Spectral and Scattering Theory for a Class of Non-Selfadjoint Operators

By Takashi KAKO*

Department of Pure and Applied Sciences, College of General Education,
University of Tokyo, Komaba, Meguro-ku, Tokyo 153

and Kenji YAJIMA*

Department of Mathematics, Faculty of Science, University of Tokyo,
Hongo, Bunkyo-ku, Tokyo 113

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§ 0. Introduction

The purpose of the present paper is to develop the spectral and scattering theory for a class of non-selfajoint operators which is obtained by the perturbations of selfadjoint operators satisfying the conditions (A-1), (A-2) and (A-3) (or (A-3)′) to be specified in § 2. Briefly speaking, (A-1) guarantees the existence of the perturbed operator, and (A-2) and (A-3) (or (A-3)′) are the localized version of the smooth perturbation which was firstly introduced by Kato [4] and later extended by Lavine [8] in the selfadjoint case. Our results include the so-called short-range perturbations as well as the small perturbations (see § 2). In particular, we apply our results to $-\Delta+q(x)$ in $L^2(\boldsymbol{R}^n)$ with a complex valued potential $q(x)$ which behaves like $o(|x|^{-1-\epsilon})$ at infinity (see § 5).

We remark here the following respects. The results of Goldstein [2] include ours formally, but the conditions in [2] seem to be difficult to prove directly in concrete cases. We also mention that our results are more general than those of Mochizuki [9] and [10] as we assume (A-3) (or (A-3)′) instead of the existence of the boundary values of the resolvents on the (whole) real axis.

The composition of the present paper is as follows. In § 1 we investigate the concept of the (local) smoothness and some important estimates derived from it. Following Mochizuki [9], [10] and Goldstein [2], we define in § 2 the spectral projections for the perturbed operators and investigate the basic properties of them. In § 3 we develop the stationary (time-independent) scattering theory by means of the wave operators. § 4 is devoted to the time-dependent formulation of the scattering theory and the study of the invariance principle for the wave operators. We give some brief applications of our results to partial differential operators in § 5.

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§ 1. Preliminaries

In this section we shall review, for the later use, the concept of the smooth perturbation. Here and hereafter Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ are assumed to be separable. The inner products and the norms are distinguished by subscripts as $(\,,\,)_{\mathfrak{H}}, (\,,\,)_{\mathfrak{K}}$ and $\|\;\|_{\mathfrak{H}}, \|\;\|_{\mathfrak{K}}$, respectively, but they will be omitted if there is no risk of confusions.

$\mathcal{C}(\mathfrak{H}, \mathfrak{K})$ is the set of all closed operators T with domain $\mathfrak{D}(T)\subset\mathfrak{H}$ and range $\mathfrak{R}(T)\subset\mathfrak{K}$. $\mathcal{C}_0(\mathfrak{H}, \mathfrak{K})$ and $\mathcal{B}(\mathfrak{H}, \mathfrak{K})$ are the subsets of $\mathcal{C}(\mathfrak{H}, \mathfrak{K})$ consisting of all T with $\mathfrak{D}(T)$ dense in $\mathfrak{H}$ and $\mathfrak{D}(T)=\mathfrak{H}$ respectively. We write $\mathcal{C}(\mathfrak{H})$ for $\mathcal{C}(\mathfrak{H}, \mathfrak{H})$, and $\mathcal{C}_0(\mathfrak{H})$ and $\mathcal{B}(\mathfrak{H})$ are defined similarly. For $T\in\mathcal{C}_0(\mathfrak{H}, \mathfrak{K})$ the adjoint $T^*\in\mathcal{C}_0(\mathfrak{K}, \mathfrak{H})$ exists. All $T\in\mathcal{B}(\mathfrak{H}, \mathfrak{K})$ are bounded and the bound of T is denoted by $\|T\|$.

For any $T\in\mathcal{C}(\mathfrak{H})$, the resolvent set and the spectrum of T are denoted by $\rho(T)$ and $\sigma(T)$, respectively. For $\zeta\in\rho(T)$ we write $R(\zeta)=(T-\zeta)^{-1}$. For $T\in\mathcal{C}_0(\mathfrak{H})$ and $\zeta\in\rho(T^*)$ we write $R^*(\zeta)=(T^*-\zeta)^{-1}$.

We denote $\Omega_{\pm}=\{\zeta\in\boldsymbol{C}^1 | Im\zeta\gtrless 0\}$. For any pair of Borel sets Δ_1 and $\Delta_2\subset\boldsymbol{R}^1$ we write $\Delta_1\Subset\Delta_2$ if the closure $\bar{\Delta}_1$ of Δ_1 is the compact subset of the interior Δ_2^0 of Δ_2. For $\Delta\subset\boldsymbol{R}^1$ and $\delta>0$, we set $\Pi_\delta^{\pm}(\Delta)=\{\zeta\in\boldsymbol{C}^1 | Re\zeta\in\Delta, 0\leqq Im\zeta\leqq\pm\delta\}$.

Definition 1.1 (Kato, Lavine). Let T be a selfadjoint operator in $\mathfrak{H}$ and $A\in\mathcal{C}_0(\mathfrak{H}, \mathfrak{K})$ with $\mathfrak{D}(A)\supset\mathfrak{D}(T)$. Then A is said to be T-smooth if one of the (common) values

$$(1.1)\qquad \sup_{\substack{\varepsilon\neq 0\\ \|\varphi\|=1}} \int_{-\infty}^{\infty} \|AR(\lambda\pm i\varepsilon)\varphi\|^2 d\lambda,$$

$$(1.2)\qquad \sup_{I, \|\varphi\|=1, \varphi\in\mathfrak{D}(A^*)} \frac{\|E(I)A^*\varphi\|^2}{|I|},\ I \text{ is an interval, } |I|=\text{length of } I,$$

$$(1.3)\qquad \frac{1}{\pi} \sup_{\|\varphi\|=1, \varepsilon\neq 0, \lambda\in\boldsymbol{R}^1} |\varepsilon|\, \|AR(\lambda\pm i\varepsilon)\varphi\|^2$$

and

$$(1.4)\qquad \frac{1}{2\pi} \sup_{\|\varphi\|=1} \int_{-\infty}^{\infty} \|Ae^{-itT}\varphi\|^2 dt$$

is finite. A is said to be T-smooth on the Borel set $\Delta\subset\boldsymbol{R}^1$ if $AE(\Delta)$ is T-smooth, where $E(\Delta)$ is the spectral measure for T.

Lemma 1.2 (Lavine). *Let T be a selfadjoint operator in $\mathfrak{H}$ and $A\in\mathcal{C}_0(\mathfrak{H}, \mathfrak{K})$ with $\mathfrak{D}(A)\supset\mathfrak{D}(T)$. Assume that there exists a constant $C>0$ such that*

$$(1.5)\qquad \|A(R(\zeta)-R(\bar{\zeta}))A^*\|\leqq C \quad \text{for } \zeta\in\boldsymbol{C}^1, Im\zeta\neq 0, Re\zeta\in\Delta.$$

Then A is T-smooth on Δ.

We shall give here the direct proof of Lemma 1.2. A simpler proof can be found in Lavine [8].

Proof. We first note that (1.5) implies that $E(\Delta)\mathfrak{N}(A^*)\subset\mathfrak{H}_{ac}(\Delta)\equiv E(\Delta)\mathfrak{H}_{ac}$, where $\mathfrak{H}_{ac}$ is the absolutely continuous subspace of $\mathfrak{H}$ with respect to T. Put $P(\lambda,\varepsilon)=\frac{1}{2\pi i}(R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon))$. Then $P(\lambda,\varepsilon)$ is non-negative. Let $M(\lambda,\varepsilon)$ be the square root of $P(\lambda,\varepsilon)$. Then (1.5) implies $AM(\lambda,\varepsilon)\in\mathscr{B}(\mathfrak{H},\mathfrak{K})$ and

$$\|AM(\lambda,\varepsilon)\|\leqq\sqrt{2\pi}C,\qquad \lambda\in\Delta,\ \varepsilon>0. \tag{1.6}$$

(See Kato [4], pp. 273). Let $\{\varphi_n\}$ be an A^*-admissible orthonormal basis of $\mathfrak{K}$ such that $\varphi_n\in\mathfrak{D}(A^*)$ for all n. Since $E(\Delta)A^*\varphi_n\in\mathfrak{H}_{ac}(\Delta)$ we get for any $u\in E(\Delta)\mathfrak{H}$

$$(AP(\lambda,\varepsilon)u,\varphi_n)=(P(\lambda,\varepsilon)u,A^*\varphi_n)\longrightarrow\frac{d}{d\lambda}(E(\lambda)u,A^*\varphi_n)^{(1)}$$

a.e. $\lambda\in\mathbf{R}^1$.

Furthermore by (1.6)

$$\int_I|(AP(\lambda,\varepsilon)u,\varphi_n)|^2d\lambda\leqq 2\pi C^2\int_I\|M(\lambda,\varepsilon)E_{ac}(\Delta)u\|^2d\lambda,$$

I is a Borel set $\subset\Delta$, and as $\varepsilon\searrow 0$

$$\|M(\lambda,\varepsilon)E_{ac}(\Delta)u\|^2\longrightarrow\frac{d}{d\lambda}(E_{ac}(\lambda)u,u)\qquad\text{a.e. }\lambda\text{ and in }L^1(\Delta).$$

Therefore by Vitali's convergence theorem we get

$$(AP(\lambda,\varepsilon)u,\varphi_n)=(P(\lambda,\varepsilon)u,A^*\varphi_n)\longrightarrow\frac{d}{d\lambda}(E(\lambda)u,A^*\varphi_n)\quad in\quad L^2(\Delta).$$

On the other hand for any $u\in E(\Delta)\mathfrak{H}$

$$\begin{aligned}(AR(\lambda\pm i\varepsilon)u,\varphi_n)&=(R(\lambda\pm i\varepsilon)u,A^*\varphi_n)\\&=\int_{\mathbf{R}^1}\frac{\chi_\Delta(\mu)}{\mu-(\lambda\pm i\varepsilon)}\frac{d}{d\mu}(E(\lambda)u,A^*\varphi_n)d\mu,\end{aligned}$$

where $\chi_\Delta(\mu)$ is the characteristic function of $\Delta\subset\mathbf{R}^1$. Then the trivial use of the boundedness of the Hilbert transform in $L^2(\mathbf{R}^1)$ shows that

$$\begin{aligned}\int_{\mathbf{R}^1}|(AR(\lambda\pm i\varepsilon)u,\varphi_n)|^2d\lambda&\leqq\int_\Delta\left|\frac{d}{d\mu}(E(\mu)u,A^*\varphi_n)\right|^2d\mu\\&=\lim_{\delta\to 0}\int_\Delta|(AP(\mu,\delta)u,\varphi_n)|^2d\mu.\end{aligned}$$

Therefore we get

(1) We denote $E((-\infty,\lambda))$ simply by $E(\lambda)$.

$$\begin{aligned}\int_{R^1}||AR(\lambda\pm i\varepsilon)u||^2d\lambda&=\sum_{n=1}^{\infty}\int_{R^1}|(AR(\lambda\pm i\varepsilon)u,\varphi_n)|^2d\lambda\\&\leqq\sum_{n=1}^{\infty}\lim_{\delta\to0}\int_{\Delta}|(AP(\mu,\delta)u,\varphi_n)|^2d\mu\\&=\lim_{\delta\to0}\sum_{n=1}^{\infty}\int_{\Delta}|(AP(\mu,\delta)u,\varphi_n)|^2d\mu\\&=\lim_{\delta\to0}\int_{\Delta}||AP(\mu,\delta)u||^2d\mu\\&\leqq2\pi C^2\lim_{\delta\to0}\int_{\Delta}||M(\mu,\delta)u||^2d\mu=2\pi C^2||u||.\end{aligned}$$

This proves the T-smoothness of A on Δ. q.e.d.

Lemma 1.3. *Let A be T-smooth on Δ_0. Then for any $\Delta_1\Subset\Delta_0$*

$$\sup_{||f||=1}\int_{\Delta_1}||AR(\lambda\pm i\varepsilon)f||^2d\lambda\leqq C(\Delta_1). \tag{1.7}$$

Furthermore $AR(\lambda\pm i\varepsilon)f$ have limits in $L^2(\Delta_1;\mathfrak{K})$ as $\varepsilon\searrow0$.

Proof. $AR(\lambda\pm i\varepsilon)f=AR(\lambda\pm i\varepsilon)E(\Delta_0)f+AR(\lambda\pm i\varepsilon)E(\Delta_0^c)f$, where $\Delta_0^c=\boldsymbol{R}^1\backslash\Delta_0$. The first term belongs to the Hardy class $H^2(\Omega_\pm)$ and

$$\int_{\Delta_1}||AR(\lambda\pm i\varepsilon)E(\Delta_0)f||^2d\lambda\leqq C||E(\Delta_0)f||^2, \tag{1.8}$$

by Lemma 1.2. The second term is analytic in some neighbourhood of Δ_1 and we see easily that

$$\int_{\Delta_1}||AR(\lambda\pm i\varepsilon)E(\Delta_0^c)f||^2d\lambda\leqq C(\Delta_1,\Delta_0)||E(\Delta_0^c)f||^2, \tag{1.9}$$

where $C(\Delta_1,\Delta_0)$ is a constant depending only on Δ_1 and Δ_0. Combining (1.8) and (1.9) we get the desired results. q.e.d.

§ 2. Construction of the perturbed spectral measure

Let T_1 be a selfadjoint operator in $\mathfrak{H}$ with the corresponding spectral measure $E_1(\cdot)$, and let $A,B\in\mathcal{C}_0(\mathfrak{H},\mathfrak{K})$. We introduce the following conditions (A-1) and (A-2).

(A-1) $\mathfrak{D}(A)\cap\mathfrak{D}(B)\supset\mathfrak{D}(T)$. The domain of the closed operator $B[R_1(\zeta)A^*]$[(2)] is equal to the whole space $\mathfrak{K}$ for some ζ, $Im\zeta\neq0$, where $R_1(\zeta)=(T_1-\zeta)^{-1}$.

Remark. By the resolvent equation for $R_1(\zeta)$, (A-1) implies that $B[R_1(\zeta)A^*]$

(2) We denote the closure of an operator A by $[A]$.

is a $\mathscr{B}(\mathfrak{K})$-valued holomorphic function in ζ, $Im\zeta\neq 0$.

We denote

(2.1) $$B[R_1(\zeta)A^*]=Q(\zeta), \qquad A[R_1(\zeta)B^*]=Q^*(\zeta)=Q(\bar{\zeta})^*.$$

(A-2) There exist a constant $\delta>0$ and an open set Δ_0 such that $1+Q(\zeta)$ has a bounded inverse $(1+Q(\zeta))^{-1}$ for any $\zeta\in\Pi_\delta^\pm(\Delta_0)$ and $(1+Q(\zeta))^{-1}$ is uniformly bounded in $\zeta\in\Pi_\delta^\pm(\Delta_0)$.

We note that (A-2) implies that $1+Q^*(\zeta)$ has the same properties as $1+Q(\zeta)$ and that both $(1+Q(\zeta))^{-1}$ and $(1+Q^*(\zeta))^{-1}$ are holomorphic in $\zeta\in\Pi_\delta^\pm(\Delta_0)$. We give here two types of sufficient conditions for (A-2).

(B-1) (Small perturbation) There exists $\gamma, 0\leqq\gamma<1$ such that $\|Q(\zeta)\|\leqq\gamma$ for $\zeta\in\Pi_\delta^\pm(\Delta_0)$.

(B-2) (Compact perturbation) $Q(\zeta)$ is compact at some ζ (hence so is $Q(\zeta)$ at any $\zeta\in\Pi_\delta^\pm(\Delta_0)$) and $Q(\zeta)$ has a boundary value $Q(\lambda\pm i0)=\lim_{\varepsilon\downarrow 0}Q(\lambda\pm i\varepsilon)$ in the operator norm topology which is continuous in $\lambda\in\Delta_0$ and $\sigma(1+Q(\lambda\pm i0))\not\ni 0, \lambda\in\Delta_0$.

The first implication; (B-1) $\rightarrow$ (A-2), is easy and the proof for the second one; (B-2) $\rightarrow$ (A-2), is seen in [5].

Now under the conditions (A-1) and (A-2) we define the perturbed "resolvent" $R_2(\zeta)$ as

(2.2) $$R_2(\zeta)=R_1(\zeta)-[R_1(\zeta)A^*](1+Q(\zeta))^{-1}BR_1(\zeta), \qquad \zeta\in\Pi_\delta^\pm(\Delta).$$

Then following Kato [4] § 2.1, we can prove that $R_2(\zeta)$ is the resolvent of some closed operator T_2. The rest of this paper is devoted to the investigation of the spectral properties of the operator T_2 under the following additional condition

(A-3) A and B are T_1-smooth on Δ_0

or

(A-3)' there exists a constant $C>0$ such that

(2.3) $$\|A(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))A^*\|_{\mathfrak{K}}, \|B(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))B^*\|_{\mathfrak{K}}\leqq C$$

for $\lambda\in\Delta_0, \varepsilon>0$.

We now introduce the following sesqui-linear forms on $\mathfrak{H}\times\mathfrak{H}$.

(2.4) $$\mathfrak{E}_2(\varepsilon, \Delta)[f, g]=\frac{1}{2\pi i}\int_\Delta((R_2(\lambda+i\varepsilon)-R_2(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda,$$

$\varepsilon\in(0, \delta)$, $\Delta\Subset\Delta_0$.

Lemma 2.1. *Let (A-1), (A-2) and (A-3) (or (A-3)') be satisfied. Then for any Borel set* $\Delta\Subset\Delta_0$, $\mathfrak{E}_2(\varepsilon, \Delta)[f, g]$ *is a family of uniformly bounded sesqui-linear forms in* $\varepsilon\in(0, \delta)$:

(2.5) $$|\mathfrak{E}_2(\varepsilon, \Delta)[f, g]| \leqq C(\Delta)\|f\|_{\mathfrak{H}}\|g\|_{\mathfrak{H}}.$$

Furthermore $\lim_{\varepsilon\downarrow 0} \mathfrak{E}_2(\varepsilon, \Delta)[f, g] \equiv \mathfrak{E}_2[f, g]$ *exists for any* $f, g \in \mathfrak{H}$ *and* $\mathfrak{E}_2(\Delta)[f, g]$ *is expressed as*

(2.6) $$\mathfrak{E}_2(\Delta)[f, g] = (E_2(\Delta)f, g)_{\mathfrak{H}}$$

by a unique bounded operator $E_2(\Delta)$.

Proof. By the resolvent equation (2.2), $\mathfrak{E}_2(\varepsilon, \Delta)[f, g]$ is expressed as

(2.7) $$\begin{aligned}\mathfrak{E}_2(\varepsilon, \Delta)[f, g] = &\frac{1}{2\pi i}\int_\Delta ((R_1(\lambda+i\varepsilon) - R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}} d\lambda \\ &- \frac{1}{2\pi i}\int_\Delta ((1+Q(\lambda+i\varepsilon))^{-1}BR_1(\lambda+i\varepsilon)f, AR_1(\lambda-i\varepsilon)g)_{\mathfrak{K}} d\lambda \\ &+ \frac{1}{2\pi i}\int_\Delta ((1+Q(\lambda-i\varepsilon))^{-1}BR_1(\lambda-i\varepsilon)f, AR_1(\lambda+i\varepsilon)g)_{\mathfrak{K}} d\lambda.\end{aligned}$$

The first term is uniformly bounded in ε and converges to $(E_1(\Delta)f, g)_{\mathfrak{H}}$ as $\varepsilon \searrow 0$. The second and the third terms are treated as follows. By Lemma 1.2, (A-3)′ implies (A-3). (A-3) and Lemma 1.3 then imply that $AR_1(\lambda\pm i\varepsilon)f$ and $BR_1(\lambda\pm i\varepsilon)g$ converge in $L^2(\Delta, \mathfrak{K})$ as $\varepsilon \searrow 0$. Since $(1+Q(\lambda\pm i\varepsilon))^{-1}$ are holomorphic and uniformly bounded in $\Pi_\delta^\pm(\Delta)$, Fatou's theorem shows that the integrands in the second and the third terms converge in $L^1(\Delta)$ as $\varepsilon \searrow 0$. Hence, remembering the estimate (1.7) and using the Schwarz inequality, we obtain the desired boundedness and the existence of the limit. $E_2(\Delta)$ is then determined uniquely by the Riesz representation theorem. q.e.d.

Remark. By Fatou's theorem, $(1+Q(\lambda\pm i\varepsilon))^{-1}BR_1(\lambda\pm i\varepsilon)f$ converge weakly in $L^2(\Delta, \mathfrak{K})$.

We call the $\mathcal{B}(\mathfrak{H})$-valued set function $E_2(\cdot)$ on the precompact Borel subset of Δ_0 the spectral measure of T_2 on Δ_0. The next theorem justifies the name.

Theorem 2.2. *Uuder the same conditions as in Lemma* 2.1 *the family of operators* $\{E_2(\Delta) | \Delta \Subset \Delta_0, \Delta$ *is a Borel set*$\}$ *satisfy*

(2.8) $$E_2(\Delta')E_2(\Delta'') = E_2(\Delta' \cap \Delta'') \qquad \Delta', \Delta'' \Subset \Delta_0$$

and

(2.9) $$E_2(\Delta)T_2 \subset T_2E_2(\Delta) \qquad \Delta \Subset \Delta_0.$$

Proof. The left hand side of (2.8) is expressed in a weak form as

(2.10) $$\begin{aligned}(E_2(\Delta')E_2(\Delta'')f, g)_{\mathfrak{H}} = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta'} d\lambda\Big[&\lim_{\delta\downarrow 0}\frac{1}{2\pi i}\int_{\Delta''}((R_2(\lambda+i\varepsilon) - R_2(\lambda-i\varepsilon)) \\ &\times (R_2(\eta+i\delta) - R_2(\eta-i\delta))f, g)_{\mathfrak{H}} d\eta\Big].\end{aligned}$$

By the use of the resolvent equation for $R_2(\zeta)$ we have

$$(2.11)\qquad (R_2(\lambda+i\varepsilon)-R_2(\lambda-i\varepsilon))(R_2(\eta+i\delta)-R_2(\eta-i\delta))$$
$$=\frac{2i\delta}{(\lambda+i\varepsilon-\eta)^2+\delta^2}R_2(\lambda+i\varepsilon)-\frac{2i\delta}{(\lambda-i\varepsilon-\eta)^2+\delta^2}R_2(\lambda-i\varepsilon)$$
$$+\frac{2i\varepsilon}{(\eta+i\delta-\lambda)^2+\varepsilon^2}R_2(\eta+i\varepsilon)-\frac{2i\varepsilon}{(\eta-i\delta-\lambda)^2+\varepsilon^2}R_2(\eta-i\varepsilon).$$

We write the L^1-limits

$$\lim_{\delta\downarrow 0}((1+Q(\eta\pm i\delta))^{-1}BR_1(\eta\pm i\delta)f, AR_1(\eta\pm i\delta)g)_{\mathfrak{K}}\quad\text{in}\quad L^1(\Delta'')$$

as $F^{\pm}(\eta)$. Then using (2.2), (2.11) and Fubini's theorem we obtain that

$$(2.12)\qquad (E_2(\Delta')E_2(\Delta'')f, g)_{\mathfrak{H}}=(E_1(\Delta'\cap\Delta'')f, g)_{\mathfrak{H}}$$
$$-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta'}d\lambda\left[\int_{\Delta''}\frac{\varepsilon}{\pi((\eta-\lambda)^2+\varepsilon^2)}(F^+(\eta)-F^-(\eta))d\eta\right]$$
$$=(E_1(\Delta'\cap\Delta'')f, g)_{\mathfrak{H}}$$
$$-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta''}d\eta\left[\int_{\Delta'}\frac{\varepsilon d\lambda}{\pi((\eta-\lambda)^2+\varepsilon^2)}\right](F^+(\eta)-F^-(\eta)).$$

On the other hand, $\int_{\Delta'}\frac{\varepsilon d\lambda}{\pi((\eta-\lambda)^2+\varepsilon^2)}$ is uniformly bounded and converges pointwisely to the characteristic function $\chi_{\Delta'}(\eta)$ as $\varepsilon\searrow 0$. Therefore using the dominated convergence theorem and (2.7), we easily obtain (2.8).

(2.9) is the direct consequence of the fact that $R_2(\zeta)T_2\subset T_2R_2(\zeta)$ and the detailed proof is omitted. q.e.d.

Remark 2.3. Theorem 2.2 shows that $E_2(\Delta)\mathfrak{H}$ is a closed subspace of $\mathfrak{H}$ which reduces T_2. We can show directly that T_2 is bounded in $E_2(\Delta)\mathfrak{H}$ if Δ is bounded, which will be proved in the next section by means of the wave operators.

§ 3. Wave operators and the similarity

In this section we shall establish the similarity between the parts of T_1 and T_2 by means of the wave operators to be constructed below. First we define two forms $\mathfrak{W}_{\pm}(\varepsilon, \Delta)$ and $\mathfrak{Z}_{\pm}(\varepsilon, \Delta)$ on $\mathfrak{H}\times\mathfrak{H}$:

$$(3.1)\qquad \mathfrak{W}_{\pm}(\varepsilon, \Delta)[f, g]=\frac{1}{2\pi i}\int_{\Delta}((R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda$$
$$-\frac{1}{2\pi i}\int_{\Delta}(B(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f,$$
$$(1+Q^*(\lambda\pm i\varepsilon))^{-1}AR_1(\lambda\pm i\varepsilon)g)_{\mathfrak{K}}d\lambda,$$

$f, g \in \mathfrak{H}$, $0<\varepsilon<\delta$, $\varDelta \Subset \varDelta_0$;

(3.2)
$$\mathfrak{Z}_{\pm}(\varepsilon, \varDelta)[f, g]=\frac{1}{2\pi i}\int_{\varDelta}((R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda$$
$$+\frac{1}{2\pi i}\int_{\varDelta}((1+Q(\lambda\pm i\varepsilon))^{-1}BR_1(\lambda\pm i\varepsilon)f,$$
$$A(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))g)_{\mathfrak{K}}d\lambda,$$

Lemma 3.1. *Let (A-1), (A-2) and (A-3) (or (A-3)') be satisfied. Then* $\mathfrak{W}_{\pm}(\varepsilon, \varDelta)[f, g]$ *and* $\mathfrak{Z}_{\pm}(\varepsilon, \varDelta)[f, g]$ *are uniformly bounded (in* ε*) sesqui-linear forms on* $\mathfrak{H}$ *for any fixed* $\varDelta \Subset \varDelta_0$. *Furthermore as* $\varepsilon \searrow 0$, $\mathfrak{W}_{\pm}(\varepsilon, \varDelta)[f, g]$ *and* $\mathfrak{Z}_{\pm}(\varepsilon, \varDelta)[f, g]$ *converge to the limits* $\mathfrak{W}_{\pm}(\varDelta)[f, g]$ *and* $\mathfrak{Z}_{\pm}(\varDelta)[f, g]$. $\mathfrak{W}_{\pm}(\varDelta)[f, g]$ *and* $\mathfrak{Z}_{\pm}(\varDelta)[f, g]$ *are expressed as*

$$\mathfrak{W}_{\pm}(\varDelta)[f, g]=(W_{\pm}(\varDelta)f, g)_{\mathfrak{H}}, \qquad \mathfrak{Z}_{\pm}(\varDelta)[f, g]=(Z_{\pm}(\varDelta)f, g)_{\mathfrak{H}}$$

with some bounded operators $W_{\pm}(\varDelta)$ *and* $Z_{\pm}(\varDelta)$.

Proof of Lemma 3.1 can be carried out in the same way as the proof of Lemma 2.1 and is omitted here. We call these operators $W_{\pm}(\varDelta)$ and $Z_{\pm}(\varDelta)$ the wave operators. The next theorem is the main result of this section. The theorem states some basic properties of $W_{\pm}(\varDelta)$ and $Z_{\pm}(\varDelta)$ and establishes the similarity between the parts of T_1 and T_2.

Theorem 3.2. *Let the conditions (A-1), (A-2) and (A-3) (or (A-3)') be satisfied. Let* $\varDelta \Subset \varDelta_0$. *Then the wave operators* $W_{\pm}(\varDelta)$ *and* $Z_{\pm}(\varDelta)$ *constructed in Lemma 3.1 satisfy;*

(3.4)
$$W_{\pm}(\varDelta)E_1(\varDelta)=E_2(\varDelta)W_{\pm}(\varDelta)=W_{\pm}(\varDelta),$$

(3.5)
$$Z_{\pm}(\varDelta)E_2(\varDelta)=E_1(\varDelta)Z_{\pm}(\varDelta)=Z_{\pm}(\varDelta),$$

and

(3.6)
$$W_{\pm}(\varDelta)Z_{\pm}(\varDelta)=E_2(\varDelta), \qquad Z_{\pm}(\varDelta)W_{\pm}(\varDelta)=E_1(\varDelta).$$

Furthermore the wave operators have the intertwining properties in the sense that they satisfy

(3.7)
$$W_{\pm}(\varDelta)T_1E_1(\varDelta)=T_2W_{\pm}(\varDelta)$$

and

(3.8)
$$Z_{\pm}(\varDelta)T_2E_2(\varDelta)=T_1Z_{\pm}(\varDelta).$$

In order to prove Theorem 3.2 we prepare the following lemma without proof. The lemma shows the various relations between the resolvents $R_1(\zeta)$ and $R_2(\zeta)$.

Lemma 3.3. *Under the same conditions as in Theorem* 3.2, *we have, in the case of bounded A and B, that*

(3.9) $$(1+R_1(\zeta)A^*B)(1-R_2(\zeta)A^*B)=(1-R_2(\zeta)A^*B)(1+R_1(\zeta)A^*B)=1;$$

(3.10) $$(1+A^*BR_1(\zeta))(1-A^*BR_2(\zeta))=(1-A^*BR_2(\zeta))(1+A^*BR_1(\zeta))=1;$$

(3.11) $$R_1(\zeta)A^*(1+Q(\zeta))^{-1}=R_2(\zeta)A^*:$$

(3.12) $$(1+Q(\zeta))^{-1}BR_1(\zeta)=BR_2(\zeta);$$

(3.13) $$R_2(\zeta)-R_2(\bar\zeta)=(1-R_2(\zeta)A^*B)(R_1(\zeta)-R_1(\bar\zeta))(1-A^*BR_2(\bar\zeta));$$

(3.14) $$R_1(\zeta)-R_1(\bar\zeta)=(1+R_1(\zeta)A^*B)(R_2(\zeta)-R_2(\bar\zeta))(1+A^*BR_1(\bar\zeta)).$$

Furthermore, replacing $R_1(\zeta)A^*$ *by* $[R_1(\zeta)A^*]$ *and making some slight modifications, we have the same types of identities in the case of general unbounded A and B.*

Proof of Theorem 3.2[(3)]. We first prove a part of (3.4). By (3.1), and (A-3),

(3.15) $$|(W_\pm(\Delta)f,g)_{\mathfrak{H}}|\leqq \limsup_{\substack{\varepsilon\in(0,\delta)\\ \varepsilon\to 0}}\left\{|(E_1(\Delta)f,g)_{\mathfrak{H}}|+c\left[\int_\Delta(P_1(\lambda,\varepsilon)f,f)_{\mathfrak{H}}d\lambda\right]^{1/2}\|g\|_{\mathfrak{H}}\right\}$$
$$\leqq c\|E_1(\Delta)f\|_{\mathfrak{H}}\,\|g\|_{\mathfrak{H}},$$

where $P_1(\lambda,\varepsilon)=\frac{1}{2\pi i}(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))$. Hence putting $f=h-E_1(\Delta)h$ in (3.15), we get that $W_\pm(\Delta)E_1(\Delta)=W_\pm(\Delta)$. We shall next prove a part of (3.5). We have

(3.16) $$(Z_\pm(\Delta)E_2(\Delta)f,g)_{\mathfrak{H}}$$
$$=\lim_{\varepsilon\downarrow 0}\int_\Delta(P_1(\lambda,\varepsilon)(1-A^*BR_2(\lambda+i\varepsilon))E_2(\Delta)f,g)_{\mathfrak{H}}d\lambda$$
$$=\lim_{\varepsilon\downarrow 0}\int_\Delta d\lambda\Big[\lim_{\delta\downarrow 0}\int_\Delta(P_1(\lambda,\varepsilon)(1-A^*BR_2(\lambda\pm i\varepsilon))$$
$$\times(R_2(\eta+i\delta)-R_2(\eta-i\delta))f,g)_{\mathfrak{H}}d\eta\Big]$$
$$=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta d\lambda\Big[\lim_{\delta\downarrow 0}\frac{1}{2\pi i}\int_\Delta((1+R_1(\lambda\mp i\varepsilon)A^*B)(R_2(\lambda+i\varepsilon)$$
$$-R_2(\lambda-i\varepsilon))(R_2(\eta+i\delta)-R_2(\eta-i\delta))f,g)_{\mathfrak{H}}d\eta\Big].$$

Hence, remembering the identity (2.11), we have

(3) We use the notations in the case of bounded A and B, but the results still hold in general by slight modifications of the proof.

(3.17) $$(Z_\pm(\Delta)E_2(\Delta)f, g)_{\mathfrak{H}} = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta d\lambda\Big[\lim_{\delta\downarrow 0}\frac{1}{2\pi i}\int_\Delta((1+R_1(\lambda\mp i\varepsilon)A^*B)S_2(\lambda,\varepsilon,\eta,\delta)f, g)_{\mathfrak{H}}d\eta\Big],$$

where

$$S_2(\lambda,\varepsilon,\eta,\delta)=\rho(\lambda,\varepsilon,\eta,\delta)R_2(\lambda+i\varepsilon)+\overline{\rho(\lambda,\varepsilon,\eta,\delta)}R_2(\lambda-i\varepsilon)$$
$$+\rho(\eta,\delta,\lambda,\varepsilon)R_2(\eta+i\delta)+\overline{\rho(\eta,\delta,\lambda,\varepsilon)}R_2(\eta-i\delta)$$

with $\rho(\lambda,\varepsilon,\eta,\delta)=\dfrac{2i\delta}{(\lambda+i\varepsilon-\eta)^2+\delta^2}$.
Therefore we get

(3.18) $$(Z_\pm(\Delta)E_2(\Delta)f, g)_{\mathfrak{H}} =(E_2(\Delta)f, g)_{\mathfrak{H}}+\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta d\lambda\frac{1}{2\pi i}\int_\Delta d\eta\frac{2i\varepsilon}{(\eta-\lambda)^2+\varepsilon^2}$$
$$\times(B(R_2(\eta+i0)-R_2(\eta-i0))f, AR_1(\lambda\pm i\varepsilon)g)_{\mathfrak{K}},$$

where $B(R_2(\eta+i0)-R_2(\eta-i0))f$ denotes the weak limit of $B(R_2(\eta+i\delta)-R_2(\eta-i\delta))f$ $(\delta\downarrow 0)$ in $L^2(\Delta;\mathfrak{K})$. Since $AR_1(\lambda\pm i\varepsilon)g$ have the strong limits $AR_1(\lambda\pm i0)g$ (symbolically) in $L^2(\Delta;\mathfrak{K})$ and the last terms of (3.18) are expressed as

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}(B(R_2(\eta+i0)-R_2(\eta-i0))f, \int_\Delta\frac{\varepsilon}{\pi((\eta-\lambda)^2+\varepsilon^2)}AR_1(\lambda\pm i\varepsilon)d\lambda)_{L^2(\Delta;\mathfrak{K})}$$

where the integration is Bochner's integral, we get the desired result:

(3.19) $$(Z_\pm(\Delta)E_2(\Delta)f, g)_{\mathfrak{H}} =(E_2(\Delta)f, g)_{\mathfrak{H}}+\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta(B(R_2(\lambda+i0)-R_2(\lambda-i0))f, AR_1(\lambda\pm i\varepsilon)g)_{\mathfrak{K}}d\lambda$$
$$=(Z_\pm(\Delta)f, g)_{\mathfrak{H}}.$$

We shall now proceed to the proof of (3.6). Using the same procedure as is used to get (3.19) from (3.16), we obtain

(3.20) $$(W_\pm(\Delta)Z_\pm(\Delta)f, g)_{\mathfrak{H}} = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta\Big[\lim_{\delta\downarrow 0}\frac{1}{2\pi i}\int_\Delta((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))$$
$$\times(R_1(\eta+i\delta)-R_1(\eta-i\delta))(1-A^*BR_2(\eta+i\delta))f, g)_{\mathfrak{H}}d\eta\Big]d\lambda$$
$$= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta\Big[\lim_{\delta\downarrow 0}\frac{1}{2\pi i}\int_\Delta((1-R_2(\lambda\mp i\varepsilon)A^*B)(\rho(\eta,\delta,\lambda,\varepsilon)R_1(\eta+i\delta)$$
$$-\overline{\rho(\eta,\delta,\lambda,\varepsilon)}R_1(\eta-i\delta))(1-A^*BR_2(\eta\pm i\delta))f, g)_{\mathfrak{H}}d\eta\Big]d\lambda$$

$$= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta ((R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda$$
$$-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta (BR_2(\lambda\pm i\varepsilon)f, A(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))g)_{\mathfrak{K}}d\lambda$$
$$-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta (B(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f, AR_2^*(\lambda\pm i\varepsilon)g)_{\mathfrak{K}}d\lambda$$
$$+\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta\Big[\lim_{\delta\downarrow 0}\frac{1}{2\pi i}\int_\Delta (R_2(\lambda\mp i\varepsilon)A^*B(\rho(\eta,\delta,\lambda,\varepsilon)R_1(\eta+i\delta)$$
$$-\overline{\rho(\eta,\delta,\lambda,\varepsilon)}R_1(\eta-i\delta))A^*BR_2(\eta\pm i\delta)f, g)_{\mathfrak{H}}d\eta\Big]d\lambda.$$

By (3.11), (A-3) (or (A-3)′) and Fatou's theorem

$$\lim_{\delta\downarrow 0}\int_\Delta (R_2(\lambda\mp i\varepsilon)A^*\{B(\rho(\eta,\delta,\lambda,\varepsilon)R_1(\eta+i\delta)-\overline{\rho(\eta,\delta,\lambda,\varepsilon)}R_1(\eta-i\delta))A^*\}$$
$$\times BR_2(\eta\pm i\delta)f, g)_{\mathfrak{H}}d\lambda$$
$$=\int_\Delta ((BR_1(\eta+i0)A^*-BR_1(\eta-i0)A^*)BR_2(\eta\pm i0)f, AR_2^*(\lambda\pm i\varepsilon)g)_{\mathfrak{K}}\frac{2i\varepsilon}{(\eta-\lambda)^2+\varepsilon^2}d\eta,$$

where $BR_2(\eta\pm i0)f$ are the weak-$L^2(\Delta,\mathfrak{K})$ limits of $BR_2(\eta\pm i\delta)f$ and $BR_1(\eta\pm i0)A^*$ are the nontangential strong limits of uniformly bounded and holomorphic operator valued functions $BR_1(\zeta)A^*$, $\zeta\in\Omega_\pm$, $Re\zeta\in\Delta_0$. Therefore

$$\lim_{\varepsilon\downarrow 0}\int_\Delta\Big\{\int_\Delta\frac{\varepsilon}{\pi((\lambda-\eta)^2+\varepsilon^2)}(\{BR_1(\eta+i0)A^*-BR_1(\eta-i0)A^*\}$$
$$\times BR_2(\eta\pm i0)f, AR_2^*(\lambda\pm i\varepsilon)g)_{\mathfrak{K}}d\eta\Big\}d\lambda$$
$$=\lim_{\varepsilon\downarrow 0}\Big(\int_\Delta\frac{\varepsilon}{\pi((\lambda-\eta)^2+\varepsilon^2)}(\{BR_1(\eta+i0)A^*-BR_1(\eta-i0)A^*\}$$
$$\times BR_2(\eta\pm i0)fd\eta, AR_2^*(\lambda\pm i\varepsilon)g\Big)_{L^2(\Delta,\mathfrak{K})}$$
$$=((BR_1(\lambda+i0)A^*-BR_1(\lambda-i0)A^*)BR_2(\lambda\pm i0)f, AR_2^*(\lambda\pm i0)g)_{L^2(\Delta;\mathfrak{K})}$$
$$=((1+Q(\lambda\mp i0))^{-1}(BR_1(\lambda+i0)A^*$$
$$-BR_1(\lambda-i0)A^*)BR_2(\lambda\pm i0)f, AR_1(\lambda\pm i0)g)_{L^2(\Delta;\mathfrak{K})}$$
$$=\lim_{\varepsilon\downarrow 0}\int_\Delta (R_2(\lambda\mp i\varepsilon)A^*B(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))A^*BR_2(\lambda\pm i\varepsilon)f, g)_{\mathfrak{H}}d\lambda.$$

Combining the above calculations we have

(3.21) $$(W_\pm(\Delta)Z_\pm(\Delta)f, g)_{\mathfrak{H}}$$
$$=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta ((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))$$

$$\times(1-A^*BR_2(\lambda\pm i\varepsilon))f, g)_{\mathfrak{H}}d\lambda$$
$$=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}((R_2(\lambda+i\varepsilon)-R_2(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda$$
$$=(E_2(\Delta)f, g)_{\mathfrak{H}}.$$

This concludes the proof of the first part of (3.6). The second part is proved in a similar way and we omit the proof. Next we complete the proof of (3.4) and (3.5). Using (3.6) we have that

$$E_2(\Delta)W_\pm(\Delta)=W_\pm(\Delta)Z_\pm(\Delta)W_\pm(\Delta)=W_\pm(\Delta)E_1(\Delta)=W_\pm(\Delta) \tag{3.22}$$

and in a similar way that

$$E_1(\Delta)Z_\pm(\Delta)=Z_\pm(\Delta). \tag{3.23}$$

Finally we shall prove the intertwining properties (3.7) and (3.8). First we show (3.7). For $f\in E_1(\Delta)\mathfrak{H}$ and $g\in\mathfrak{D}(T_2^*)$, we have

$$\begin{aligned}(3.24)\quad &(W_\pm(\Delta)T_1f, g)_{\mathfrak{H}}\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))T_1f, g)_{\mathfrak{H}}d\lambda\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\Big[\int_{\Delta}\lambda((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda\\ &\quad+i\varepsilon\int_{\Delta}((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)+R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda\Big]\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}\{\lambda((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}\\ &\quad+i\varepsilon((R_1(\lambda+i\varepsilon)+R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}\}d\lambda.\end{aligned}$$

On the other hand we have

$$\begin{aligned}(3.25)\quad &(T_2W_\pm(\Delta)f, g)_{\mathfrak{H}}\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f_1, T_2^*g)_{\mathfrak{H}}d\lambda\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}((R_2(\lambda+i\varepsilon)-R_2(\lambda-i\varepsilon))(1+A^*BR_1(\lambda\pm i\varepsilon))f, T_2^*g)_{\mathfrak{H}}d\lambda\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}\{\lambda((R_2(\lambda+i\varepsilon)-R_2(\lambda-i\varepsilon))(1+A^*BR_1(\lambda\pm i\varepsilon))f, g)_{\mathfrak{H}}\\ &\quad+i\varepsilon((R_2(\lambda+i\varepsilon)+R_2(\lambda-i\varepsilon))(1+A^*BR_1(\lambda\pm i\varepsilon))f, g)_{\mathfrak{H}}\}d\lambda\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta}\{\lambda((1-R_2(\lambda\mp i\varepsilon)A^*B)(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}\}d\lambda\\ &\quad+\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{2\pi}\int_{\Delta}((R_1(\lambda+i\varepsilon)+R_1(\lambda-i\varepsilon))f, g)_{\mathfrak{H}}d\lambda.\end{aligned}$$

Comparing (3.24) and (3.25), we get the desired result. (3.8) is proved as follows

$$Z_{\pm}(\Delta)T_2E_2(\Delta)=Z_{\pm}(\Delta)T_2W_{\pm}(\Delta)Z_{\pm}(\Delta) \tag{3.26}$$
$$=Z_{\pm}(\Delta)W_{\pm}(\Delta)T_1E_1(\Delta)Z_{\pm}(\Delta)$$
$$=E_1(\Delta)TE_1(\Delta)Z_{\pm}(\Delta)=T_1Z_{\pm}(\Delta).$$

This completes the proof of the theorem. q.e.d.

§ 4. Scattering theory and the invariance principle

We shall give here the time dependent expressions for $Z_{\pm}(\Delta)$ and $W_{\pm}(\Delta)$ constructed by the stationary method in § 3, and the invariance principle for them. To state and prove the theorems we prepare some immediate consequences of the results obtained in the preceding sections. First we note that Theorem 3.2 implies

$$W_{\pm}(\Delta)R_1(\zeta)E_1(\Delta)Z_{\pm}(\Delta)=R_2(\zeta)E_2(\Delta), \qquad \zeta\in \textstyle\prod_{\delta}^{\pm}(\Delta_0) \tag{4.1}$$

$$Z_{\pm}(\Delta)^*R_1(\zeta)E_1(\Delta)W_{\pm}(\Delta)^*=R_2^*(\zeta)E_2(\Delta)^*, \qquad \zeta\in \textstyle\prod_{\delta}^{\pm}(\Delta_0). \tag{4.1*}$$

Therefore $R_2(\zeta)E_2(\Delta)$ (or $R_2^*(\zeta)E_2(\Delta)^*$) can be extended to $\boldsymbol{C}^1\backslash\Delta$ holomorphically and $\sigma(T_2|_{E_2(\Delta)\mathfrak{H}})\subset\bar{\Delta}$ (or $\sigma(T_2^*|_{E_2(\Delta)^*\mathfrak{H}})\subset\bar{\Delta}$).

Let $\mathfrak{M}(\Delta)$ be the algebra of all bounded Borel measurable functions on Δ. Then for any $\phi(\lambda)\in\mathfrak{M}(\Delta)$ we can define the operator $\phi(T_2)\in\mathscr{B}(E_2(\Delta)\mathfrak{H})$ ($\phi(T_2^*)\in\mathscr{B}(E_2(\Delta)^*\mathfrak{H})$) by

$$(\phi(T_2)f,g)=\lim_{\varepsilon\downarrow 0}\int_{\Delta}(\phi(\lambda)(R_2(\lambda+i\varepsilon)-R_2(\lambda-i\varepsilon))f,g)d\lambda, \tag{4.2}$$
$$f\in E_2(\Delta)\mathfrak{H},\ g\in E_2(\Delta)^*\mathfrak{H},$$

$$\left((\phi(T_2^*)f,g)=\lim_{\varepsilon\downarrow 0}\int_{\Delta}(\phi(\lambda)(R_2^*(\lambda+i\varepsilon)-R_2^*(\lambda-i\varepsilon))f,g)d\lambda,\right.$$
$$\left. f\in E_2(\Delta)^*\mathfrak{H},\ g\in E_2(\Delta)\mathfrak{H}.\right)$$

We consider $\phi(T_2)E_2(\Delta)$ (or $\phi(T_2^*)E_2(\Delta)^*$) as an element of $\mathscr{B}(\mathfrak{H})$. Relations (4.1) and (4.2) show that

$$W_{\pm}(\Delta)\phi(T_1)E_1(\Delta)Z_{\pm}(\Delta)=\phi(T_2)E_2(\Delta). \tag{4.3}$$

$$Z_{\pm}(\Delta)^*\phi(T_1)E_1(\Delta)W_{\pm}(\Delta)^*=\phi(T_2^*)E_2(\Delta). \tag{4.3*}$$

$$(\phi(T_2)E_2(\Delta))^*=\bar{\phi}(T_2^*)E_2(\Delta)^*. \tag{4.4}$$

The correspondence $\mathfrak{M}(\Delta)\ni\phi\rightarrow\phi(T_2)\in\mathscr{B}(E_2(\Delta)\mathfrak{H})$ induces a homomorphism from the algebra $\mathfrak{M}(\Delta)$ to $\mathscr{B}(E_2(\Delta)\mathfrak{H})$. If $\phi(\lambda)$ is real-valued (or $|\phi(\lambda)|=1$), $\phi(T_2)$ is a selfadjoint (or unitary) operator in $E_2(\Delta)\mathfrak{H}$ with respect to the inner product $(Z_{\pm}(\Delta)\cdot,$

$Z_{\pm}(\Delta)\cdot)$. Similar relations hold for $\phi(T_2^*)$.

By (4.1), (4.2) and the use of Fourier transform we get that for any $A\in C_0(\mathfrak{H},\mathfrak{K})$ with $\mathfrak{D}(A)\subset\mathfrak{D}(T)$

$$\pm\int_0^{\pm\infty}\|Ae^{-\varepsilon|t|-itT}E(\Delta)f\|^2dt=\int_{R^1}\|AR(\lambda\pm i\varepsilon)E(\Delta)f\|^2d\lambda \tag{4.5}$$

where $(T,R(\zeta),E(\Delta))$ stands for any one of $(T_i,R_i(\zeta),E_i(\Delta))$ $(i=1,2,)$ and $(T_2^*, R_2^*(\zeta),E_2(\Delta)^*)$. Hence if A and B satisfy the conditions (A-1), (A-2) and (A-3) (or (A-3)′) we can conclude that $CR(\zeta)E(\Delta)f$ belongs to the Hardy class $H^2(\Omega_{\pm})$ and

$$\int_{R^1}(CR_i(\lambda+i\varepsilon)E_i(\Delta)f,DR_j(\lambda-i\varepsilon)E_j(\Delta)g)d\lambda=0,\quad i,j=1,2, \tag{4.6}$$
$$\varepsilon>0,\ f,g\in\mathfrak{H}$$

where each one of C and D stands for any one of A and B. Furthermore (4.6) remains valid even if any one of $R_i(\lambda\pm i\varepsilon)$ $(i=1,2)$ is replaced by $R_2^*(\lambda\pm i\varepsilon)$.

We finally remark that by the relation (4.6) and the theory of Fourier transform $W_{\pm}(\Delta)$ and $Z_{\pm}(\Delta)$ are represented as

$$\begin{aligned}(W_{\pm}(\Delta)f,g)&=(f,g)+\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}(BR_1(\lambda\pm i\varepsilon)E_1(\Delta)f,AR_2^*(\lambda\pm i\varepsilon)\\&\quad\times E_2^*(\Delta)g)d\lambda\\&=(f,g)\pm\int_0^{\pm\infty}(Be^{-itT_1}E_1(\Delta)f,Ae^{-itT_2^*}E_2(\Delta)^*g)dt,\\&\qquad f\in E_1(\Delta)\mathfrak{H},\ g\in E_2(\Delta)^*\mathfrak{H},\end{aligned} \tag{4.7}$$

$$\begin{aligned}(Z_{\pm}(\Delta)f,g)&=(f,g)-\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}(BR_2(\lambda\pm i\varepsilon)E_2(\Delta)f,AR_1(\lambda\pm i\varepsilon)\\&\quad\times E_1(\Delta)g)d\lambda\\&=(f,g)\mp\int_0^{\pm\infty}(Be^{-itT_2}E_2(\Delta)f,Ae^{-itT_1}E_1(\Delta)g)dt,\\&\qquad f\in E_2(\Delta)\mathfrak{H},\ g\in E_1(\Delta)\mathfrak{H}.\end{aligned} \tag{4.8}$$

We can now state the theorems.

Theorem 4.1. *Let the conditions* (A-1), (A-2) *and* (A-3) (*or* (A-3)′) *be satisfied. Then the strong limits in the following formulas exist and*

$$W_{\pm}(\Delta)=\text{s}-\lim_{t\to\pm\infty}e^{itT_2}E_2(\Delta)e^{-itT_1}E_1(\Delta). \tag{4.9}$$

$$Z_{\pm}(\Delta)=\text{s}-\lim_{t\to\pm\infty}e^{itT_1}E_1(\Delta)e^{-itT_2}E_2(\Delta). \tag{4.10}$$

Remark 4.2. If A is compact as an operator from $\mathfrak{D}(A)$ to $\mathfrak{K}$, we can omit

$E_1(\Delta)$ in the expression of $Z_\pm(\Delta)$ in (4.10).

Theorem 4.3. (*Invariance principle*). *Let the conditions* (*A*-1), (*A*-2) *and* (*A*-3) (*or* (*A*-3)′) *be satisfied. Let* $\phi(\lambda)$ *be a real valued Borel measurable function on* Δ *such that*

$$\int_0^\infty \left| \int_\Delta f(\lambda) e^{-it\phi(\lambda)-is\lambda} d\lambda \right|^2 ds \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \tag{4.11}$$

for any $f \in L^2(\Delta)$. *Then the strong limits in the following formulas exist and*

$$W_\pm(\Delta) = \text{s}-\lim_{t\to\pm\infty} e^{it\phi(T_2)} E_2(\Delta) e^{-it\phi(T_1)} E_1(\Delta). \tag{4.12}$$

$$Z_\pm(\Delta) = \text{s}-\lim_{t\to\pm\infty} e^{it\phi(T_1)} E_1(\Delta) e^{-it\phi(T_2)} E_2(\Delta). \tag{4.13}$$

Remark 4.4. Some sufficient conditions on $\phi(\lambda)$ which imply (4.11) may be found in Kato-Kuroda [5].

Proof of Theorem 4.1 is entirely the same as the one of Theorem 3.9 of Kato [4] (pp. 270) and it is omitted here.

Proof of Theorem 4.3. We shall give here the proof for $W_+(\Delta)$. Others are proved similarly. Since Theorem 3.2 and (4.3) imply

$$e^{it\phi(T_2)} W_+(\Delta) = W_+(\Delta) e^{it\phi(T_1)},$$

we get

$$(W_+(\Delta) e^{-it\phi(T_1)} E_1(\Delta) f, e^{-it\phi(T^*_2)} E_2(\Delta)^* g) = (W_+(\Delta) f, g)$$

by (4.4). Hence replacing f, g by $e^{is\phi(T_1)} E_1(\Delta) f$ and $e^{-is\phi(T_2^*)} E_2(\Delta)^* g$ in (4.7), we get for any $f, g \in \mathfrak{H}$

$$\begin{aligned}(W_+(\Delta) f, g) &= (e^{is\phi(T_2)} E_2(\Delta) e^{-is\phi(T_1)} E_1(\Delta) f, g) \\ &\quad + \int_0^\infty (B e^{-itT_1} e^{-is\phi(T_1)} E_1(\Delta) f, A e^{-itT_2^*} e^{-is\phi(T_2^*)} E_2(\Delta)^* g) dt.\end{aligned}$$

Therefore by Schwarz's inequality

$$\begin{aligned}&|(\{W_+(\Delta) - e^{is\phi(T_2)} E_2(\Delta) e^{-is\phi(T_1)} E_1(\Delta)\} f, g)| \\ &\leqq \left(\int_0^\infty \|B e^{-itT_1 - is\phi(T_1)} E_1(\Delta) f\|^2 dt \right)^{1/2} \left(\int_0^\infty \|A e^{-itT_2^* - is\phi(T_2^*)} E_2(\Delta)^* g\|^2 dt \right)^{1/2}.\end{aligned} \tag{4.12}$$

The second factor of the right hand side of (4.12) is dominated by

$$\left(\sup_{\varepsilon\in(0,\delta)} \int_{R^1} \|A R^*(\lambda + i\varepsilon) e^{-is\phi(T_2^*)} E_2(\Delta)^* g\|^2 d\lambda \right)^{1/2} \leqq C\|g\|, \tag{4.13}$$

where C is a constant which depends only on Δ and ϕ. Since we have, for any

$f\in E_1(\Delta)\mathfrak{H}$ and any Borel set $I\subset\Delta$, that

$$\int_I BE_1(d\lambda)f=\int_I B(R_1(\lambda+i0)-R_1(\lambda-i0))fd\lambda,$$

where the integral in the right hand side is Bochner's integral and

$$B(R_1(\lambda+i0)-R_1(\lambda-i0))f=\lim_{\varepsilon\downarrow 0} B(R_1(\lambda+i\varepsilon)-R_1(\lambda-i\varepsilon))f$$

in $L^2(\Delta,\mathfrak{K})$, we get easily that

$$\int_0^\infty \|Be^{-itT_1-is\phi(T_1)}E_1(\Delta)f\|^2dt \tag{4.14}$$

$$=\int_0^\infty\left\|\int_\Delta e^{-it\lambda-is\phi(\lambda)}BE_1(d\lambda)f\right\|^2dt$$

$$=\int_0^\infty\left\|\int_\Delta e^{-it\lambda-is\phi(\lambda)}B(R_1(\lambda+i0)-R_1(\lambda-i0))fd\lambda\right\|^2dt.$$

On the other hand it is easily seen that (4.10) implies the same statement for $f\in L^2(\Delta:\mathfrak{K})$. Therefore the first factor of the right hand side of (4.12) converges to zero as $s\to\infty$. Combining the results we get the desired result:

$$s-\lim_{t\to\infty} e^{it\phi(T_2)}E_2(\Delta)e^{-it\phi(T_1)}E_1(\Delta)=W_+(\Delta). \qquad \text{q.e.d.}$$

§ 5. Applications to partial differential equations

Let $\mathfrak{H}$ be $L^2(\boldsymbol{R}^3)$ and T_1 be $-\Delta=-\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}$ with domain $D(T_1)=W_2^2(\boldsymbol{R}^3)$, the Sobolev space of order 2. Let A, B be the closed extensions of the operators from $\mathfrak{H}$ to $\mathfrak{H}^8$, the direct sum of eight copies of $\mathfrak{H}$, given as

$$Af=\Big(a_0(x)f(x), a_1(x)f(x), a_2(x)\frac{\partial f}{\partial x_1}(x), \cdots, a_4(x)\frac{\partial f}{\partial x_3}(x), a_5(x)f(x), \cdots, a_7(x)f(x)\Big) \tag{5.1}$$

and

$$Bf=\Big(b_0(x)f(x), b_1(x)f(x), b_2(x)f(x)\cdots, b_4(x)f(x), b_5(x)\frac{\partial f}{\partial x_1}(x), \cdots, b_7(x)\frac{\partial f}{\partial x_3}(x)\Big) \tag{5.2}$$

for $f\in\mathfrak{D}(T_1)\subset\mathfrak{D}(A), \mathfrak{D}(B)$, where $|a_j(x)|, |b_j(x)|<c(1+|x|)^{-(1+\varepsilon)/2}$, $\varepsilon>0, j=1, \cdots, 7$ and $a_0(x), b_0(x)\in L^3(\boldsymbol{R}^3)$. Then we have the next lemma.

Lemma 5.1. *Under the above conditions on A and B, we have* (*A*-1), (*B*-2) *and* (*A*-3)′ (*hence* (*A*-3)) *in* § 2, *where* Δ *in* (*B*-2) *is taken as the complement of some closed null set.*

The proof of this lemma may be found in [5, § 7] and [1, Appendix A].

In the case that n is general and T_1 is a general elliptic operator with constant coefficients, we can treat the perturbation by a lower order differential operator whose coefficients are bounded and behave like $o(|x|^{-1-\varepsilon})$ at infinity (see [1], [6], [7] and also [12]). We may also apply the recent results of Schechter [12] for real potentials, where he relaxed among others the conditions on the local singularity of the potentials.

Finally we remark that using the results of Ikebe-Saito [3], or Saito [11], we may choose T_1 to be $-\Delta+q(x)$ in $L^2(\boldsymbol{R}^n)$ with $q(x)$ a real long-range potential and A^*B to be a complex short-range potential.

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