# Some Counterexamples in the Theory of Embedding Manifolds in Codimension Two

By Yukio MATSUMOTO*

Department of Mathematics, College of General Education, University of Tokyo, Komaba, Meguro-ku, Tokyo 153

(Received June 27, 1975)

## § 1. Introduction.

A famous theorem due to Browder, Casson, Sullivan, Haefliger and Wall (referred to as BCSHW) states that a homotopy equivalence $f: M \to W$ of a closed $m$-manifold $M$ to a compact $(m+q)$-manifold $W$ is homotopic to a piecewise linear (PL) embedding provided that codimension $q$ is greater than or equal to three [14, § 11.3.4]. This theorem naturally leads to a question asking whether the corresponding result holds in the case the codimension $q=2$. It is proven by Cappell and Shaneson that for any closed even dimensional (say, $m$-dimensional) manifold $M$ with *finite* fundamental group of certain type, there are infinitely many $(m+2)$-manifolds $W$, simple homotopy equivalent to $M$, such that any simple homotopy equivalence $f: M \to W$ cannot be homotopic to a PL embedding [3]. Consequently, the BCSHW theorem fails in codimension two.

In this paper, we shall give a rather simple example which shows this failure.

THEOREM 1. *Let $m$ be a positive integer with $m \equiv 2$ (mod 4). Then there exists a compact orientable PL $(m+2)$-manifold $W^{m+2}$ with the following properties: $W^{m+2}$ has the homotopy type of a product of spheres $S^1 \times S^{m-1}$, but no homotopy equivalence $f: S^1 \times S^{m-1} \to W^{m+2}$ is homotopic to a PL embedding.*

Here a PL embedding means a not necessarily locally flat one. It should be noted that our example has an *infinite* fundamental group contrasting with Cappell and Shaneson's examples[1].

Our method of construction also gives a remarkable example of a knotted torus.

THEOREM 2. *Let $m \equiv 2$ (mod 4) and suppose $m \geqq 6$. There exists a locally flat PL embedding*

* The author is partially supported by the Fūjukai Foundation.

1) For $m \geqq 6$ the fundamental group of our example is an infinite cyclic group. The existence of such an example contradicts an announcement in [2, p. 578, line 10].

$$i: S^1\times S^{m-2}\to S^1\times S^m$$

*which induces an isomorphism of fundamental groups but cannot be extended to any PL embedding of* $S^1\times D^{m-1}$ *into* $S^1\times D^{m+1}$.

Note that, in knotted spheres, any PL embedding $S^n\to S^{n+2}$ can be extended to a PL embedding of disks $D^{n+1}\to D^{n+3}$ by conical extension.

## § 2. The obstruction to finding a locally flat spine.

In this section, we recall some results of condimension two surgery [7]. Proofs will be omitted. For a detailed account, refer to [7].

Suppose that a compact oriented PL manifold $W^{m+2}$ of dimension $m+2$ has the same simple homotopy type as an oriented, connected, finite Poncaré complex $X$ of formal dimension $m\geqq 5$. Then one of the most fundamental problems would be to find a closed PL $m$-submanifold $M^m$ of $W^{m+2}$ such that the inclusion mapping $M^m\to W^{m+2}$ is a simple homotopy equivalence. If such a submanifold exists, we shall call it a *spine* of $W^{m+2}$. Although in this paper we are mainly concerned with PL embeddings which are not locally flat in general, we first formulate an obstruction theory to finding a *locally flat* spine.

Given an $m+2$-manifold $W^{m+2}$ satisfying the above condition, one can find a locally flat closed $m$-submanifold $L^m$ which is exterior $\left[\frac{m}{2}\right]$-connected[2] [5, Lemma 3.4]. Furthermore, the isomorphism class of the fundamental group $\pi_1(W-L)$ is independent of the choice of $L^m$. The inclusion $W-L\subsetneq W$ induces an onto homomorphism $\pi\to\pi'$ of fundamental groups whose kernel is generated by a central element $t$. The element $t$ is represented by the 'positive' $S^1$-fiber of the associated $S^1$-bundle with a $D^2$-bundle neibourhood $N$ of $L^m$. The $i$-th relative homotopy groups $\pi_i(W-L, N-L)$ has the structure of a (left) $\boldsymbol{Z}\pi$-module $\left(i\geqq\left[\frac{m}{2}\right]+1\right)$. Here note that the ring $\boldsymbol{Z}\pi$ (or $\boldsymbol{Z}\pi'$) is equipped with an involution '$-$' defined by $g\longmapsto g^{-1}$.

Now suppose that $m$ is even: $m=2n$. A key observation in [7] was that the $\boldsymbol{Z}\pi$-module $G=\pi_{n+1}(W-L, N-L)$ carries a $(-1)^n t$-hermitian form $\lambda$: $G\times G\to\boldsymbol{Z}\pi$ in the sense of Bourbaki, ALGEBRE, ch. 9, § 3, n° 1) and a 'quadratic form' $\mu$: $G\to\boldsymbol{Z}\pi/\{a-(-1)^n\bar{a}t;\ a\in\boldsymbol{Z}\pi\}$ which are related as $\lambda(x,x)=\mu(x)+(-1)^n\overline{\mu(x)}t$, $\mu(x+y)=\mu(x)+\mu(y)+\lambda(x,y)$. The form $(G,\lambda,\mu)$ is not necessarily non-singular, but it becoms non-singular over $\boldsymbol{Z}\pi'$ after being tensored with $\boldsymbol{Z}\pi'$. Thus we obtain a $(-1)^n t$-hermitian form which is defined over $\boldsymbol{Z}\pi$ and becomes non-singular over $\boldsymbol{Z}\pi'$. We call a form of this type[3] a $(-1)^n$-*Seifert form over* $\pi\to\pi'$. One can define the concept of *stably null-cobordant Seifert forms*, analogously to that of

2) A *taut* submanifold in the sense of Thomas and Wood [13].

3) The form of essentially the same type was independently discovered by M Freedman [4].

split hermitian inner product spaces [9] or kernels [14]. Thus we can obtain the 'Witt group' of $(-1)^n$-Seifert forms. It is an abelian group and is denoted by $P_{2n}(\pi \to \pi')$.

Returning to our geometrical situation, the 'Witt class' of the $(-1)^n$-Seifert from $(G, \lambda, \mu)$ associated with $L^{2n}$ does not depend on the choice of $L^{2n}$ and only depends on $W^{2n+2}$. Moreover, denoting the class by $\eta(W^{2n+2}) \in P_{2n}(\pi \to \pi')$, $W^{2n+2}$ admits a locally flat spine if and only if $\eta(W^{2n+2})=0$, provided that $2n \geqq 6$.

Our result in [7] is slightly more general. It concerns not only the absolute case but also the relative case and can be stated as follows:

THEOREM 2.1. [7, Theorem 5.10] *Let $(W^{2n+2}, K^{2n-1})$ be a pair consisting of a compact oriented PL $2n+2$-manifold $W$ and a locally flat, oriented, closed $(2n-1)$-submanifold $K^{2n-1}$ in the boundary $\partial W$. Suppose that the pair $(W^{2n+2}, K^{2n-1})$ has the same simple homotopy type as an oriented, connected, finite Poincaré pair $(X^{2n}, Y^{2n-1})$ of formal dimension $2n \geqq 6$. Then there is canonically defined a unique obstruction element $\eta(W, K)$ in the group $P_{2n}(\pi \to \pi')$ which vanishes if and only if $W$ admits a locally flat proper $2n$-submanifold $M^{2n}$ with $\partial M=K$ such that the inclusion map $M^{2n} \subsetneq W^{2n+2}$ is a simple homotopy equivalence. Here the homomorphism $\pi \to \pi'$ is associated with an exterior $n$-connected, locally flat $2n$-submanifold $L^{2n}$ with $\partial L=K$, and is defined to be $\pi_1(W-L) \to \pi_1(W)$.*

As in the absolute case, such $M^{2n}$ will be called a spine of $(W, K)$.

Next we shall state a realization theorem of the obstruction. Let $K^{2n-1}$ be an oriented, closed PL $(2n-1)$-manifold with $\pi_1(K) \cong \pi'$. Let $\pi \to \pi'$ be an onto homomorphism whose kernel is generated by a central element $t$. By a result of Wall [14, p. 125], such a homomorphism can be realized by an oriented $S^1$-bundle $E \to K$ with the total space $E$, so that the induced homomorphism $\pi_1(E) \to \pi_1(K)$ is identical with $\pi \to \pi'$. (The identification sends $t$ to the element of $\pi_1(E)$ represented by an oriented fiber.) Let $N$ be the total spece of the $D^2$-bundle associated with $E \to K$.

Now suppose that we are given a locally flat spine $K_1$ of $N$. Let $K_0$ be the zero cross-section of $N \to K$. Then the pair $(N \times [0,1], K_0 \times \{0\} \cup K_1 \times \{1\})$ is simple homotopy equivalent to $(K \times [0,1], K \times \{0\} \cup K \times \{1\})$, hence the obstruction $\eta(N \times [0,1], K_0 \times \{0\} \cup K_1 \times \{1\}) \in P_{2n}(\pi \to \pi')$ is well-defined by Theorem 2.1.

We have the following

THEOREM 2.2. *Suppose $2n \geqq 6$. Let $\eta_0$ be any prescribed element of $P_{2n}(\pi \to \pi')$. Then there exists a locally flat spine $K_1^{2n-1}$ of $N$ such that $\eta(N \times [0,1], K_0 \times \{0\} \cup K_1 \times \{1\})=\eta_0$.*

Theorem 2.2 follows from [7, Lemma 5.2] and the relative $s$-cobordism theorem.

## § 3. An example of a $(-1)$-Seifert from over $(\boldsymbol{Z} \to 1) \times \boldsymbol{Z}$

Let $(\boldsymbol{Z} \to 1) \times \boldsymbol{Z}$ denote the projection $\boldsymbol{Z} \times \boldsymbol{Z} \to \boldsymbol{Z}$ of the direct product of infinite cyclic groups to the second factor. These groups are considered to be multiplicative and let $t$ (or $s$) be the positive generator of the first (or the second) factor of $\boldsymbol{Z}\times\boldsymbol{Z}$.

In the category whose objects are onto homomorphisms of groups $\pi\to\pi'$, the kernels of which are generated by a preferred central element $t$, we define a morphism

$$(\pi_1\to\pi_1')\Longrightarrow(\pi_2\to\pi_2')$$

to be a pair $(h,h')$ of homomorphisms which makes the diagram commute:

$$\begin{array}{ccc}\pi_1 & \xrightarrow{\ h\ } & \pi_2\\ \downarrow & & \downarrow\\ \pi_1' & \xrightarrow{\ h'\ } & \pi_2'\end{array}$$

($h$ is assumed to preserve the preferred element $t$.)

Then there is the 'inclusion' morphism

$$(\boldsymbol{Z}\to 1)\Longrightarrow(\boldsymbol{Z}\to 1)\times\boldsymbol{Z}$$

defined by the pair of inclusions to the first factors: $\boldsymbol{Z}\subsetneq\boldsymbol{Z}\times\boldsymbol{Z}$, $1\subsetneq 1\times\boldsymbol{Z}$. By the functorial property of the $P$-groups [7, p. 301], we have a homomorphism

$$i_*:P_{2n}(\boldsymbol{Z}\to 1)\longrightarrow P_{2n}((\boldsymbol{Z}\to 1)\times\boldsymbol{Z})$$

induced by the 'inclusion' morpism.

The aim of this section is to show that if $m=4k+2$, the homomorphism $i_*$ is not surjective. For this, it will suffice to give a $(-1)$-Seifert form over $(\boldsymbol{Z}\to 1)\times\boldsymbol{Z}$ whose 'Witt class' in $P_{4k+2}((\boldsymbol{Z}\to 1)\times\boldsymbol{Z})$ is not contained in the image of $i_*$. Such an example is given as follows:

$$\begin{aligned}(G,\lambda,\mu):G=\Lambda x_1\oplus\Lambda x_2,&\\ \lambda(x_1,x_2)=-\overline{\lambda(x_2,x_1)}t=-s^{-1}&\\ \mu(x_1)=s-1,\qquad \mu(x_2)=-1,&\\ \Lambda=\boldsymbol{Z}[t,t^{-1},s,s^{-1}].&\end{aligned}$$

*Remark.* We found this form in a study of a 'spineless' 4-manifold [8]. Note that the matrix $(\lambda(x_i,x_j))$ of the form is given by

$$\begin{bmatrix}(s-1)-(s^{-1}-1)t, & -s^{-1}\\ st & , -1+t\end{bmatrix}$$

LEMMA 3.1. *The element $\eta_0$ of $P_{4k+2}((\boldsymbol{Z}\to 1)\times\boldsymbol{Z})$ represented by the above form is not in the image of $i_*$.*

*Proof.* We construct two left inverses $\rho_+$, $\rho_-$ of the homomorphism $i_*$ as follows: Let $(G, \lambda, \mu)$ be a $(-1)$-Seifert form over $(\boldsymbol{Z}\to 1)\times\boldsymbol{Z}$. Then by substitution $s=1$, it gives rise to a $(-1)$-Seifert form over $\boldsymbol{Z}\to 1$, and this defines the homomorphism

$$\rho_+ : P_{4k+2}((\boldsymbol{Z}\to 1)\times\boldsymbol{Z}) \longrightarrow P_{4k+2}(\boldsymbol{Z}\to 1).$$

The substitution $s=-1$ gives another homomorphism

$$\rho_- : P_{4k+2}((\boldsymbol{Z}\to 1)\times\boldsymbol{Z}) \longrightarrow P_{4k+2}(\boldsymbol{Z}\to 1).$$

Since both homomorphisms are left inverses of $i_*$, if the element $\eta_0$ were in the image of $i_*$, we would have $\rho_+(\eta_0)=\rho_-(\eta_0)$. However, this is not the case. To show this, we define the Murasugi signature [10]:

$$\sigma_M : P_{4k+2}(\boldsymbol{Z}\to 1) \longrightarrow \boldsymbol{Z}.$$

Let $(G, \lambda, \mu)$ be a $(-1)$-Seifert form over $\boldsymbol{Z}\to 1$. By substitution $t=-1$, the $(-1)t$-hermitian form $\lambda$ gives rise to a symmetric bilinear form over $\boldsymbol{Q}$ defined on $G\otimes_{\Gamma}\boldsymbol{Q}$, where $\Gamma=\boldsymbol{Z}[t, t^{-1}]$. $\sigma_M(G, \lambda, \mu)$ is defined to be the signature of this symmetric bilinear form. It is not difficult to see the above definition gives a well defined homomorphism $\sigma_M$.

Now let us compute $\sigma_M\rho_\pm(\eta_0)$ using the matrix which was given in *Remark* in page 52:

$$\sigma_M\rho_+(\eta_0)=\mathrm{sign}\begin{pmatrix} 0, & -1 \\ -1, & -2 \end{pmatrix}=0,$$

$$\sigma_M\rho_-(\eta_0)=\mathrm{sign}\begin{pmatrix} -4, & 1 \\ 1, & -2 \end{pmatrix}=-2.$$

Therefore, $\rho_+(\eta_0)\neq\rho_-(\eta_0)$. This completes the proof of 3.1.

*Remark.* Lemma 3.1 was announced in [8]. This lemma implies that Shaneson's splitting formula which was proven for Wall groups [12] does not hold in our $P$-groups.

Let $h:\pi\to\pi'$ be an onto homomorphism whose kernel is generated by a preferred central element $t$. Then a morphism $b:(\pi\to\pi')\Rightarrow(id:\pi'\to\pi')$ is defined by $b=(h, id)$. The morphism induces a homomorphism $b_*:P_{2n}(\pi\to\pi')\to P_{2n}(\pi'\to\pi')$. Since $P_{2n}(\pi'\to\pi')$ is identical with the Wall group $L_{2n}(\pi')$, we have obtained a homomorphism

$$b_*:P_{2n}(\pi\to\pi') \longrightarrow L_{2n}(\pi').$$

See [7, p. 309]. In particular, we have

$$b_* : P_{4k+2}((\boldsymbol{Z}\to 1)\times \boldsymbol{Z}) \longrightarrow L_{4k+2}(\boldsymbol{Z}).$$

LEMMA 3.2. *Let $\gamma_0$ be the element of Lemma* 3.1 *then* $b_*(\gamma_0)=0$.

*Proof.* Let $(G, \lambda, \mu)$ be the $(-1)$-Seifert form over $(\boldsymbol{Z}\to 1)\times \boldsymbol{Z}$ which was given earlier and represents $\gamma_0$. Then, since $(G, \lambda, \mu)\underset{\Lambda}{\otimes}\boldsymbol{Z}[s, s^{-1}]$ is obtained by substitution $t=1$, $b_*(\gamma_0)$ is represented by the $(-1)$-hermitian form given by

$$(G', \lambda', \mu') : G' = \Lambda' x_1 \oplus \Lambda' x_2,$$

$$\lambda'(x_1, x_2) = -\overline{\lambda'(x_2, x_1)} = -s^{-1},$$

$$\mu'(x_1) = s-1, \quad \mu'(x_2) = -1,$$

$$\Lambda' = \boldsymbol{Z}[s, s^{-1}].$$

From this, it follow that $(G', \lambda', \mu')$ is a kernel, for $x_1 - x_2$ generates a subkernel in the sense of Wall [14, Lemma 5.3]. This proves $b_*(\gamma_0)=0$.

## §4. Proof of Theorem 1.

In case $m=2$, Theorem 1 was proven in [8], So we may suppose that $m=4k+2\geqq 6$. Let $S^1\times S^{m-2}$ be a product of 1- and $(m-2)$-spheres. Since dim $(S^1\times S^{m-2})=m-1\geqq 5$, one can find, by Theorem 2.2, a locally flat spine $K_1^{m-1}$ of $S^1\times S^{m-2}\times D^2$ such that the obstruction $\gamma(S^1\times S^{m-2}\times D^2\times[0,1], S^1\times S^{m-2}\times\{0\}\times\{0\}\cup K_1\times\{1\})=\gamma_0$, where $\gamma_0$ is the element of $P_{4k+2}((\boldsymbol{Z}\to 1)\times \boldsymbol{Z})$ given in §3. We claim that $K_1^{m-1}$ is PL homeomorphic with $S^1\times S^{m-2}$. To prove this, construct a locally flat, oriented, proper $m$-submanifold $V^m$ of $S^1\times S^{m-2}\times D^2\times[0,1]$ such that $\partial V = S^1\times S^{m-2}\times\{0\}\times\{0\}\cup K_1^{m-1}\times\{1\}$. $V^m$ may be assumed to be exterior $\left[\frac{m}{2}\right]$-connected [5, Lemma 3.4]. Since the inclusion $V^m\to S^1\times S^{m-2}\times D^2\times[0,1]$ is $\left[\frac{m}{2}\right]$-connected [7, Lemma 1.3], we have

$$H^2(V^m : \boldsymbol{Z})\cong 0.$$

Therefore, the normal 2-disk bundle of $V^m$ in $S^1\times S^{m-2}\times D^2\times[0,1]$ is trivial, and we obtain a normal map in the sense of Browder [1]: $(V^m, \partial V^m)\to(S^1\times S^{m-2}\times[0,1], S^1\times S^{m-2}\times\{0\}\cup S^1\times S^{m-2}\times\{1\})$. By [7], the surgery obstruction for this normal map in the usual sense is $b_*(\gamma_0)\in L_{4k+2}(\boldsymbol{Z})$. However, Lemma 3.2 states that $b_*(\gamma_0)=0$. Thus we can perform surgery on $V^m$ to make it an $s$-cobordism. This proves our assertion that $K_1^{m-1}\cong S^1\times S^{m-2}$, [6].

Next, glue a copy of $S^1\times D^{m-1}\times D^2$ $(\cong S^1\times D^{m+1})$ to $S^1\times S^{m-2}\times D^2\times[0,1]$ by identifying each point $(\theta, p, \xi)$ of $S^1\times\partial D^{m-1}\times D^2(\subset S^1\times D^{m-1}\times D^2)$ with the point $(\theta, p, \xi)\times\{0\}$ of $S^1\times S^{m-2}\times D^2\times[0,1]$ to form a manifold $Y$ which is again PL-homeomorphic with $S^1\times D^{m+1}$. The new manifold $Y$ contains in its boundary the locally flat $(m-1)$-manifold $K_1^{m-1}\times\{1\}(\subset S^1\times S^{m-2}\times D^2\times\{1\}\subset\partial Y)$. The pair

$(Y, K_1^{m-1}\times\{1\})$ has the homotopy type of the pair $(S^1\times D^{m-1}, \partial(S^1\times D^{m-1}))$, and, by Theorem 2.1, the obstruction $\eta(Y, K_1^{m-1}\times\{1\})$ can be defined in $P_m((\boldsymbol{Z}\to 1)\times\boldsymbol{Z})$. By naturality of the obstruction [7, p. 305],

$$\begin{aligned}\eta&(Y, K_1^{m-1}\times\{1\})\\&=\eta(S^1\times S^{m-2}\times D^2\times[0,1],\ S^1\times S^{m-2}\times\{0\}\times\{0\}\cup K_1^{m-1}\times\{1\})\\&=\eta_0.\end{aligned}$$

Since $Y\cong S^1\times D^{m+1}$, $\partial Y\cong S^1\times S^m$ and $K_1^{m-1}\times\{1\}\cong S^1\times S^{m-2}$, we have the following lemma:

LEMMA 4.1. *Let* $m\equiv 2 \pmod 4$. *There is a locally flat embedding* $i: S^1\times S^{m-2}\to S^1\times S^m$ *which satisfies the following:*

(1) *$i$ induces an isomorphism of fundamental groups.*

(2) *Considering $i$ to be an embedding into the bounadry of* $S^1\times D^{m+1}$, *the pair* $(S^1\times D^{m+1}, i(S^1\times S^{m-2}))$ *is homotopy equivalent to* $(S^1\times D^{m-1}, \partial(S^1\times D^{m-1}))$.

(3) *The obstruction* $\eta(S^1\times D^{m+1}, i(S^1\times S^{m-2}))\in P_m((\boldsymbol{Z}\to 1)\times\boldsymbol{Z})$ *is equal to* $\eta_0$ *of* § 3.

We continue the proof of Theorem 1. Let $V^m$ be the locally flat submanifold of $S^1\times S^{m-2}\times D^2\times[0,1]$ constructed earlier in this section. By gluing $S^1\times D^{m-1}\times\{0\}(\subset S^1\times D^{m-1}\times D^2)$ to $V^m(\subset S^1\times S^{m-2}\times D^2\times[0,1])$ along $S^1\times\partial D^{m-1}\times\{0\}$, we obtain a locally flat, exterior $\left[\frac{m}{2}\right]$-connected submanifold $\tilde{V}$ of $Y$. The normal 2-disk bundle of $\tilde{V}$ in $Y$ is trivial. Let $\Phi:\tilde{V}\times D^2\to Y$ be a normal frame. Restricting $\Phi$ to the boundary, we have normal frame of $\partial\tilde{V}\subsetneq\partial Y$, that is, a normal framing $\phi: S^1\times S^{m-2}\times D^2\to S^1\times S^m$ of $i$, $i$ being the embedding of Lemma 4.1. Take another copy of the product $S^1\times D^{m-1}\times D^2$ and glue it to $Y(\cong S^1\times D^{m+1})$ by identifying each point $(\theta, p, \xi)$ of $S^1\times\partial D^{m-1}\times D^2(\subset S^1\times D^{m-1}\times D^2)$ with the point $\phi(\theta, p, \xi)$ of $S^1\times S^m$ $(\cong\partial Y)$. We have obtained a suitably oriented $(m+2)$-manifold $W^{m+2}$:

$$W^{m+2}=S^1\times D^{m-1}\times D^2\underset{\phi}{\cup} Y.$$

$W^{m+2}$ is homotopy equivalent to $S^1\times S^{m-1}$. Moreover, in $W^{m+2}$, there is a locally flat exterior $\left[\frac{m}{2}\right]$-connected, framed $m$-submanifold $L^m$ defined to be $S^1\times D^{m-1}\times\{0\}\underset{i}{\cup}\tilde{V}(\subset S^1\times D^{m-1}\times D^2\underset{\phi}{\cup} Y=W^{m+2})$. It is easy to see that the associated homomorphism with $L^m$, $\pi_1(W-L)\to\pi_1(W)$, is isomorphic with $(\boldsymbol{Z}\to 1)\times\boldsymbol{Z}$. By Theorem 2.1, the obstruction $\eta(W)$ to finding a locally flat spine is defined as an element of $P_m((\boldsymbol{Z}\to 1)\times\boldsymbol{Z})$, and by naturality of the obstruction,

$$\begin{aligned}\eta(W)&=\eta(Y, i(S^1\times S^{m-2}))\\&=\eta(Y, K_1^{m-1}\times\{1\})\\&=\eta_0\end{aligned}$$

It remains to prove that $W^{m+2}$ admits no PL spine. Suppose, on the contrary, that $W^{m+2}$ admits a PL spine $M^m$. Let $j: M^m \to W^{m+2}$ be the PL embedding which is a homotopy equivalence. We apply Noguchi's obstruction theory [11] to the map $j$. Let $C_k$ denote the knot cobordism group of locally flat PL $k$-knots in $k+2$-spheres. Then his theory tells us that if $H_i(M^m; C_{m-i-1}) \cong 0$ for any $i$ such that $m-2 \geqq i \geqq p$, then any PL embedding $f: M^m \to W^{m+2}$ is homotopic to a $p$-flat embedding, see [11, p. 204]. In our situation, $M^m$ is homotopy equivalent to $S^1 \times S^{m-1}$ with $m \equiv 2 \pmod 4$. Since $C_k \cong 0$ for any even integer $k \geqq 0$, we have

$$H_i(M^m; C_{m-i-1}) \cong 0$$

for any $i$ such that $m-2 \geqq i \geqq 1$. Therefore, $j: M^m \to W^{m-2}$ is homotopic to a 1-flat PL embedding $j'$. Then by the definition of 1-flatness, the PL spine $j'(M^m)$ may be assumed to be lacally flat except at one point, at which the pair $(W^{m+2}, j'(M^m))$ is locally a cone over a knotted $(m-1)$-sphere in a $(m+1)$-sphere. By naturality of $\gamma$-obstruction and the fact that the knot cobordism group $C_{m-1}$ is isomorphic to $P_m(\boldsymbol{Z} \to 1)$ ([7, Proposition 6.2]), the above observation implies that $\gamma(W^{m+2})$ $(=\gamma_0)$ is contained in the image of

$$i_*: P_m(\boldsymbol{Z} \to 1) \to P_m((\boldsymbol{Z} \to 1) \times \boldsymbol{Z}).$$

This contradicts Lemma 3.1. We have completed the proof of Theorem 1.

By the same method, one can prove a more general result:

THEOREM 4.2. *Let $m$ be a positive integer with $m \equiv 2 \pmod 4$. Let $M^m$ be a closed $m$-manifold with $H_*(M^m; \boldsymbol{Z}) \cong H_*(S^1 \times S^{m-1}; \boldsymbol{Z})$. Then there exists a compact orientable $(m+2)$-manifold $W^{m+2}$ which is simple homotopy equivalent to $M^m$ but admits no PL spine.*

## § 5. Proof of Theorem 2.

Let $i: S^1 \times S^{m-2} \to S^1 \times S^m$ be the locally flat PL embedding of Lemma 4.1. We shall show that $i$ cannot be extended to any PL embedding $S^1 \times D^{m-1} \to S^1 \times D^{m+1}$. Suppose, on the contrary, that $i$ is extended to a PL embedding $\phi: S^1 \times D^{m-1} \to S^1 \times D^{m+1} = Y$. Then a submanifold $M^m$ of $W^{m+2}$ defined by

$$\begin{aligned} M^m &= S^1 \times D^{m-1} \times \{0\} \cup_i \phi(S^1 \times D^{m-1}) \\ &\subset S^1 \times D^{m-1} \times D^2 \cup_\phi Y = W^{m+2} \end{aligned}$$

would be a PL spine of $W^{m+2}$. This contradicts what is proven in § 4. This completes the proof.

## References

[1] W. Browder, *Surgery on simply-connected manifolds*, Ergebnisse der Mathematik, Band 65, Springer-Verlag, 1972.

[2] S. E. Cappell and J. L. Shaneson, Nonlocally flat embeddings, smoothings, and group actions, *Bull. Amer. Math. Soc.*, **79** (1973), 577–582.

[3] S. E. Cappell and J. L. Shaneson, Totally spineless manifolds, *Topology* (to appear).

[4] M. Freedman, Surgery on codimension 2 submanifolds, (to appear).

[5] M. Kato and Y. Matsumoto, Simply connected surgery of submanifolds in codimension two I, *J. Math. Soc. Japan*, **24** (1972), 586–608.

[6] M. Kervaire, Le théorème de Barden-Mazur-Stallings, *Comment. Math. Helv.* **40** (1965), 31–42.

[7] Y. Matsumoto, Knot cobordism groups and surgery in condimension two, *J. Fac. Sci. Univ. Tokyo, Sect. IA*, **20** (1973), 253–317.

[8] Y. Matsumoto, A 4-manifold which admits no spine, *Bull. Amer. Math. Soc.*, **81** (1975), 467–470.

[9] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Ergebnisse der Mathematik, Band 73, Springer-Verlag, 1973.

[10] K. Murasugi, On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, **117** (1965), 387–422.

[11] H. Noguchi, Obstruction to locally flat embeddings of combinatorial manifolds, *Topology*, **5** (1966), 203–213.

[12] J. L. Shaneson, Wall's surgery obstruction groups for $Z \times G$, *Ann. of Math.*, **90** (1969), 296–334.

[13] E. Thomas and J. Wood, On manifolds representing homology classes in codimension 2, *Inventiones Math.*, **25** (1974), 63–89.

[14] C.T.C. Wall, *Surgery on compact manifolds*, Academic Press, 1970.