

## Some Remarks on Unit-Boundaries for a Banach Algebra

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### § 1. Introduction.

A complex function algebra (or a uniform algebra) is an algebra of continuous complex functions on a compact Hausdorff space  $X$ , which is uniformly closed and contains constant functions and separates points of  $X$ .

Endowed with supremum norm

$$\|f\| = \sup_{x \in X} |f(x)|$$

the function algebra  $A$  becomes a commutative Banach algebra with identity, or  $B$ -algebra. Identifying each  $x$  in  $X$  with the evaluation mapping at  $x$ , we shall regard  $X$  as a closed subset of the maximal ideal space  $M(A)$  of  $A$ . The Silov boundary  $\Gamma(A)$  then becomes a closed set of  $X$ .

The author has defined in [1] the concept of unit-boundary for a function algebra, but now it is possible to define it for a  $B$ -algebra (Definition in § 2). In doing this, it seems to be able to extend almost all the known results about unit-boundaries to the case of a  $B$ -algebra. Our main result is to determine a unit-boundary for a  $B$ -algebra by a dense subset of it (Theorem 1).

In § 2, we shall prove the existence of a minimal unit-boundaries for a  $B$ -algebra (Proposition 1) and show an example which has no minimum one (Example 1).

In § 3, we shall prove Theorem 1.

In § 4, we shall show that the concept of unit-boundary has a connection with the concept of rationally convex hull, when a  $B$ -algebra is finitely generated. Applying this result we shall give a simple proof for a eminent result obtained as an application of Oka-Weil approximation theorem.

In § 5, we shall give a representation of some unit-boundaries for the polydisc algebra  $A(\mathbb{D}^n)$  by continuous functions from the unit disc  $\mathbb{D}$  to the unit polydisc  $\mathbb{D}^n$  (Theorem 2), another expression of the result obtained by W. Rudin [7]. And we shall show by this theorem that there is a function algebra which is analytic on  $M(A)$ , but not on some minimal unit-boundary for  $A$  (Example 4).

## § 2. Unit-boundaries for a B-algebra.

Let  $\hat{x}$  be the Gelfand transform of an element  $x$  in a B-algebra  $A$ ,  $\hat{x}(S)$  the set of values of the continuous function  $\hat{x}$  on a subset  $S$  in  $M(A)$ ,  $\sigma_A(x)$  the spectrum of  $x$  in  $A$ ,  $Z(\hat{x})$  the set  $\{m \in M(A) : \hat{x}(m) = 0\}$  and  $S^i$  the interior of a set  $S$  in a topological space.

DEFINITION. A closed subset  $F$  of  $M(A)$  is called a unit-boundary for a B-algebra  $A$  iff  $F$  satisfies the following condition; each element  $x$  in  $A$  whose Gelfand transform  $\hat{x}$  does not attain the value zero on  $F$  is regular.

Remark 1.  $M(A)$  is a unit-boundary for  $A$ .

In fact, for an element  $x$  in  $A$  to be regular, it is necessary and sufficient that  $\sigma_A(x) (= \hat{x}(M(A)))$  should not contain the value zero.

Remark 2. The definition for a closed set  $F$  of  $M(A)$  to be a unit-boundary can be stated as follows;

$F$  is a closed subset with  $\hat{x}(F) = \sigma_A(x)$  for each  $x$  in  $A$ .

Remark 3. Every unit-boundary contains the Silov boundary.

In fact,  $I(A)$  is the minimum boundary for  $A$  and every unit-boundary is a boundary for  $A$ . (A boundary for  $A$  is a closed set on which every element  $\hat{x}$  attains its maximum).

By Remark 1 and Remark 3, it is evident that a B-algebra whose Silov boundary coincides with  $M(A)$  has no unit-boundary except  $M(A)$ , for example a log-modular algebra on  $M(A)$  or a maximal subalgebra of  $C(M(A))$ .

PROPOSITION 1. Let  $\mathfrak{F}$  be the set of all unit-boundaries for a B-algebra  $A$ . Then  $\mathfrak{F}$  is not empty and becomes a inductively ordered set for set inclusion, i.e. if  $\{F_\alpha\}_{\alpha \in \mathfrak{A}}$  is a totally ordered set, then the set  $\bigcap_{\alpha \in \mathfrak{A}} F_\alpha$  also belongs to  $\mathfrak{F}$ .

Proof. By Remark 1,  $M(A)$  is always in  $\mathfrak{F}$ . We assume that a function  $\hat{x}$  does not attain zero on  $\bigcap_{\alpha \in \mathfrak{A}} F_\alpha$ , i.e.  $Z(\hat{x}) \cap (\bigcap_{\alpha \in \mathfrak{A}} F_\alpha) = \emptyset$ . If  $Z(\hat{x}) \cap F_\alpha \neq \emptyset$  for every  $\alpha$ , then by the finite intersection property for compact sets,  $Z(\hat{x}) \cap (\bigcap_{\alpha \in \mathfrak{A}} F_\alpha) \neq \emptyset$ . This contradicts the assumption. Hence for some  $\alpha$ ,  $Z(\hat{x}) \cap F_\alpha = \emptyset$ , i.e.  $\hat{x}$  is regular.

Example 1. Let

$$X = \{(z, w) : |z| \leq 1, |w| \leq 1\},$$

$$X_\delta = \{(z, w) : ||z| - |w|| \leq \delta\} \cap X,$$

$$Y_\delta = \{(z, w) : 1 - \delta \leq |z| \leq 1 \text{ or } 1 - \delta \leq |w| \leq 1\} \cap X \text{ for } \delta > 0,$$

$$A = \{f \in C(X) : f \text{ is holomorphic in } X^i\},$$

Then  $A$  is a function algebra, called the polydisc algebra  $A(J^2)$ , the space  $X$  is

the maximal ideal space and the set  $T^2 = \{(z, w): |z|=1, |w|=1\}$  is the Silov boundary for  $A$ . If  $f$  in  $A$  does not attain the value 0 on  $X_\delta$  (or  $Y_\delta$ ), then  $1/f$  is also in  $A$ , i.e.  $X_\delta$  (or  $Y_\delta$ ) is a unit-boundary.

Especially the sets  $X_0 = \{(z, w): |z|=|w|\} \cap X$  and  $Y_0 = \{(z, w): |z|=1 \text{ or } |w|=1\} \cap X$  are unit-boundaries whose intersection is  $T^2$  which is not a unit-boundary for  $A$ .

This example shows that there is a  $B$ -algebra which has many unit-boundaries and can not have the minimum one.

We shall see from Theorem 2 in §5 that the polydisc algebra  $A(D^2)$  has many minimal unit-boundaries represented by some continuous functions from  $D$  to  $D^2$ .

*Example 2.* Let

$$\begin{aligned} X &= \{(z, w): |z|^2 + |w|^2 \leq 1\}, \\ X_\delta &= \{(z, w): 1 - \delta \leq |z|^2 + |w|^2 \leq 1\} \text{ for } 1 > \delta > 0, \\ A &= \{f \in C(X): f \text{ is holomorphic in } X^i\}. \end{aligned}$$

Then  $A$  is a function algebra, the space  $X$  the maximal ideal space and the set  $X_0 = \{(z, w): |z|^2 + |w|^2 = 1\}$  the Silov boundary for  $A$ . As in Example 1,  $X_\delta$  is a unit-boundary for  $A$  for every  $1 > \delta > 0$ , so that the set  $\bigcap_{1 > \delta > 0} X_\delta$  becomes a unit-boundary, i.e. the Silov boundary  $X_0$  is a unit-boundary.

This example shows that there is a  $B$ -algebra with the minimum unit-boundary identical with the Silov boundary. In §5, we shall show that there is a function algebra (Example 5) with the minimum unit-boundary which is neither  $I(A)$  nor  $M(A)$ .

*Example 3.* Let

$$\begin{aligned} X &= \{z: |z| \leq 1\}, \\ A &= \{f \in C(X): f \text{ is holomorphic in } X^i\}. \end{aligned}$$

This function algebra is called disc algebra. The space  $X$  is  $M(A)$  and  $T = \{z: |z|=1\}$  is  $I(A)$ . It is evident that  $A$  has the only one unit-boundary,  $M(A)$ , which is different from  $I(A)$ .

In §4, we shall recognize that any  $B$ -algebra with one generator can not have any unit-boundary except  $M(A)$ .

### §3. A condition for a closed set to be a unit-boundary

**THEOREM 1.** *Let  $A$  be a  $B$ -algebra and  $L$  a dense subset of  $A$ . If a closed set  $F$  in  $M(A)$  satisfies the condition that every element  $y$  in  $L$  with  $\hat{y}(F) \neq 0$  is regular in  $A$ , then  $F$  is a unit-boundary for  $A$ .*

*Remark 4.* It is well known that a boundary for a dense subset of  $A$  is also a boundary for  $A$ .

**DEFINITION.** Let  $B$  be a commutative normed algebra with identity. An

element  $z$  in  $B$  is called a topological divisor of zero iff there exists a sequence  $\{z_n\}$  in  $B$  such that  $\|z_n\|=1$  for all  $n$  and  $\|z \cdot z_n\| \rightarrow 0$  ( $n \rightarrow \infty$ ).

The set of all topological divisors of zero in  $B$  will be denoted by  $Z_{||\cdot||}(B)$  (or simply  $Z_{||\cdot||}$ ). A topological divisor of zero is automatically singular.

LEMMA 1. *Let  $x$  be singular in a  $B$ -algebra. If  $x$  is a limiting point of a sequence  $\{a_n\}$  of regular elements, then  $x$  is a topological divisor of zero.*

(This lemma is a part of Theorem 2.6 in C.E. Rickart [2]).

LEMMA 2. *Let  $X$  be a compact Hausdorff space and  $B$  a subalgebra of  $C(X)$  which contains the constants and separates points of  $X$ . An element  $x$  is a topological divisor of zero in  $B$  iff it vanishes at some point of the Silov boundary for  $B$ .*

*Proof.* Let  $\bar{B}$  be the closure of  $B$  in  $C(X)$ . Then  $\bar{B}$  is a function algebra on  $X$ . Noticing that  $I(B)=I(\bar{B})$  and  $Z_{||\cdot||}(B)=Z_{||\cdot||}(\bar{B}) \cap B$ , we have only to prove for  $\bar{B}$ . It has already been proved for a function algebra, see pp. 74-75 in §11 of I. Gelfand, D. Raikov and G. Silov [3].

Regarding a  $B$ -algebra  $A$  as a normed algebra with the spectral norm  $\|\cdot\|_\rho$ , instead of the original one, we shall denote the set of all topological divisors of zero in  $A$  by  $Z_\rho(A)$  (or simply  $Z_\rho$ ), distinguished from  $Z_{||\cdot||}$ . In general the two sets  $Z_{||\cdot||}$ ,  $Z_\rho$  are different. But we shall show that a singular element which is a limiting point of a sequence of regular elements in the original norm is contained in the intersection of  $Z_{||\cdot||}$  and  $Z_\rho$ .

LEMMA 3. *Let  $x$  be an element of  $Z_{||\cdot||}$  of a  $B$ -algebra  $A$ . If there is a sequence  $\{a_n\}$  of regular elements such that  $\|a_n - x\| \rightarrow 0$  ( $n \rightarrow \infty$ ), then  $x$  is also an element of  $Z_\rho$ .*

*Proof.* As  $x$  is in  $Z_{||\cdot||}$ , it is singular. We shall show that the sequence  $\{a_n^{-1}\}$  is unbounded in the spectral norm  $\|\cdot\|_\rho$ . Let us assume that the sequence is bounded, i.e. there exists a positive number  $M$  such that for each  $n$ .

$$\|a_n^{-1}\|_\rho = \max_{m \in M(A)} |\widehat{(a_n^{-1})}(m)| < M.$$

Then we obtain the following estimate;

$$\|1 - (a_n^{-1}) \cdot x\|_\rho \leq \|a_n^{-1}\|_\rho \cdot \|a_n - x\|_\rho < M \cdot \|a_n - x\| \rightarrow 0 \quad (n \rightarrow \infty)$$

Therefore there is a number  $n_0$  such that  $\|1 - (a_{n_0}^{-1}) \cdot x\|_\rho < 1$ . Namely  $x$  is regular in  $A$ . This contradicts the singularity of  $x$ .

Let  $b_n$  be  $(\|a_n^{-1}\|_\rho)^{-1} \cdot a_n^{-1}$ . Then  $\|b_n\|_\rho = 1$ ,  $\|b_n \cdot a_n\|_\rho \rightarrow 0$  ( $n \rightarrow \infty$ )

and

$$\|b_n \cdot x\|_\rho = \|b_n \cdot a_n - b_n(a_n - x)\|_\rho \rightarrow 0 \quad (n \rightarrow \infty).$$

Hence  $x$  is a topological divisor of zero with respect to the norm  $|||_{\rho}$ .

*Proof of Theorem 1.* Let  $x$  be an element of  $A$  with  $\hat{x}(F) \neq 0$ . As  $\hat{x}$  is continuous on  $F$ , there is a positive number  $\delta$  such that  $\min_{m \in F} |\hat{x}(m)| = \delta$ .

By the density of  $L$  in  $A$ , we can choose  $y_n$  in  $L$  such that  $||y_n - x|| < 1/n$ . Let  $n_0$  be greater than  $1/\delta$ . As, for each  $n > n_0$ ,

$$|\hat{y}_n(m)| > \min |\hat{x}(m)| - 1/n > 0$$

$y_n$  is regular in  $A$  by the assumption. Therefore  $x$  is a limiting point of the sequence  $\{y_n\}$  of regular elements. If  $x$  is singular, then  $x$  is in  $Z_{|||}(A)$  by Lemma 1 and applying Lemma 3,  $x$  is also in  $Z_{\rho}(A)$ . Hence, by Lemma 2, the continuous function  $\hat{x}$  attains the value 0 on  $\Gamma(A)$ . On the other hand, the closed set  $F$  is a boundary for  $L$ . By Remark 4,  $F$  is a boundary for  $A$  and consequently contains  $\Gamma(A)$ . Thus  $\hat{x}$  attains the value 0 on  $F$ . This contradicts the assumption. Hence  $x$  is regular in  $A$ , i.e.  $F$  is a unit-boundary for  $A$ .

#### § 4. Unit-boundaries and Rationally convex hulls

When a  $B$ -algebra is finitely generated, the concept of unit-boundary is closely related to that of rationally convex hull in  $C^n$ .

We can associate with each compact set  $X$  in  $C^n$ , its polynomially convex hull, denoted by  $\text{hull}(X)$ , and its rationally convex hull,  $\text{R-hull}(X)$ .

DEFINITION.  $\text{hull}(X)$  for a compact set  $X$  in  $C^n$  is the set of all  $p$  in  $C^n$  satisfying the relation

$$|f(p)| \leq \max_{x \in X} |f(x)|$$

for every polynomial  $f(z_1, z_2, \dots, z_n)$ . When  $X = \text{hull}(X)$ , we say that  $X$  is polynomially convex in  $C^n$ .

$\text{R-hull}(X)$  is the set of all  $p$  in  $C^n$  satisfying the relation

$$|g(p)| \leq \max_{x \in X} |g(x)|$$

for every rational functional  $g$  which is holomorphic about  $X$ . If  $X = \text{R-hull}(X)$ , we say that  $X$  is rationally convex.

These hulls are compact and satisfy

$$(*) \quad X \subset \text{R-hull}(X) \subset \text{hull}(X)$$

Remark 5. Let  $X$  be a compact set in  $C^n$ . Then  $\text{R-hull}(X)$  is the set of all  $p$  in  $C^n$  such that  $f(p) \in f(X)$  for all polynomial function  $f$ .

(This remark is the result (1.1) in G. Stolzenberg [4]).

Let  $A$  be a  $B$ -algebra with finite generators  $x_1, \dots, x_n$ . Then under the na-

tural projection  $\pi$  of  $M(A)$  into  $C^n$  defined by

$$\pi(m) = (\hat{x}_1(m), \dots, \hat{x}_n(m))$$

$M(A)$  is homeomorphic to a polynomially convex set in  $C^n$ , called the joint spectrum  $\sigma(x_1, \dots, x_n) (= \pi(M(A)))$ .

By the relation (\*), we obtain for each closed set  $F$  in  $M(A)$  the following inclusions;

$$\pi(F) \subset R\text{-hull}(\pi(F)) \subset \text{hull}(\pi(F)) \subset \pi(M(A))$$

*Remark 6.* The  $\text{hull}(\pi(F))$  coincides with  $\pi(M(A))$  for a closed set  $F$  in  $M(A)$  iff  $F$  contains the Silov boundary.

On the other hand, as for the rationally convex hull, even if  $F$  contains the Silov boundary,  $R\text{-hull}(\pi(F))$  is not always  $\pi(M(A))$ . By Theorem 1 and Remark 5, we notice here that, if  $A$  is finitely generated, to find unit-boundaries for  $A$  is reduced to the problem to find in a polynomially convex set  $X$  closed sets whose rationally convex hulls are identical with  $X$ , see N. Fujita [5].

For  $C^1$ , every compact set  $X$  is rationally convex and  $\text{hull}(X)$  is obtained by adjoining to  $X$  all the bounded components of its complement and this is the reason why a  $B$ -algebra with one generator can not have any unit-boundary except  $M(A)$ . But for  $C^n$  ( $n > 1$ ), polynomial convexity and rational convexity are no longer topological properties.

Let  $X$  be a compact set in  $C^n$ ,  $P(X)$  the completion of the algebra of all polynomials in  $n$  variables in the maximum norm over  $X$  and  $R(X)$  the completion of the algebra of all rational functions, analytic about  $X$ . In T. W. Gamelin [6], it is shown that Oka-weil's theorem implies the following proposition. Using the concept of unit-boundary, we shall give its simple proof.

PROPOSITION 2.

$$P(X) = R(X) \text{ iff } \text{hull}(X) = R\text{-hull}(X).$$

*Proof.* If  $X$  satisfies  $\text{hull}(X) = R\text{-hull}(X)$ , then  $X$  is a unit-boundary for  $P(X)$  from Remark 5. Let  $Q$  be the family of all rational functions of a form  $1/p$ , where  $p$  is a polynomial not vanishing on  $X$ . Then  $z_1, \dots, z_n$  and  $Q$  generate  $R(X)$ . As  $X$  is a unit-boundary for  $P(X)$ , every element in  $Q$  belongs to  $P(X)$ . Hence  $P(X) = R(X)$ .

Conversely, if these two function algebras are equal, then their maximal ideal spaces are also equal, i.e.  $\text{hull}(X) = R\text{-hull}(X)$ .

For the case that  $A$  is not finitely generated, we shall introduce a new hull in  $M(A)$  to distinguish whether  $A$  has a unit-boundary except  $M(A)$  or not.

DEFINITION. Let  $A$  be a  $B$ -algebra. For a closed subset  $S$  in  $M(A)$ ,  $A_R\text{-hull}(S)$  is the set of all  $m$  in  $M(A)$  such that  $\hat{x}(m) \in \hat{x}(S)$  for every  $x$  in  $A$ .

PROPOSITION 3. Let  $A$  be a  $B$ -algebra. If  $\Gamma(A) \subsetneq M(A)$  and the interior of

the set  $A_R\text{-hull}(\Gamma(A)/\Gamma(A))$  is not empty, then there is a unit-boundary except  $M(A)$ .

*Proof.* Let  $K$  be  $A_R\text{-hull}(\Gamma(A)/\Gamma(A))$  and  $F$  be  $M(A)/K^\perp$ . Then  $F$  is a proper closed set of  $M(A)$ . Let  $\hat{x}$  be a function not vanishing on  $F$ . As  $\Gamma(A) \cap K^\perp = \emptyset$ ,  $F$  contains  $\Gamma(A)$ . Hence  $\hat{x}(\Gamma(A)) \neq 0$ . Consequently  $\hat{x}$  does not vanish on  $A_R\text{-hull}(\Gamma(A))$ , which shows  $\hat{x}(M(A)) \neq 0$ , i.e.  $x$  is regular. Thus  $F$  is a unit-boundary for  $A$ .

Actually there is a  $B$ -algebra satisfying the condition of Proposition 3 (Example 5 in § 5).

### § 5. A representation theorem of unit-boundaries for the polydisc algebra.

**DEFINITION** A loop in a topological space  $X$  is a continuous map  $\alpha: I = [0, 1] \rightarrow X$  with  $\alpha(0) = \alpha(1)$ .

**DEFINITION** Let  $C^*$  be the set of all non zero complex numbers, with the topology induced from  $C$ . If  $\gamma$  is a loop in  $C^*$ , there is a real continuous function  $\varphi$  on  $I$  such that

$$\gamma(s)/|\gamma(s)| = \exp\{2\pi i \cdot \varphi(s)\}$$

The integer  $\varphi(1) - \varphi(0)$  is the index (or winding number) of  $\gamma$ , denoted by  $\text{ind}(\gamma)$ .

Let  $E(s)$  be the function  $\exp\{2\pi i \cdot s\}$ ,  $0 \leq s \leq 1$ . Then  $E$  is a loop in  $T (= \{z: |z|=1\})$ .

**THEOREM 2.** Suppose  $\Phi = (\varphi_1, \dots, \varphi_n)$  is a continuous map of the closed unit disc  $\Delta$  into the closed polydisc  $\Delta^n$  which carries  $T$  into  $T^n$ , such that  $\text{ind}(\varphi_j \circ E) > 0$  for  $1 \leq j \leq n$ . Then  $\Phi(\Delta) \cup T^n$  is a unit-boundary for the polydisc algebra  $A(\Delta^n)$ .

(This theorem is obtained by rewriting in term of unit-boundary Theorem 4.7.2, pp. 87-90 in W. Rudin [7]).

We shall show two examples of function algebras  $A_1$  and  $A_2$ :  $A_1$  is analytic on  $M(A_1)$ , but not on some minimal unit-boundary for  $A_1$  and conversely  $A_2$  is analytic on some minimal unit-boundary for  $A_2$ , but not on  $M(A_2)$ .

*Example 4.* Let  $A_1$  be the polydisc algebra  $A(\Delta^2)$ . Then  $A_1$  is analytic on  $M(A_1)$ , i.e.  $\Delta^2$ . As the functions  $\varphi_j(\lambda) = \lambda$  ( $j=1, 2$ ) satisfy the condition of Theorem 2, the union of the diagonal  $K = \{(\lambda, \lambda): \lambda \in \Delta\}$  and  $T^2$  is a unit-boundary for  $A_1$ . The function  $f(z, w) = z - w$  is in  $A_1$ . The set  $K' = \{\lambda, \lambda\}: \lambda \in \Delta^i\}$  is open in  $K \cup T^2$  and the function  $f$  is constant on  $K'$ , but not on  $K \cap T^2$ . Hence  $A_1$  is not analytic on  $K \cup T^2$ .

*Example 5.* Let  $E$  be a totally disconnected compact perfect set in the complex plane, each point of which is a density point and  $F$  a Cantor set of measure zero on  $T$ . There is a homeomorphism  $\tau$  of  $E$  onto  $F$ . Let  $A_0(E)$  be the algebra of all continuous functions on the Riemann sphere  $S^2$  which are holomor-

phic on  $S^2/E$ . Then  $A_0(E)$  is a function algebra, its maximal ideal space  $S^2$  and  $E$  its Silov boundary. At the same time  $E$  is the minimum unit-boundary, i.e.  $f(E)=f(S^2)$  for every  $f$  in  $A_0(E)$ .

Let  $X$  be the compact set obtained from the disjoint union of  $S^2$  and  $\Delta$ . Identifying  $E$  with  $F$  via homomorphism  $\tau$ , let  $A_2$  be the algebra of all continuous function  $f$  on  $X$  such that the restriction  $f|S^2$  is in  $A_0(E)$  and  $f|\Delta$  is in  $A(\Delta)$ . Then  $A_2$  is a function algebra on  $X$ , maximal ideal space is  $X$  and the Silov boundary is  $T$ . Furthermore  $A_2$  has the minimum unit-boundary  $\Delta$  different from the Silov boundary and from the maximal ideal space.

Since  $A_2$  is analytic on  $\Delta$ , but not on  $M(A_2)$ , it is the required one. (The function algebra  $A_2$  was used in pp. 230-231 of G. Leibowitz [8] in another purpose).

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