A Class of Solutions of Bogolioubov System of Equations for Classical Statistical Mechanics of Hard Core Particles

By Yoichiro TAKAHASHI

Department of Mathematics, College of General Education, University of Tokyo
Komaba, Meguro-ku, Tokyo 153

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§ 0. Introduction

Recently the construction problem of the dynamical systems of classical statistical mechanics (under equilibrium) has been solved for sufficiently wide classes of pair potentials by Sinai [9], Lanford [5], and Presutti et al [6]. But the equivalence between the time evolutions in their sense and in the sense of Bogolioubov [1] is still obscure. The latter is described by a system of equations (BBGKY hierarchy)

$$(1)\qquad \frac{\partial}{\partial t} f(\underline{x}) = Af(\underline{x}) + \sum_{i=1}^{N} \int_{\boldsymbol{R}^d \times \boldsymbol{R}^d} (\Phi'(q_i - q_0), \operatorname{grad}_{p_i} f(x_0, \underline{x}))dx_0 + \bar{C}f(\underline{x})$$

for correlation functions f. Here $\underline{x}=(x_1, \cdots, x_N)$, $x_i=(q_i, p_i) \in \boldsymbol{R}^d \times \boldsymbol{R}^d$ $(i=1, \cdots, N)$, A denotes the formal infinitesimal generator of the N-particle Hamiltonian flow under a given pair potential Φ, and $\bar{C}$ is an operator which appears only when the potential has hard core and is then given as follows:

$$(2)\qquad \bar{C}f(\underline{x}) = \sum_{i=1}^{N} \iint dudp_0 (p_i - p_0, u)[f(x_0', x_1, \cdots, x_i', \cdots, x_N) - f(x_0, x_1, \cdots, x_N)]$$

where $|u|=2r_0$ (r_0 being the radius of a particle), $q_i - q_0 = u$, $x_0'=(q_0', p_0')$, $x_i'=(q_i, p_i')$, $q_i' - q_0' = -u$, $p_i' = p_i - (p_i - p_0, u)\cdot u/(2r_0)^2$, $p_0' = p_0 - p_i' + p_i$ and the integral is taken for $(u, p_0) \in \{u;\ |u|=2r_0\} \times \boldsymbol{R}^d$ such that $(p_i - p_0, u) > 0$, and that $x_0', x_1, \cdots, x_i', \cdots, x_N$ is a configuration of particles (See also [2], [5]).

We shall deduce the system of equations rigorously and prove that the correlation function f_t of the measure $F_t \cdot \mu$ for some bounded continuous function F on the configuration space solves the equation, where $F_t = F \circ T^t$ and T^t is the dynamical system of infinitely many particles under a Gibbsian measure μ. But the potential considered here is restricted to the pure hard core potential

$$\Phi(q) = \begin{cases} \infty & \text{if } |q| < r_0 \\ 0 & \text{otherwise} \end{cases} \qquad r_0 > 0$$

and the density is assumed sufficiently low, so that the flow T^t is locally finite in the sense that almost every configuration ξ satisfies the following condition: for any $t>0$, any compact subset K of $\boldsymbol{R}^d$ and any compact subset L of $\boldsymbol{R}^d \times \boldsymbol{R}^d$, there exists a compact subset V of $\boldsymbol{R}^d$ such that the configuration at a time $s \in [-t, t]$ restricted to the area K, $T^s\eta|_K$, coincides with $T_V^s(\eta|_V)|_K$ whenever $\eta \cap L^c = \xi \cap L^c$. Here T_V^s is the flow obtained by freezing the particles outside of the vessel V and making the boundary ∂V reflecting wall.

Finally we shall show a result on the stationary solutions of the equation (1) under a considerably strong assumption, Maxwellian distribution in velocities.

§ 1. Preliminary

Let Q be the configuration space of hard core particles with radius $r_0>0$ over $\boldsymbol{R}^d$ $(d \geqq 1)$. In other words Q is the totality of countable subsets ξ of $R = \boldsymbol{R}^d \times \boldsymbol{R}^d$ with the following two properties: (a) $\xi \cap (K \times \boldsymbol{R}^d)$ is a finite set for each compact set $K \subset \boldsymbol{R}^d$. (b) For any distinct points $x_i = (q_i, p_i) \in \xi$ $(i=1, 2)$

$$|q_1 - q_2| \geqq 2r_0.$$

For any integer $n \geqq 1$ and a function φ_n: $R \rightarrow \boldsymbol{R}$, let us denote

$$\xi_n(\varphi_n) = \sum \varphi_n(x_1, \cdots, x_2)$$

where the sum is taken over mutually distinct points $x_1 = (q_i, p_i)$, $i = 1, \cdots, n$ of ξ. Identifying a configuration ξ with the Radon measure ξ_1, one can endow Q with the induced vague topology. The topological Borel σ-algebra $\mathcal{B}$ coincides with the σ-algebra generated by the σ-algebras $\mathcal{B}_K$, K being compact subsets of $\boldsymbol{R}^d$, where

$$\mathcal{B}_K = \sigma(\xi_K)$$

and ξ_K is the restriction of ξ to the set $K \times \boldsymbol{R}^d$.

DEFINITION. Let μ be a probability measure on Q and $n \geqq 1$. If the *n-th (factorial) moment measure*

$$C_c(R^n) \ni \varphi_n \dashrightarrow \int \mu(d\xi)\xi_n(\varphi_n)$$

exists and is absolutely continuous with respect to the Lebesgue measure in $(\boldsymbol{R}^d \times \boldsymbol{R}^d)^n$, then the density $\rho(x_1, \cdots, x_n) = \rho_n(x_1, \cdots, x_n)$ is called *n-th correlation function* of μ.

If the n-th moment of μ exists, then one can define a σ-finite measure M_n on the product space $Q \times R^n$ by

$$M_n(d\xi dx_1 \cdots dx_n) = \mu(d\xi)\xi_n(dx_1 \cdots dx_n)$$

Let $M_n^{x_1,\cdots,x_n}$ be the conditional measure of M_n given $x_1,\cdots,x_n$. It is defined for $\rho_n(x_1\cdots x_n)dx_1\cdots dx_n$—a. e. by Rohlin's theorem. Then one can identify $M_n^{x_1,\cdots,x_n}$ on $Q\times(x_1,\cdots,x_n)$ with the measure $\mu^{x_1,\cdots,x_n}$, projected to Q.

DEFINITION. The measure $\mu^{\underline{x}}$, $\underline{x}=(x_1,\cdots,x_n)$ is called *Palm measure.*

It is easy to see that

$$\mu^{x_1,\cdots,x_n}=(\mu^{x_1,\cdots,x_{n-1}})^{x_n}$$

and that $\mu^{x_1,\cdots,x_n}$ is invariant under the permutations of points $x_1,\cdots,x_n$ whenever it is defined. Furthermore, if the $(n+m)$-th correlation function of μ exists, then the m-th correlation function of $\mu^{x_1,\cdots,x_n}$ is $\rho_{n+m}(x_1,\cdots,x_n,\cdot)/\rho_n(x_1,\cdots,x_n)$. When all the correlation functions $\rho=\rho_n$, $n\geqq 1$ exist and satisfy the estimate

$$\rho_n\leqq C^n \quad \text{for some constant C,} \tag{1}$$

then they determine the measure μ. In fact the following formula due to Ruelle [8] holds

$$\int\mu(d\xi)e^{-\xi_1(\varphi)}=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int dx_1\cdots dx_n\rho(x_1,\cdots,x_n)\prod_{i=1}^{n}(e^{-\varphi(x_i)}-1) \tag{2}$$

for any bounded Borel function φ on R with compact support.

We note that, if μ is a Gibbsian measure for a potential Φ (in the sense of Dobrushin [3] or Ruelle [7, 8]), then

$$\rho(x)\int\mu^{\underline{x}}(d\xi)F(\xi)=\int\mu(d\xi)e^{-U(\underline{x}|\xi)}F(\underline{x}\cdot\xi) \tag{3}$$

for any bounded Borel function F on Q, where

$$U(\underline{x}|\xi)=\sum_{i=1}^{n}\left(\frac{\beta|p_i|^2}{2m}-\log z\right)+\sum_{\substack{i,j=1\\ i\neq j}}^{n}\Phi(q_i-q_j)+\sum_{i=1}^{n}\sum_{(q',p')\in\xi}\Phi(q_i-q'),$$

and $\underline{x}=(x_1,\cdots,x_n)$, $x_i=(q_i,p_i)$ $(i=1,\cdots,N)$ (β and z being inverse temperature and activity). The proofs can be found in [10] and are omitted here.

§ 2. The equations for a bounded domain

Let us consider the motion of finite particles with radius $r_0>0$ and centers in a bounded domain V whose boundary ∂V is assumed smooth surface in $\boldsymbol{R}^d$ $(d\geqq 1)$. The particles move uniformly according to their velocities till one of them collides elastically with another or with the wall of the vessel

$$V^{(r_0)}=\{q\in\boldsymbol{R}^d;\ d(q,V)\leqq r_0\}$$

and then they begin to move uniformly again and so on. Let us introduce the transformations C_{ij} and D_i. Let $N \geqq 1$ and

$$V_N = \{\underline{x} = (x_1, \cdots, x_N);\ x_i = (q_i, p_i) \in V \times \boldsymbol{R}^d,\ |q_i - q_j| \geqq 2r_0\}.$$

For $i = 1, \cdots, N$, $\underline{x}' = D_i \underline{x}$ is defined for those $\underline{x} = (x_1, \cdots, x_N)$ such that $q_i \in \partial V$ and $(p_i, \nu(q_i)) > 0$ where $\nu(q_i)$ is the unit outer normal to ∂V at q_i and the value $x' = (x_1', \cdots, x_N')$ is given as follows: $x_j' = x_j$ for $j \neq i$ and

$$(1) \qquad x_i' = (q_i, p_i'), \qquad p_i' = p_i - 2(p_i, \nu(q_i)).$$

For $1 \leqq i,\ j \leqq N,\ i \neq j,\ \underline{x}' = C_{ij}\underline{x}$ is defined for those $x = (x_1, \cdots, x_N)$ such that $|q_i - q_j| = 2r_0$ and

$$(2) \qquad (p_i - p_j, q_i - q_j) < 0$$

and the value $\underline{x}'$ is given as follows: $x_k' = x_k$ for $k \neq i, j$, $x_i' = (q_i, p_i')$, $x_j' = (q_j, p_j')$,

$$(3) \qquad \begin{aligned} p_i' &= p_i - (2r_0)^{-2}(p_i - p_j, q_i - q_j)(q_i - q_j), \quad \text{and} \\ p_j' &= p_j + (2r_0)^{-2}(p_i - p_j, q_i - q_j)(q_i - q_j). \end{aligned}$$

Consequently the formal infinitesimal generator A_n of the n particle motion T_n^t is the operator

$$(4) \qquad A_n \varphi_n(x_1, \cdots, x_n) = \sum_{i=1}^{n} p_i \frac{\partial \varphi_n}{\partial q_i}(x_1, \cdots, x_n), \quad \frac{\partial}{\partial q_i} = \text{grad}\ q_i$$

for C^1-functions φ_n such that $\varphi_n = \varphi_n \circ C_{ij}$ and $\varphi_n = \varphi_n \circ D_i$ for each $1 \leqq i,\ j \leqq n$.

Fixing a finite configuration of particles $\underline{x} = (x_1, \cdots, x_N) \in V_N$ let us add an extra particle at x_0 and consider the motion of $(N+1)$-particles

$$T_{N+1}^t(x_0, \underline{x}) = T_{N+1}^t(x_0, x_1, \cdots, x_n).$$

The set

$$V(\underline{x}) = V \cap (\bigcup_{i=1}^{N} D(x_i))^c, \qquad D(x_i) = \{q \,|\, d(q, q_i) < 2r_0\}$$

is the area where one can find the additional particle. Let $\sigma_k(x_0)$ be the k-th collision time after time 0 ($k = 1, 2, \cdots$). The collision may ocur with the wall of the vessel as well as among the $N+1$ particles. Now let σ_0 be the first collision time for the N-particle motion $T_N^t \underline{x}$ after time 0, and $\tau(x_0)$ be the first collision time of the added particle. In the following we shall assume

$$\sigma_0 > 0, \qquad \tau(x_0) > 0, \qquad \text{and} \qquad \sigma_1(x_0) > 0.$$

LEMMA 1. *If* $\sigma_0 > t > 0$, *then*

$$V(\underline{x}) = E_0(t) \cup \bigcup_{i=1}^{N} E_i(t) \cup E_\infty(t) \cup E'(t),$$

where

$$E'(t)=\{x_0 \in V(\underline{x})\,;\ \sigma_2(x_0)\leqq t\},$$

$$E_0(t)=\{x_0 \in V(\underline{x})\,;\ \tau(x_0)>t\},$$

$$E_i(t)=\{x_0 \in V(\underline{x})\,;\ \tau(x_0)\leqq t<\sigma_2(x_0) \quad \text{and the first collision of the added particle is with the } i\text{-th particle}\}$$

$$(i=1, \cdots, N)$$

and

$$E_\infty(t)=\{x_0 \in V(\underline{x})|\tau(x_0)\leqq t<\sigma_2(x_0) \quad \text{and the first collision of the added particle is with the wall}\}.$$

Proof. Obvious since $\sigma_1(x_0)=\tau(x_0)$ if $\sigma_0>t\geqq\tau(x_0)$.

Let us introduce new cordinates (τ, u, p_0) in $E_i(t)$. For $1\leqq i\leqq N$, let $\tau=\tau(x_0)$ be the first collision time and u be defined by the relation

$$(5) \qquad q_0+\tau p_0=q_i+\tau p_i+u, \qquad u \in \partial D=\{u \in \boldsymbol{R}^d;\ |u|=2r_0\}$$

if the added particle collides with the i-th particle. Then in term of the new coordinate (τ, u, p_0), the condition $x_0 \in E_i(t)$ is equivalent to the following three conditions:

(a) $\tau\leqq t$

(b) $q_j+sp_j-u-q_i-\tau p_i-(\tau-s)p_0 \notin D \qquad (0\leqq s\leqq\tau)$

(c) $q_j+sp_j-u-q_1-\tau p_i(s-\tau)p_0' \notin D \qquad (\tau\leqq s\leqq t)$

$q_j+sp_j-q_i-\tau p_i-(s-\tau)p_i' \notin D \qquad (\tau\leqq s\leqq t)$

It is easy to see the existence of a function $t_i(\underline{x}, p_0, u)>0$ such that $x_0 \in E_i(t)$ if and only if

$$(6) \qquad \tau\leqq\min\{t_i(\underline{x}, p_0, u), t\}.$$

Finally we note that

$$(7) \qquad dx_0=dq_0dp_0=|(p_i-p_0, q_i-q_0)|d\tau du dp_0.$$

Similarly one can introduce a coordinate $(\tau, u, p_0) \in (0, \infty)\times\partial V\times\boldsymbol{R}^d$ in the set $E_\infty(t)$ and define $t_\infty(\underline{x}, p_0, u)$ for which (6) holds.

LEMMA 2. *For any finite configuration $\underline{x} \in V_N$ such that $\sigma_0>0$,*

$$\int_{E'(t)} dx_0=o(t) \qquad as \qquad t\to 0.$$

Proof. It follows from (6) that $x_0 \in E'(t)$ if and only if

$$0 < t_i(\underline{x}, p_0, u) < \tau \leqq t$$

for some $i=1, \cdots, N, \infty$. Consequently

$$\lim_{t\to 0} \frac{1}{t} \int_{E'(t)} dx_0 = 0.$$

LEMMA 3. *Let $F(x_0, x_1, \cdots, x_N)$ be a bounded Borel function on V_N, which is continuous in $q_0, q_1, \cdots, q_N$. Put*

$$F_0(t, \underline{x}) = F_0(t, x_1, \cdots, x_M) = \int_{V(\underline{x})} dx_0 F(T^t_{N+1}(x_0, \underline{x})), \quad (t \geqq 0).$$

Then

$$\lim_{t\searrow 0} \frac{1}{t}\{F_0(t, \underline{x}) - F_0(0, T^t_N \underline{x})\} = \sum_{i=1}^N \bar{C}_{V,i} F(\underline{x}) + \bar{D} F(\underline{x})$$

$$\equiv \sum_{i=1}^N \int_{\partial D(x_i)} du \int_{\{p_0;\,(p_i-p_0,\, q_i-q_0)<0\}} dp_0 (p_i - p_0, q_i - q_0)\{F(C_{0i}(x_0, \underline{x}) - F(x_0, \underline{x})\}$$

$$+ \int_V du \int_{\{p_0;\,(p_0, \nu)>0\}} dp_0 (p_0, \nu)\{F(x_0, \underline{x}) - F(D_0(x_0, \underline{x}))\}$$

where $\nu = \nu(u)$, and du is the surface element of ∂V or of $\partial D(x_i) = \{q;\, |q_i - q| = 2r_0\}$.

Proof. We may assume that $\sigma_0 > 0$. Then

$$\frac{1}{t}\{F_0(t, \underline{x}) - F_0(0, T^t_N \underline{x})\}$$

$$= \sum_{i=0,1,\cdots,N,\infty} \frac{1}{t} \int_{E_i(t)} dx_0 \{F(T^t_{N+1}(x_0, \underline{x})) - F(x_0, T^t_N \underline{x})\}$$

$$+ \frac{1}{t} \int_{E'(t)} dx_0 \{F(T^t_{N+1}(x_0, \underline{x})) - F(x_0, T^t_N \underline{x})\}.$$

By Lemma 2, the last term tends to zero as $t \to 0$. For $1 \leqq i \leqq N$,

$$T^t_{N+1}(x_0, \underline{x}) = (\underbrace{T^{t-\tau}_1 \times \cdots \times T^{t-\tau}_1}_{N+1}) C_{0i} (\underbrace{T^\tau_1 \times \cdots \times T^\tau_1}_{N+1})(x_0, \underline{x})$$

on the set $E_i(t)$. Consequently

$$\lim_{t\to 0} \frac{1}{t} \int_{E_i(t)} dx_0 F(T^t_{N+1}(x_0, \underline{x}))$$

$$= \lim_{t\to 0} \frac{1}{t} \int_{\partial D(x_i)} du \int_{\{p_0;\,(p_i-p_0,\, q_i-q_0)<0\}} dp_0 |(p_i - p_0, q_i - q_0)|$$

$$\times\int_0^{\min\{t,t_i(\underline{x},p_0,u)\}} d\tau F((T_1^{t-\tau}\times\cdots\times T_1^{t-\tau})C_{0i}(T_1^{\tau}\times\cdots\times T_1^{\tau})(x_0,x))$$

$$=\int_{\partial D(x_i)} du\int_{\{p_0\,;\,(p_i-p_0,q_i-q_0)<0\}} dr_0|(p_i-p_0,q_i-q_0)|F(C_{0i}(x_0,\underline{x}))$$

and

$$\lim_{t\to 0}\frac{1}{t}\int_{F_i(t)} dx_0F(x_0,T_N^t\underline{x})$$

$$=\int_{\partial D(x_i)} du\int_{\{p_0\,;\,(p_i-p_0,q_i-q_0)<0\}} dp_0|(p_i-p_0,q_i-q_0)|F(x_0,\underline{x})$$

For $i=\infty$, the proof is similar and the case $i=0$ is simpler.

COROLLARY. *Let $F(x_1,\cdots,x_N)$ satisfy the similar assumption as in Lemma* 3, *and put*

$$f_n(t,x_1,\cdots,x_n)=\int_{V_{N-n}} dx_{n+1}\cdots dx_NF(T_N^t(x_1,\cdots,x_N)).$$

Then f_n, $1\leqq n\leqq N$ satisfy the following integral equation

$$(8)\qquad f_n(t,\underline{x})=f_n(0,T_n^t\underline{x})+\int_0^t\sum_{i=1}^n C_if_{n+1}(s,\ \cdot\ )(T_n^{t-s}\underline{x})ds$$

if the function F satisfies $F\circ D_i=F$ for each i and if one can interpret T^t as its right continuous version.

Remark. The above argument given for the canonical ensemble implies that the equation (8) remains valid for correlation functions corresponding to grand canonical ensemble.

§ 3. Reduction of the equations

Let V be a subset of $\boldsymbol{R}^d$ and Φ_V be the set of Borel functions φ on the set $\bigcup_{N=0}^{\infty} V_N$ such that

$$(1)\qquad \sup_{\underline{x}\in V_N}|\varphi(\underline{x})|\leqq C^N$$

for some constant C. Here V_0 is a one point set (of empty configuration) and the value of functions φ on V_0 is always assumed to be one. Let

$$(2)\qquad L_V\varphi(\underline{x})=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{V(\underline{x})_n}\varphi(\underline{x}\underline{y})d^n\underline{y}$$

and

$$(3) \qquad L_V^{-1}\varphi(\underline{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{V(\underline{x})_n} \varphi(\underline{x}\underline{y}) d^n \underline{y}.$$

These operators are introduced by Ruelle [8] and satisfy the following properties if V is a bounded measurable set:

(a) L_V and L_V^{-1} maps Φ_V into itself.

(b) $L_V L_V^{-1} = L_V^{-1} L_V = id.$

(c) If $V = V_1 \cup V_2$ is a measurable partition, then, $L_{V_1} L_{V_2} = L_V$.

Let T_V^t be the finite particle motion in V with reflecting boundary obtained by freezing the particles outside of the set V. Thus, for $\underline{x} \in \bigcup V_N$ and $\xi \in Q$ such that $\xi_1(V) = 0$,

$$T_V^t(\underline{x}\cdot\xi) = (T_{V(\xi)}^t \underline{x})\cdot\xi.$$

Here $\underline{x}\cdot\xi$ is the union of the set ξ and $\underline{x}$ considered as a finite subset of $\boldsymbol{R}^d$. It is proved by Sinai [9] that the flow of infinitely many particles

$$(4) \qquad \lim T_V^t \xi = T^t \xi, \qquad \mu\text{—a. e.}$$

exists along a suitable increasing sequence of compact sets $V \subset \boldsymbol{R}^d$, where μ is any limiting Gibbsian measure at sufficiently low density. Furthermore (4) remains valid if μ is replaced by its Palm measures $\mu^{\underline{x}}$, $\underline{x} = (x_1, \cdots, x_N)$ $(N \geqq 1)$ since the flow is locally finite in the sense stated in § 0.

Now let us introduce operators

$$(5) \qquad U_V^t f = L_V T_{V'}^t L_V^{-1} f$$

acting on the space $\Phi = \Phi_{\boldsymbol{R}^d}$ for compact subsets V of $\boldsymbol{R}^d$, where

$$V' = V^{(-r_0)} \equiv \{q \in V;\, d(q, V^c) \geqq r_0\}.$$

LEMMA 1. *If $f \in \Phi$, then*

$$(6) \qquad U^t f = \lim U_V^t f$$

exists and the convergence is uniform on each compact subset of $\bigcup(\boldsymbol{R}^d)_N$. Furthermore

$$(7) \qquad U^t f(\underline{x}) = \rho(\underline{x})\mu^{\underline{x}}(F \circ T^t)$$

provided that, for some bounded measurable function F on Q,

$$(8) \qquad f(\underline{x}) = \rho(\underline{x})\mu^{\underline{x}}(F).$$

Proof. Let σ_V be the density function of the Gibbsian measure projected to $\mathscr{B}_V$ with respect to the measure $d\underline{x}$ on $\bigcup_{N=0}^{\infty} V_N$ which is defined as

$$\frac{1}{N!}d^N x = \frac{1}{N!}dq_1 dp_1 \cdots dq_N dp_N$$

on each V_N. Assume (8) and then note that

(a) $L_V^{-1} f(x) = \sigma_V(x)\mu(F|\mathcal{B}_V)$,
(b) $L_V(\sigma_V G)(x) = \rho(x)\mu^x_-(G)$ for each nonnegative $\mathcal{B}_V$-measurable function G, and
(c) $\mu^x_-(\mu(G|\mathcal{B}_V)) = \mu^x_-(G)$ if x is contained in the set V'.

It then follows

$$L_V T^t_{V'} L_V f(x) = \rho(x)\mu^x_-(\mu(F|\mathcal{B}_V)\circ T^t_{V'})$$
$$= p(x)\mu^x_-(\mu(F\circ T^t_{V'}|\mathcal{B}_V)) = \rho(x)\mu^x_-(F\circ T^t_{V'})$$

if x is contained in V'. Consequently (7) holds in virtue of (4). Finally (6) follows from the denseness of functions of form (8) in the space Φ.

COROLLARY. *The operators* U^t, $-\infty < t < +\infty$, *form a one parameter group of operators on the space of the functions* f *for which* (8) *holds for some bounded measurable functions* F *on the configuration space* Q.

Remark. The above proof remains valid if one adds a smooth potential with finite range and takes $V^{(-r_1)}$ instead of V' for a suitable number r_1 greater than r_0.

Let $\mathcal{F}$ be the family of bounded continuous functions F on Q such that

(9) $$F(\xi) = F(\xi')$$

if $\xi \cap \eta^c = \{(q_i, p_i); i=1, 2\}$ and $\xi' \cap \eta^c = \{(q_i', p_i'); i=1, 2\}$ for some η and the relation (3) of §2 holds with $i=1$ and $j=2$.

LEMMA 2. *Let* $F \in \mathcal{F}$ be $\mathcal{B}_K$*-measurable for some compact set* K *and* f *be the correlation function of the measure* $F\cdot\mu$ *where* μ *is a Gibbsian measure at low density so that the flow* T^t *is locally finite. Then, for a compact set* V *such that* K *is contained in the set* $V'' = \{q \in V;\, d(q, V^c) \geqq 4r_0\}$, *the following integral equation holds*

(10) $$U^t_V f(x) = f(T^t x) + \int_0^t \bar{C}\, U^s_V f(T^{t-s} x)\, ds \qquad (t \geqq 0)$$

if the trajectory $T^s x$, $0 \leqq s \leqq t$, *is contained in the set* $V'' \times \boldsymbol{R}^d$.

Proof. Let $f_V(t, x) = U^t_V f(x)$ and $f_{V,y}(t, x) = L_{V'} T^t_{V'} L_V^{-1} f(xy)$ for x in V' and y in $V \backslash V'$. Then

$$f_V(t, x) = L_{V\backslash V'} L_{V'} T^t_{V'} L_V^{-1} f(x) = \int^{V\backslash V'} f_{V,y}(t, x)\, dy$$

and

$$f_{V,y}(t,x)=\int^{V'} F(T^t_{V'(y)}(xz))\sigma_V(y\cdot T^t_{V'(y)}(xz)dz.$$

Since K is contained in $V'(y)$ for any F is $\mathcal{B}_K$-measurable, thus it follows from Corollary to Lemma 3

$$f_{V,y}(t,x)=f_{V,y}(0,T^t_{V'(y)}x)+\int_0^t \bar{C}_{V'(y)}f_{V,y}(t-s,\cdot)(T^s_{V'(y)}x)ds \tag{11}$$

where $\bar{C}_V$ is the operator obtained from $\bar{C}$ by restricting the area of integration inside of the set V. If $T^s x$ is contained in $V''\times \boldsymbol{R}^d$ for $0\leqq s\leqq t$, then,

$$T^s_{V'(y)}x=T^s x$$

and

$$\bar{C}_{V'(y)}f_{V,y}(t-s,\cdot)(T^s_{V'(y)}x)=\bar{C}f_{V,y}(t-s,\cdot)(T^s x)$$

Hence (10) is obtained from (11) by integrating it.

THEOREM. *If f is the correlation function of a measure $F\cdot\mu$ with $F\in F$, then,*

$$U^t f(x)=f(T^t x)+\int_0^t \bar{C}U^s f(T^{t-s}x)ds. \tag{12}$$

Proof. The assertion follows from the bounded convergence theorem for those f which satisfy the assumption of Lemma 2 because

$$|U^t_V f(x)|\leqq ||F||_\infty\cdot\rho(x)$$

for any compact set V. Let $F\in\mathcal{F}$ and f_K be the correlation function corresponding to $F_K=\mu(F|\mathcal{B}_K)$. Then (12) follows again from the bounded convergence theorem by Lemma 1.

§ 4. Stationary solutions

Let μ be a Gibbsian measure.

THEOREM. *Let f be the correlation function of a probability measure $F\cdot\mu$ on Q with $F\in\mathcal{F}$ which is Maxwellian in velocities and continuously differentiable in q-variables. If f is a stationary solution, i.e., if*

$$Af+\bar{C}f=0, \tag{1}$$

then it is the correlation function of μ itself up to scalar multiplication. In other words, F is a constant function.

Proof. Let g_V be the density function of the measure $F\cdot\mu$ projected to $\mathscr{B}_V$. Then

$$(2) \qquad g_V = L_V^{-1} f.$$

Let us denote, for a given function $h(x)$,

$$\bar{h}(q) = \bar{h}(q_1, \cdots, q_N) = \int \cdots \int h(x_1, \cdots, x_N) dp_1, \cdots, dp_N$$

Since f is Maxwellian in velocities, it follows from (1)

$$(4) \qquad \operatorname{grad}_{q_i} \bar{f}(\underline{q}) = \int du \bar{f}(q_i + u, \underline{q}).$$

Now it is immediate to see from (3) by differentiating (2) that $\operatorname{grad}_{q_i} \bar{f}(\underline{q})$ is a function of the variables q_j's in V'^c and of the number of q_j's in V', but is independent of the positions of particles in the set V'. This means that the measure obtained by normalizing $F\cdot\mu$ is a limiting canonical Gibbsian measure. On the other hand, it is a (limiting grandcanonical) Gibbsian measure for the potential

$$U(x|\xi) = -\log [F(x\cdot\xi)/F(\xi)]$$

on Q in the sense that the formula (3) of § 3 holds. It is not difficult to observe that any Gibbsian measure cannot be a canonical Gibbsian measure for a potential Φ unless it is a Gibbsian measure for Φ. Hence F is constant function and f is the correlation function of μ up to scalar multiplication.

Remark. The proof shows that a Gibbsian measure in the generalized sense (3) of § 3 is a Gibbsian measure for the hard core potential if its correlation function satisfies the stationary equation (1). This fact can be proved for general finite range potentials under suitable regularity assumptions.

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