On a Set Including the Spectra of Positive Operators

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In 1938 A. N. Kolmogorov raised the problem of determining the set of eigenvalues of all the stochastic matrices of order n. A partial answer to this problem was given in 1945 by N. A. Dmitriev and E. B. Dynkin [1], [2]. But it is F. I. Karpelevich [4] who solved the problem completely in 1951[1].

In connection with these results, we prove a theorem which is an extension of Corollary of Proposition 11 in I. Sawashima and F. Niiro [6].

THEOREM. *Let c and r be positive numbers and k be a natural number. Then there exists a compact set C in the complex plane such that*

i) $C\subset\{\alpha\colon |\alpha|\leqq 1\}$,

ii) $C\cap\Gamma$[2] *consists of a finite number of elements,*

iii) $\sigma(T)$[3] $\subset C$ *for any positive bounded linear operator T in any Banach lattice E which satisfies*

$$r(T)=1, \tag{1}$$

$$\{\alpha\colon 0<|\alpha-1|<r\}\subset\rho(T) \tag{2}$$

and

$$\sup_{0<|\alpha-1|<r}|\alpha-1|^{k}\|R(\alpha,T)\|\leqq c. \tag{3}$$

To prove the theorem we give two lemmas.

LEMMA 1. *Let E be a Banach lattice and T be a positive operator of $\mathfrak{L}(E)$*[4]

1) For these, see Gantmacher [3].

2) Γ is the unit circle $\{\alpha: |\alpha|=1\}$.

3) $\rho(T)$, $r(T)$, $\rho(T)$ and $R(\alpha, T)$ is the spectrum, the spectral radius, the resolvent set and the resolvent operator of T respectively.

4) $\mathfrak{L}(E)$ is the set of bounded linear operators in E.

with $r(T)=1$ *and* 1 *be a pole of* $R(\alpha, T)$. *If* r *is a positive number such that* $\{\alpha: 0<|\alpha-1|<r\}\subset\rho(T)$ *and* m *is the smallest natural number satisfying*

$$\left|1-\exp\left(\frac{2\pi i}{m+1}\right)\right|<r, \tag{4}$$

then

$$\sigma(T)\cap\Gamma\subset\bigcup_{n=1}^{m}\bigcup_{k=0}^{n-1}\exp\left(\frac{2k\pi i}{n}\right).$$

Proof. By Theorem 4.10 in H. P. Lotz [5] we see that, any element $\alpha_0\neq 1$ of $\sigma(T)\cap\Gamma$ has the form $\exp(2k\pi i/n)$ where $k\leqq n-1$, $n, k\in \boldsymbol{N}$ and k is relatively prime to n and also that $\exp(2\pi i/n)$ is in $\sigma(T)$. This implies $n\leqq m$ and the lemma is proved.

LEMMA 2. *Let* c *and* r *be positive numbers and* k *be a nonnegative number and* T *be a positive bounded linear operator in a Banach lattice such that*

$$r(T)\leqq 1,$$

$$\{\alpha: 0<|\alpha-1|<r\}\subset\rho(T)$$

and

$$\sup_{0<|\alpha-1|<r}|\alpha-1|^k\|R(\alpha, T)\|\leqq c.$$

Then for any positive number α *in the open interval* $(1, 1+r)$ *we have the inequality*

$$\|T^n\|\leqq\frac{\alpha^{n+1}c}{(\alpha-1)^k}\quad\text{for}\quad n\in\boldsymbol{N}. \tag{5}$$

Proof. Let $1<\alpha<1+r$. Since $r(T)\leqq 1$, we have the expansion

$$R(\alpha, T)=\sum_{n=0}^{\infty}\frac{T^n}{\alpha^{n+1}}.$$

Therefore for positive element f of E we have

$$\left\|\frac{T^n}{\alpha^{n+1}}f\right\|\leqq\|R(\alpha, T)f\|.$$

From the well known property of positive operator that

$$|Tf|\leqq T|f|\quad\text{for any } f\in E$$

and that

$$R(\alpha, T)\text{ is a positive operator for }\ \alpha>1,$$

we get the inequality in the lemma easily.

Proof of the theorem. Put

$$A=\bigcup_{n=1}^{m}\bigcup_{k=0}^{n-1}\exp\left(\frac{2k\pi i}{n}\right) \tag{6}$$

where m is the smallest natural number satisfying (4).

To prove the theorem it is sufficient to prove the assertion: *For any* $\alpha_0\in\Gamma\setminus A$[5] *there exists a positive number* $b(\alpha_0)$ *independent of both* T *and* E *such that*

$$\|R(\alpha_0, T)\|\leqq b(\alpha_0)$$

for any positive bounded linear operator T *in any Banach lattice* E *which satisfies* (1), (2) *and* (3) *in the theorem.* For, if the assertion is proved, then the set

$$C=\{\alpha: |\alpha|\leqq 1\}\setminus\bigcup_{\alpha_0\in\Gamma\setminus A}\left\{\alpha: |\alpha-\alpha_0|<\frac{1}{b(\alpha_0)}\right\}$$

satisfies i), ii) and iii) in the theorem.

We shall show in the remaining part of the proof that the assumption of the non-existence of such $b(\alpha_0)$ yields a contradiction. Assume that

$$\alpha_0\in\Gamma\setminus A \tag{7}$$

and that there exists a sequence of positive operators T_n in Banach lattices E_n which satisfy (1), (2), (3) and

$$\|R(\alpha_0, T_n)\|>n \qquad \text{for } n\in N. \tag{8}$$

Let $\hat{E}=\{\{f_n\}: f_n\in E_n, \sup_n\|f_n\|<\infty, n\in N\}$. With linear structure and order defined coordinatewise and with norm defined by $\|\{f_n\}\|=\sup_n\|f_n\|$, $\hat{E}$ is a Banach lattice. By (5) in Lemma 2 with $n=1$, we can define a bounded operator $\hat{T}$ in $\hat{E}$ by

$$\hat{T}\{f_n\}=\{T_nf_n\}.$$

It is clear that $\hat{T}$ is a positive operator of $\mathfrak{L}(\hat{E})$. By lemma 2 we get

$$\|\hat{T}^n\|\leqq\frac{\alpha^{n+1}c}{(\alpha-1)^k} \qquad \text{for } n\in N \text{ and } \alpha\in(1, 1+r)$$

which yields

$$r(\hat{T})\leqq 1.$$

Let

$$0<|\alpha-1|<r.$$

Then it is easy to show

$$\alpha\in\rho(\hat{T}),$$

$$R(\alpha, \hat{T})\{f_n\}=\{R(\alpha, T_n)f_n\}$$

5) Lemma 1 implies that $\alpha_0\in\rho(T)$ for such T.

and also

$$|\alpha-1|^k\|R(\alpha, \hat{T})\|\leqq c.$$

If

$$r(\hat{T})=1,$$

then $\hat{T}$ satisfies (1), (2) and (3) in the theorem. Therefore, by Lemma 1 and (7), we get

$$\alpha_0\in\rho(\hat{T}).$$

If

$$r(\hat{T})<1,$$

then clearly

$$\alpha_0\in\rho(\hat{T}).$$

Since E_n is isomorphic as as a Banach lattice to the subspace $\{\{f_l\}\in\hat{E}: f_l=0 \text{ for } l\neq n\}$ of $\hat{E}$, we shall embed E_n into $\hat{E}$. Then E_n is a $\hat{T}$-invariant subspace of $\hat{E}$ and the restriction of $\hat{T}$ on E_n is the operator T_n. Since α_0 is in $\rho(\hat{T})\cap\Gamma$ and $r(\hat{T})\leqq 1$, the restriction of $R(\alpha_0, \hat{T})$ on E_n is $R(\alpha_0, T_n)$ and $\|R(\alpha_0, T_n\|\leqq\|R(\alpha_0, \hat{T})\|$. This contradicts (8) and the theorem is proved.

The following corollary is clear from the proof of the theorem.

COROLLARY 1. *Let* c *and* r *be positive numbers and* k *be a nonnegative integer. If* A *is the set defined in* (6), *then for* $\alpha_0\in\Gamma\backslash A$ *there exists a positive number* $b(\alpha_0)$ *such that*

$$\|R(\alpha_0, T)\|\leqq b(\alpha_0)$$

for any positive bounded linear operator T *in any Banach lattice* E *which satisfies* $r(T)\leqq 1$, (2) *and* (3).

A direct consequence of Corollary 1 is the following.

COROLLARY 2. *Let* c *and* r *be positive numbers and* k *be a nonnegative integer. Then there exists a compact set* C *in the complex plane such that* C *satisfies* i) *and* ii) *in the theorem and* iii′), *where* iii′) *is the assertion that we obtain from* iii) *in the theorem by replacing condition* (1) *by* $r(T)\leqq 1$.

Remark. Condition (3) in the theorem is indispensable.

Counter-example: Let E_n be the two dimensional vector space with the usual order and T_n be the positive oprator in E_n defined by the matrix

$$T_n=\begin{pmatrix}1 & n\\ 0 & 0\end{pmatrix}.$$

Then $\{\alpha: 0<|\alpha-1|<1\}\subset\rho(T_n)$. However T_n is not norm bounded and the construction of $\hat{T}$ in the proof of the theorem fails.

In contrast to the counter-example, we have the following.

PROPOSITION. *Let r be a positive number and n a natural number. Then*

$$\sup_{T\in S_{n,r}} \sup_{0<|\alpha-1|<r/3} |\alpha-1|\,\|R(\alpha, T)\|<\infty,$$

where $S_{n,r}$ is the set of all stochastic matrices T[6] *of order n which satisfy*

$$\{\alpha: 0<|\alpha-1|<r\}\subset\rho(T).$$

Proof. By a well known theorem of algebra we see that the roots of the characteristic equation

$$\det(\lambda I-T)=0$$

are continuous on the compact set of all stochastic matrices of order n. Therefore $S_{n,r}$ is compact. Since $\|R(\alpha, T)\|$ is a continuous function of (α, T) on the compact set

$$B=\left\{(\alpha, T): |\alpha-1|=\frac{r}{2},\ T\in S_{n,r}\right\}, \tag{9}$$

we have

$$\sup_{(\alpha, T)\in B}\|R(\alpha, T)\|<\infty.$$

By Lemma 4 in I. Sawashima and F. Niiro [6], we see that 1 is a pole of $R(\alpha, T)$ of order 1. Consequently we have

$$R(\alpha, T)=\sum_{k=-1}^{\infty} A_k(\alpha-1)^k \quad \text{for} \quad 0<|\alpha-1|<r \text{ and } T\in S_{n,r},$$

where

$$A_k=\frac{1}{2\pi i}\int_{|\beta-1|=r/2}\frac{R(\beta, T)}{(\beta-1)^{k+1}}\,d\beta.$$

Therefore, for $0<|\alpha-1|<r/3$ and $T\in S_{n,r}$, we have

$$(\alpha-1)R(\alpha, T)=\frac{1}{2\pi i}\int_{|\beta-1|=r/2}\sum_{k=-1}^{\infty}\frac{(\alpha-1)^{k+1}}{(\beta-1)^{k+1}}R(\beta, T)\,d\beta$$

since the series in the right hand side converges uniformly by (9). Then, we get

$$|\alpha-1|\,\|R(\alpha, T)\|\leqq\frac{r}{2}\sum_{k=-1}^{\infty}\left(\frac{2}{3}\right)^{k+1}\sup_{(\alpha, T)\in B}\|R(\alpha, T)\|$$
$$=\frac{3}{2}r\sup_{(\alpha, T)\in B}\|R(\alpha, T)\|.$$

Thus the proposition is proved.

6) We regard that T are positive bounded linear linear operators in the space ot n-dimensional vector lattice with supremum norm.

Taking into account of the problem of A. N. Kolmogorov and the theorem in our paper, we are interested in the following.

PROBLEM I) *Is it possible to express the set C in the theorem explicitly by c, r and k?*

Since the set $C\cap\Gamma$ in the theorem is included in the set A of (6) determined only by r, the following problem seems reasonable.

PROBLEM II) *Under what conditions which E and T satisfy in addition to* (1), (2) *and* (3) *in* iii), *is the set C in the theorem detemined only by r and k?*

If we alter Problem II) slightly, then we have the following.

PROBLEM III) *We consider instead of* (3) *in the theorem the following condition:*

$$1 \text{ is a pole of } R(\alpha, T) \text{ of order } k. \tag{3'}$$

Under what conditions which E and T satisfy in addition to (1), (2) *and* (3'), *does there exist positive numbers c and r_1 determined only by r and such that*

$$\sup_{0<|\alpha-1|<r_1} |\alpha-1|^k\|R(\alpha, T)\| \leqq c$$

for any positive bounded linear operator T in any Banach lattice E?

In the case where T is any Markov[7] operator in the usual n-dimensional vector space, Problems I) and II) are trivial. Indeed according to F. I. Karpelevich [4] C is independent of c, r and k. Our proposition provides an affirmative answer to Problem III) in this case. However, as is shown easily from the counter-example in p. 96, Problem III) is solved negatively without any further assumption for T even in the case where E is the usual two dimensional vector space. We don't know if Problem III) is solved in the case where T is any Markov operator in the space $C(X)$ or in the case where T is any Markov operator in any of the usual finite dimensional vector spaces.

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7) A positive linear operator in the finite dimensional vector space is Markov if and only if it is defined by a stochastic matrix.