m$. We put $a_{m+1,0}=1$ and $a_{m+1,\nu}=0, \nu\geqq1$. We denote by $\hat{\mathcal{X}}_{m-\nu+1}$ the quotient of the affine space $\boldsymbol{C}^{(m-\nu+1)(n+1)}, 1\leqq\nu\leqq m$, of $(m-\nu+1)\times(n+1)$ matrices

$$A=\begin{pmatrix}a_{\nu0}, & a_{\nu1}, & \cdots, & a_{\nu n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{m0}, & a_{m1}, & \cdots, & a_{mn}\end{pmatrix}$$

by the natural action of the multiplicative group $(\boldsymbol{C}^*)^{m-\nu+1}$ of diagonal matrices. Let $\hat{\mathcal{Y}}_{m-\nu+1}$ be the union of analytic subsets $D[i_1,\cdots,i_{n+1}]=0$ for $\nu\leqq i_1<\cdots<i_{n+1}\leqq m+1$. Then $\hat{\mathcal{X}}_m-\hat{\mathcal{Y}}_m$ will be called the "*full configuration space of m hyperplanes* S". We have the natural projection

$$\pi^m_{m-1}: \hat{\mathfrak{X}}_m - \hat{\mathfrak{Y}}_m \longrightarrow \hat{\mathfrak{X}}_{m-1} - \hat{\mathfrak{Y}}_{m-1},$$

with a general fiber $\hat{F}_m$, where $\hat{F}_m$ denotes the space of hyperplanes S_1 in general position to $(S_2, \cdots, S_{m+1})$, homeomorphic to the projective space $\boldsymbol{P}^n$ minus the hyperplanes defined by $D[1, i_1, \cdots, i_n]=0$ for any $2 \leqq i_1 < \cdots < i_n \leqq m+1$. Consider the generic line L in $\hat{F}_m$ containing $\mathfrak{R}_0$ and fixing $a_{i\nu}, 2 \leqq i \leqq m, 1 \leqq \nu \leqq n$, and move S_1 from $\mathfrak{R}_0$, so that L is isomorphic to $\boldsymbol{C} - \cup_J g_1[J]$ for $2 \leqq j_1 < \cdots < j_n$. Then the standard argument shows $\pi_1(\hat{\mathfrak{X}}_m - \hat{\mathfrak{Y}}_m, \mathfrak{R}_0)$ *is generated by* $\gamma[i_1, i_2, \cdots, i_{n+1}], 1 \leqq i_1 < \cdots < i_{n+1} \leqq m+1$.

It seems interesting to find the fundamental relations for $\pi_1(\hat{\mathfrak{X}}_m - \hat{\mathfrak{Y}}_m, \mathfrak{R}_0)$ with respect to the above generators.

4. Connection formula and monodromy.

We are going to compute the variation formula by $\sigma_-[i_0, I]$ after $\alpha[i_0, I]$ for $1 \leqq i_1 < \cdots < i_n \leqq m+1$. At $t_{i_0} = g_{i_0}[I] - \varepsilon$ in $\mathfrak{X}_m$, by taking the coordinates $X_1 = (-1)^n f_{i_1}, \cdots, X_\nu = (-1)^{n-\nu+1} f_{i_\nu}, \cdots, X_n = -f_n$, we may assume that $(-1)^n f_{i_0} = X_0$ has the following form:

$$-X_0 = \sum_{\nu=1}^{n} b_\nu x_\nu + b_0 \tag{4,1}$$

where $b_\nu > 0$ (See the figures 6, 7), so that the vanishing cycle $\mathfrak{L} = \varDelta[I]$ attached to the point $t_{i_0} = g_{i_0}[I]$ is defined as follows: $X_1 \leqq 0, \cdots, X_n \leqq 0$ and $X_0 \leqq 0$. We

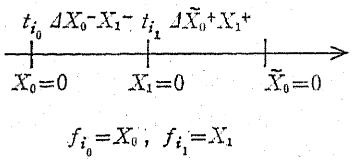

Fig. 6 ($n=1$)

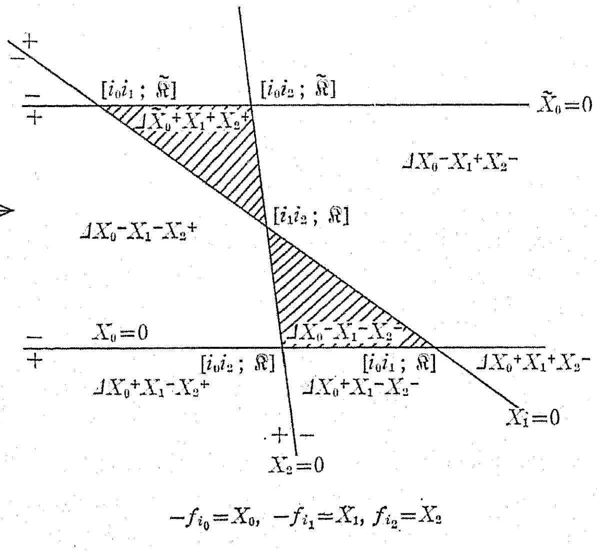

Fig. 7 ($n=2$)

denote by $\Delta X_0^{\varepsilon_0} X_1^{\varepsilon_0} \cdots X_n^{\varepsilon_n}$ the visible cycle contained in the domain $\varepsilon_0 X_0 \geqq 0$, $\varepsilon_1 X_1 \geqq 0, \cdots, \varepsilon_n X_n \geqq 0, \varepsilon_j = \pm 1$ which has a non-empty intersection with $\Delta(I; \mathfrak{R})$, and with the orientation $dX_1 \wedge \cdots \wedge dX_n > 0$. We exclude the case $\varepsilon_0 = -1, \varepsilon_1 = \cdots = \varepsilon_n = 1$ where $\Delta X_0^{\varepsilon_0} X_1^{\varepsilon_1} \cdots X_n^{\varepsilon_n}$ is empty. An elementary but tedious calculation shows

PROPOSITION 4.

$$
\begin{aligned}
&\Delta X_0^{\varepsilon_0} X_1^{\varepsilon_1} \cdots X_n^{\varepsilon_n} \\
&= (1-\varepsilon_0)/2 \cdot T_1^{(\varepsilon_1+1)/2} \cdots T_n^{(\varepsilon_n+1)/2} \mathcal{Z}_-[i_1, \cdots, i_n; \mathfrak{R}] \\
&\quad + \varepsilon_0 (1-\varepsilon_1)/2 \cdot T_2^{(\varepsilon_2+1)/2} \cdots T_n^{(\varepsilon_n+1)/2} \mathcal{Z}_-[i_0, i_2, \cdots, i_n; \mathfrak{R}] \\
&\quad + \cdots + \\
&\quad + \varepsilon_0 \varepsilon_1, \cdots, \varepsilon_{n-1} (1-\varepsilon_n)/2 \cdot \mathcal{Z}_-[i_0, \cdots, i_{n-1}; \mathfrak{R}]
\end{aligned}
\tag{4, 2}
$$

where T_ν denotes the ν-th difference operator:

$$
T_\nu \mathcal{Z}_-[i_1, \cdots, i_\nu, \cdots, i_n; \mathfrak{R}] = \mathcal{Z}_-[i_1, \cdots, i_\nu + 1, \cdots, i_n; \mathfrak{R}] - \mathcal{Z}_-[i_1, \cdots, i_\nu, \cdots, i_n; \mathfrak{R}]. \tag{4, 3}
$$

In particular the vanishing cycle $\mathcal{L}$ is equal to

$$
\begin{aligned}
\Delta X_0^- X_1^- \cdots X_n^- = {} & \mathcal{Z}_-[i_1, \cdots, i_n; \mathfrak{R}] - \mathcal{Z}_-[i_0, i_2, \cdots, i_n; \mathfrak{R}] \\
& + \cdots + (-1)^n \cdot \mathcal{Z}_-[i_0, i_1, \cdots, i_{n-1}; \mathfrak{R}].
\end{aligned}
\tag{4, 4}
$$

Suppose that X_0 has been transformed into $\tilde{X}_0$ by the displacement $\sigma_-[i_0, i_1, \cdots, i_n]$ where $-\tilde{X}_0 = \sum_{\nu=1}^n b_\nu x_\nu + \tilde{b}_0, \tilde{b}_0 < 0$. Then in the same way as Proposition 4 and by putting $\tilde{\mathfrak{R}} = \sigma_-[i_0, I]\mathfrak{R}$,

PROPOSITION 5.

$$
\begin{aligned}
&\Delta \tilde{X}_0^{\varepsilon_0} X_1^{\varepsilon_1} \cdots X_n^{\varepsilon_n} \\
&= (1-\varepsilon_0)/2 \cdot T_1^{(1+\varepsilon_1)/2} \cdots T_n^{(1+\varepsilon_n)/2} \mathcal{Z}_-[i_1, \cdots, i_n; \tilde{\mathfrak{R}}] \\
&\quad + \varepsilon_0 (1-\varepsilon_1)/2 \cdot T_2^{(1+\varepsilon_2)/2} \cdots T_n^{(1+\varepsilon_n)/2} \mathcal{Z}_-[i_0, i_2, \cdots, i_n; \tilde{\mathfrak{R}}] \\
&\quad + \cdots + \\
&\quad + \varepsilon_0 \varepsilon_1 \cdots \varepsilon_{n-1} (1-\varepsilon_n)/2 \cdot \mathcal{Z}_-[i_0, i_1, \cdots, i_{n-1}; \tilde{\mathfrak{R}}] \\
&\quad + \varepsilon_0 \varepsilon_1 \cdots \varepsilon_n \sum_{\nu=0}^{n} (-1)^{n-\nu} \mathcal{Z}_-[i_0, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_n; \tilde{\mathfrak{R}}].
\end{aligned}
\tag{4, 5}
$$

In particular the transformed vanishing cycle $\tilde{\mathcal{L}}$ is equal to

$$
\Delta \tilde{X}_0^+ X_1^+ \cdots X_n^+ = \sum_{\nu=0}^{n} \mathcal{Z}_-[i_0, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_n; \tilde{\mathfrak{R}}](-1)^{n-\nu}. \tag{4, 6}
$$

Now we apply Pham's generalized Picard-Lefschetz formula for the cycles $\Delta X_0^{\varepsilon_0} X_1^{\varepsilon_1} \cdots X_n^{\varepsilon_n}$ and $\Delta \tilde{X}_0^{\varepsilon_0} X_1^{\varepsilon_1} \cdots X_n^{\varepsilon_n}$. The variation formulae of $V(\sigma_\pm[i_0, I])$ of δ in $H_n(M, \mathcal{S}_{-\infty})$ are given as follows:

Proposition 6.

$$(4,7)\quad \begin{aligned} &V(\sigma_-[i_0, I]): \Delta X_0^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_n^{\varepsilon_n} \longrightarrow \Delta \tilde{X}_0^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_n^{\varepsilon_n} \\ &-\varepsilon_0\varepsilon_1\cdots\varepsilon_n \exp(2\pi\sqrt{-1}(1-\varepsilon_0)/2\cdot\lambda_{i_0})\Delta\tilde{X}_0^+X_1^+\cdots X_n^+, \end{aligned}$$

except for $\varepsilon_0=\varepsilon_1=\cdots=\varepsilon_n=-1$. *In this case we have*

$$\Delta X_0^- X_1^- \cdots X_n^- \longrightarrow (-1)^n \exp(2\pi\sqrt{-1}\lambda_{i_0})\Delta\tilde{X}_0^+X_1^+\cdots X_n^+.$$

The remaining visible cycles are left unchanged.

Proof. See F. Pham [8] p. 128.

In the same manner we get

Proposition 7.

$$(4,8)\quad \begin{aligned} &V(\sigma_+[i_0, I]): \Delta \tilde{X}_0^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_n^{\varepsilon_n} \longrightarrow \Delta X_0^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_n^{\varepsilon_n} \\ &+(-1)^n\varepsilon_0\varepsilon_1\cdots\varepsilon_n\cdot\exp\left(2\pi\sqrt{-1}\sum_{\nu=1}^{n}\frac{(\varepsilon_\nu+1)\lambda_{i_\nu}}{2}\right)\Delta X_0^-X_1^-\cdots X_n^- \end{aligned}$$

except for $\varepsilon_0=\varepsilon_1=\cdots=\varepsilon_n=1$. *In this case we get*

$$\Delta\tilde{X}_0^+X_1^+\cdots X_n^+ \longrightarrow (-1)^n \exp\left(2\pi\sqrt{-1}\sum_{\nu=1}^{n}\lambda_{i_\nu}\right)\Delta X_0^-X_1^-\cdots X_n^-.$$

Corollary (Local monodromy formula)

$$(4,9)\quad \begin{aligned} &V(\beta[i_0, I]): \Delta X_0^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_n^{\varepsilon_n} \longrightarrow \Delta X_0^{\varepsilon_0}X_1^{\varepsilon_1}\cdots X_n^{\varepsilon_n} \\ &+(-1)^n\varepsilon_0\varepsilon_1\cdots\varepsilon_n \exp\left(2\pi\sqrt{-1}\sum_{\nu=1}^{n}\lambda_{i_\nu}\right)\cdot \\ &\cdot\left\{\left(-\exp\left(2\pi\sqrt{-1}\cdot\frac{(1-\varepsilon_0)\lambda_{i_0}}{2}\right)+\exp\left(2\pi\sqrt{-1}\sum_{\nu=1}^{n}\frac{\lambda_{i_\nu}(\varepsilon_\nu-1)}{2}\right)\right\}\Delta X_0^-X_1^-\cdots X_n^-, \end{aligned}$$

except for $\varepsilon_0=\varepsilon_1=\cdots=\varepsilon_n=-1$, *in which case we have*

$$\Delta X_0^-X_1^-\cdots X_n^- \longrightarrow \exp\left(2\pi\sqrt{-1}\sum_{\nu=0}^{n}\lambda_{i_\nu}\right)\Delta X_0^-X_1^-\cdots X_n^-.$$

Finally we have come to the following conclusion:

Theorem (*Formula of global monodromy*). *The set of paths* $\alpha[i_0, I]^{-1}\beta[i_0, I]$ $\alpha[i_0, I]$ $1\leqq i_0<i_1<\cdots<i_n$ *generate the group* $\pi_1(\hat{\mathcal{X}}^m-\hat{\mathcal{Y}}^m, \mathfrak{K}_0)$. *Their representation on the space* $H_n(M, S_{-\omega})$ *is given by* $V(\alpha[i_0, I])^{-1}\cdot V(\beta[i_0, I])\cdot V(\alpha[i_0, I])$ *where* $V(\alpha[i_0, I])$ *and* $V(\beta[i_0, I])$ *are given by* (3, 4) *and* (4, 2)~(4, 9).

Appendix. Differential equations satisfied by *J*.

We denote by $\varphi(I), I=(i_1, \cdots, i_n), 1\leqq i_1<\cdots<i_n\leqq m$ the form $d\log f_{i_1}\wedge\cdots\wedge$

$d\log f_{i_n}$ and by $\tilde{\varphi}(I;\lambda)$ the integral

$$\text{(A, 1)} \qquad \int_\gamma U\cdot\varphi(I)$$

with $U=\exp(f_0)\cdot f_1^{\lambda_1}\cdots f_m^{\lambda_m}$ and $\gamma\in H_n(M, S_{-\omega})$. The differentiation with respect to $a_{i\nu}, 0\leq\nu\leq n, 1\leq i\leq m,$

$$\text{(A, 2)} \qquad \frac{\partial}{\partial a_{i\nu}}\int U\cdot\varphi(I)=\int\frac{\partial}{\partial a_{i\nu}}(U\cdot\varphi(I))=\int U\cdot\Delta_{\frac{\partial}{\partial a_{i\nu}}}\varphi(I)$$

shows

$$\text{(A, 3)} \qquad \begin{aligned} d\tilde{\varphi}(I;\lambda)=&\sum_{i_0\notin I}\sum_{\nu=0}^{n} da_{i_0\nu}\otimes\partial/\partial a_{i_0\nu}\tilde{\varphi}(I;\lambda)\\ &+\sum_{\sigma=1}^{n}\sum_{\nu=0}^{n} da_{i_\sigma\nu}\otimes\partial/\partial a_{i_\sigma\nu}\tilde{\varphi}(I;\lambda)\\ =&\int U\Big\{\sum_{i_0\notin I}\sum_{\nu=0}^{n} da_{i_0\nu}\otimes\frac{\lambda_{i_0}x_\nu}{f_{i_0}}\varphi(I)+\sum_{\sigma=1}^{n}\sum_{\nu=0}^{n} da_{i_\sigma\nu}\\ &\otimes\frac{(\lambda_{i_\sigma}-1)x_\nu}{f_{i_\sigma}}\Big\}\varphi(I)+d\log D[i_1,\cdots,i_n]\otimes\tilde{\varphi}(I;\lambda), \end{aligned}$$

where $D[i_1,\cdots,i_n]$ denotes the determinant $(-1)^n D[i_1,\cdots,i_n,m+1]$

$$\begin{vmatrix} a_{i_1 1}, & \cdots, & a_{i_1 n}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ a_{i_n 1}, & \cdots, & a_{i_n n}\end{vmatrix}.$$

Because of the formulae

$$0=\int U\cdot\Gamma_\omega\left(\frac{df_{j_2}}{f_{j_2}}\wedge\cdots\wedge\frac{df_{j_n}}{f_{j_n}}\right)=\int U\cdot\Gamma_\omega\left(\frac{x_\nu}{f_{j_1}}\,\frac{df_{j_2}}{f_{j_2}}\wedge\cdots\wedge\frac{df_{j_n}}{f_{j_n}}\right)$$

and by expansion of partial fractions for $x_\nu/f_{j_0}f_{j_1}\cdots f_{j_n}$, $x_\nu/f_{j_1}\cdots f_{j_n}$, $1/f_{j_0}f_{j_1}\cdots f_{j_n}$ and $1/f_{j_1}f_{i_2}\cdots f_{j_n}$, we have

$$\begin{aligned} &\sum_{\nu=0}^{n} da_{i_0\nu}\otimes\partial/\partial a_{i_0\nu}\tilde{\varphi}(I;\lambda)\\ &=\lambda_{i_0}\sum_{\kappa=0}^{n}(-)^\kappa\cdot\frac{D[i_1,\cdots,i_n]}{D[i_0,i_1,\cdots,i_n]D[i_0,\cdots,\overset{\kappa}{\vee},\cdots,i_n]}\cdot\tilde{\varphi}(i_0,\cdots,\overset{\kappa}{\vee},\cdots,i_n)\\ &\qquad\cdot\;{}^{\kappa}\!\!>\begin{vmatrix} da_{i_0 0}, & da_{i_0 1}, & \cdots, & da_{i_0 n}\\ a_{i_0 0}, & a_{i_0 1}, & \cdots, & a_{i_0 n}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ a_{i_n 0}, & a_{i_n 1}, & \cdots, & a_{i_n n}\end{vmatrix} \end{aligned}$$

and

$$\sum_{\sigma=1}^{n}\sum_{\nu=0}^{n} da_{i_\sigma\nu}\otimes\partial/\partial a_{i_\sigma\nu}\varphi(i_1,\cdots,i_n;\lambda)$$

$$=\sum_{\sigma=1}^{n}(-1)^{\sigma}\frac{D[0,i_1,\cdots,\overset{\sigma}{\vee},\cdots,i_n]}{D[i_1,\cdots,i_n]^2}\cdot\begin{vmatrix} da_{i_\sigma 0},\cdots,da_{i_\sigma n}\\ a_{i_1 0},\cdots,\ a_{i_1 n}\\ \vdots \qquad \vdots\\ a_{i_n 0},\cdots,\ a_{i_n n}\end{vmatrix}\varphi(I;\lambda)$$

$$+\sum_{\kappa=1}^{n}\sum_{\sigma=1}^{n}(-1)^{\kappa+\sigma}\frac{D[0,i_1,\cdots,\overset{\sigma}{\vee},\cdots,i_n]}{D[i_1,\cdots,i_n]D[0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n]}\cdot\begin{vmatrix} da_{i_\sigma 1},\cdots,da_{i_\sigma n}\\ a_{i_1 1},\cdots,\ a_{i_1 n}\\ \\ a_{i_n 1},\cdots,\ a_{i_n n}\end{vmatrix}<^{\kappa}$$

$$\cdot\sum_{i_0=1}^{m}\sum_{\kappa\in I}\lambda_{i_0}\varphi(i_0,i_1,\cdots,\overset{\kappa}{\vee},\cdots i_n;\lambda)$$

$$+\sum_{i_0\notin I}\sum_{\kappa=1}^{n}\sum_{\sigma=1}^{n}(-1)^{\kappa+\sigma}\lambda_{i_0}\frac{D[i_0,i_1,\cdots,\overset{i_\sigma}{\vee},\cdots,i_n]}{D[i_0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n]\cdot D[i_0,i_1,\cdots,i_n]}$$

$$\cdot\,{}^{\kappa}\!\!>\begin{vmatrix} da_{i_\sigma 0}, da_{i_\sigma 1},\cdots,da_{i_\sigma n}\\ a_{i_0 0},\ a_{i_0 1},\cdots,\ a_{i_0 n}\\ a_{i_1 0},\ a_{i_1 1},\cdots,\ a_{i_1 n}\\ \cdots\cdots\cdots\cdots\\ \\ \cdots\cdots\cdots\cdots\\ a_{i_n 0},\ a_{i_n 1},\cdots,\ a_{i_n n}\end{vmatrix}\cdot\varphi(i_0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n;\lambda)$$

so that the Jacobi identity concerning "determinants" simplifies the above in the following manner:

$$\text{(A,4)}\qquad d\varphi(I;\lambda)=d\left(\frac{D[0,i_1,\cdots,i_n]}{D[i_1,\cdots,i_n]}\right)\varphi(I;\lambda)$$
$$+\sum_{i_0\notin I}\sum_{\kappa=0}^{n}\left\{\lambda_{i_0}(-1)^{\kappa}\cdot d\log\frac{D[i_0,i_1,\cdots,i_n]}{D[i_0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n]}\right.$$
$$\left.-d\log\frac{D[0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n]}{D[i_1,\cdots,i_n]}\right\}\varphi(i_0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n;\lambda).$$

This defines *the Gauss-Manin connection in the space of full configurations of* S, $\hat{\mathfrak{X}}_m-\hat{\mathfrak{Y}}_m$, and a generalization of the classical Jordan-Pochhammer linear differential equations. In particular, in case of $\mu=0$, this is reduced to

$$\text{(A,5)}\qquad d\varphi(I;\lambda)=\sum_{i_0\notin I}\sum_{\kappa=0}^{n}\left\{\lambda_{i_0}(-1)^{\kappa}\cdot d\log\frac{D[i_0,i_1,\cdots,i_n]}{D[i_0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n]}\right.$$
$$\left.\cdot\varphi(i_0,i_1,\cdots,\overset{\kappa}{\vee},\cdots,i_n;\lambda)\right\}$$

which has *logarithmic poles along* $\hat{\mathfrak{Y}}_m$ *in the sense of P. Deligne.*

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