Notes on K_nQ and WQ

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0. In the first section of this paper we shall compute K_nQ ($n \geq 2$) of the field Q of rational numbers, by following, *verbatim*, techniques in [1], §2 (cf. **1.1**, for definition of K_nQ).

Section 2 will compute the Witt group WQ of the anisotropic quadratic modules over Q, by using methods very similar to those of section 1. The constructions in section 2 also follow, *ad verbum*, after those in [1], §5.

1.1. Let F be any field. By definition (cf. [1]), K_1F is just the multiplicative group $F^{\cdot}$ written additively, with the canonical isomorphism $l\colon F^{\cdot} \to K_1F$. Then

$$K_*F = (K_0F, K_1F, K_2F, \cdots)$$

is a graded ring which is defined to be the quotient of the tensor algebra

$$(Z, K_1F, K_1F \otimes K_1F, \cdots)$$

by the ideal generated by all $l(a) \otimes l(1-a)$, with $a \neq 0, 1$. Thus, each K_nF, $n \geq 2$, is the quotient of the n-fold tensor product $K_1F \otimes \cdots \otimes K_1F$ by the subgroup generated by all $l(a_1) \otimes \cdots \otimes l(a_n)$ such that $a_i + a_{i+1} = 1$ for some i.

First let us recollect two fundamental properties of the ring K_*F. ([1], §1, Lemmas 1.1 and 1.2)

Lemma A. *For every $l(a)$, $l(b) \in K_1F$, the identity*

$$l(a)l(b) = -l(b)l(a)$$

is valid in K_2F.

Lemma B. *The identity $l(a)^2 = l(a)l(-1)$ is valid for every $l(a) \in K_1F$.*

1.2. Let Q be the field of rational numbers, and let $1 = p_0 < p_1 < p_2 < \cdots < p_d < \cdots$ be the increasing sequence of all rational prime numbers.

Keeping a natural number $n (\geq 2)$ fixed, write L_d for the subgroup of K_nQ generated by those products $l(a_1) \cdots l(a_n)$ such that each a_i is one of the numbers ± 1, $\pm p_1, \cdots, \pm p_d$. Thus $L_0 \subseteq L_1 \subseteq \cdots \subseteq L_d \subseteq \cdots$ with union K_nQ. We also denote by $L(\leq p_d)$, $L(< p_d)$ for L_d, L_{d-1} respectively.

First, note that $L_1=L_0$ and L_0 is isomorphic with $Z/2Z$. In fact, since $2+(-1)=1$, we have $l(2)^2=l(2)l(-1)=0$, whence $L_1=L_0$.

Now, L_0 is generated by $l(-1)^n$. Since $2l(-1)=0$, we have $2l(-1)^n=0$. Hence L_0 is of order at most 2. To show $l(-1)^n\neq 0$, consider an n-linear map

$$K_1Q\times\cdots\times K_1Q\ni(l(a_1),\ \cdots,\ l(a_n))\rightarrow\prod_{i=1}^{n}\frac{1-sign(a_i)}{2}\in Z/2Z.$$

Obviously the right hand side is zero whenever $a_i+a_{i+1}=1$. Therefore this map induces a homomorphism $K_nF\rightarrow Z/2Z$ which carries $l(-1)^n$ to 1. This proves $l(-1)^n\neq 0$. (cf. [1], Theorem 1. 4)

Now, we need to study L_d for $d\geq 2$, so let $p_d=p\geq 3$. Then each element $\bar{b}$ of the residue class field $F_p=Z/pZ$ is represented by a unique $b\in Z$ such that $-\frac{p}{2}<b<\frac{p}{2}$. We fix, henceforth, such a system of representatives modulo p.

Lemma 1. *There exists one and only one homomorphism*

$$h_p:\ K_{n-1}(Z/pZ)\ \rightarrow\ L(\leq p)/L(<p)$$

which carries each product $l(\bar{b}_2)\cdots l(\bar{b}_n)$ *to the residue class of* $l(p)l(b_2)\cdots l(b_n)$ mod $L(<p)$.

Proof. Consider a map

$$(l(\bar{b}_2),\ \cdots,\ l(\bar{b}_n))\rightarrow l(p)l(b_2)\cdots l(b_n)\ \mathrm{mod}\ L(<p)$$

from $K_1(Z/pZ)\times\cdots\times K_1(Z/pZ)$ to $L(\leq p)/L(<p)$. To show that this map is linear, for example, as a function of $\bar{b}_2$, suppose that

$$b_2\equiv b_2'b_2''\qquad \mathrm{mod}\ p,$$

where $b_2',\ b_2''\in Z$ such that $-\frac{p}{2}<b_2',\ b_2''<\frac{p}{2}$. Write

$$b_2=pq+b_2'b_2'',$$

then, since $-\frac{p}{2}<b_2,\ b_2',\ b_2''<\frac{p}{2}$, we have $|q|<p$.

Since the assertion is obvious for $q=0$, so we suppose $q\neq 0$. Then

$$1=\frac{pq}{b_2}+\frac{b_2'b_2''}{b_2},$$

whence

$$\{l(p)+l(q)-l(b_2)\}\{l(b_2')+l(b_2'')-l(b_2)\}=0.$$

Multiplying on the right by $l(b_3)\cdots l(b_n)$ and then reducing mod $L(<p)$, we have

$$l(p)\{l(b_2')+l(b_2'')-l(b_2)\}l(b_3)\cdots l(b_n)\equiv 0 \mod L(<p),$$

since only prime numbers $< p$ can be prime factors of q. Hence the map

$$(l(\bar{b}_2), \cdots, l(\bar{b}_n)) \to l(p)l(b_2)\cdots l(b_n) \mod L(<p)$$

gives rise to a homomorphism from $(n-1)$-fold tensor product $K_1(Z/pZ)\otimes\cdots\otimes K_1(Z/pZ)$ to $L(\leq p)/L(<p)$. To show that this homomorphism gives rise to a homomorphism

$$l(\bar{b}_2)\cdots l(\bar{b}_n) \to l(p)l(b_2)\cdots l(b_n) \mod L(<p)$$

from $K_{n-1}(Z/pZ)$ to $L(\leq p)/L(<p)$, it is enough to check that the image is zero whenever $\bar{b}_j+\bar{b}_{j+1}=1$ $(2\leq j\leq n-1)$. But, since $-\frac{p}{2}<b_j, b_{j+1}<\frac{p}{2}$, $\bar{b}_j+\bar{b}_{j+1}=\bar{1}$ simply means that b_j+b_{j+1} is equal to 1 or $-p+1$. If $b_j+b_{j+1}=1$, then $l(p)l(b_2)\cdots l(b_n)=0$. If $b_j+b_{j+1}=-p+1$, then

$$\{l(b_j)-l(-p+1)\}\{l(b_{j+1})-l(-p+1)\}=0.$$

Multiplying on the left by $l(p)$ and then noticing that $l(p)l(-p+1)=0$, we have thus $l(p)l(b_j)l(b_{j+1})=0$, whence $l(p)l(b_2)\cdots l(b_n)=0$.

Lemma 2. *There exists a homomorphism ∂_p: $K_nQ\to K_{n-1}(Z/pZ)$ which carries the product $l(p)l(u_2)\cdots l(u_n)$ to $l(\bar{u}_2)\cdots l(\bar{u}_n)$ for all rational p-adic units $u_2, \cdots, u_n$. This homomorphism ∂_p annihilates every product of the form $l(u_1)\cdots l(u_n)$.*

Lemma 2 is just a special case of Lemma 2.1 in [1], §2.

Lemma 3. *The homomorphism ∂_p: $K_nQ\to K_{n-1}(Z/pZ)$ $(p\geq 3)$ gives rise to an isomorphism*

$$L(\leq p)/L(<p)\cong K_{n-1}(Z/pZ).$$

Proof. By Lemma 2, ∂_p induces a homomorphism

$$\partial_p:\ L(\leq p)/L(<p) \to K_{n-1}(Z/pZ)$$

and it is trivial to see that the composition

$$K_{n-1}(Z/pZ) \xrightarrow{h_p} L(\leq p)/L(<p) \xrightarrow{\partial_p} K_{n-1}(Z/pZ)$$

is the identity, whence h_p is injective. Hence we need only to show that the image of h_p generates $L(\leq p)/L(<p)$.

Now, consider any generator of $L(\leq p)/L(<p)$,

$$l(p)l(a_2)\cdots l(a_n),$$

where each a_i is one of those numbers $\pm 1, \pm p_1, \cdots, \pm p_{d-1}$ $(p_{d-1}<p)$.

If $\frac{p}{2}<a_2<p$, write $a_2=p+b$ with $-\frac{p}{2}<b<0$. It follows that $\frac{a_2}{b}+\frac{-p}{b}=1$, whence

$$\{l(a_2)-l(b)\}\{l(p)+l(-1)-l(b)\}=0.$$

Therefore the product $l(p)l(a_2)$ is a sum of terms

$$l(p)l(b)-l(-1)l(a_2)+l(-1)l(b)+l(b)l(a_2)-l(b)^2.$$

Multiplying on the right by $l(a_3)\cdots l(a_n)$, we have

$$l(p)l(a_2)\cdots l(a_n)\equiv l(p)l(b)l(a_3)\cdots l(a_n) \mod L(<p),$$

because b contains only prime factors $<p$.

Remark. Since $K_{n-1}(Z/pZ)=0$ for $n\geq 3$ (cf. [1], §1, Example 1.5), Lemma 3 shows that $L_d=L_{d-1}$ in K_nQ $(n\geq 3)$. But, to prove $L_d=L_{d-1}$ in K_nQ $(n\geq 3)$ it is sufficient to show that the image of h_p generates L_d/L_{d-1}. So we need Lemma 2 only for the calculation of K_2Q.

Lemma 4. *There exists an isomorphism*

$$L_d\cong Z/2Z\oplus \coprod_{2<p\leq p_d} K_{n-1}(Z/pZ) \qquad (d=0, 1, 2, \cdots),$$

that is

$$L_d\cong Z/2Z\oplus \coprod_{2<p\leq p_d} K_1(Z/pZ) \qquad \textit{in } K_2Q$$

and

$$L_d\cong Z/2Z \qquad \textit{in } K_nQ\ (n\geq 3).$$

Proof. Denote $M_d=Z/2Z\oplus \coprod_{2<p\leq p_d} K_{n-1}(Z/pZ)$. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{d-1} & \longrightarrow & L_d & \longrightarrow & K_{n-1}(Z/p_dZ) & \longrightarrow & 0 \quad \text{(exact)} \\ & & \downarrow{\scriptstyle \phi_{d-1}} & & \downarrow{\scriptstyle \phi_d} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & M_{d-1} & \longrightarrow & M_d & \longrightarrow & K_{n-1}(Z/p_dZ) & \longrightarrow & 0 \quad \text{(exact)} \end{array}$$

is commuative, where ϕ_d is induced by $L_0\cong Z/2Z$ and ∂_p's $(p\leq p_d)$. Therefore, an induction on d proves the Lemma.

Passing to the direct limit $K_nQ=\varinjlim L_d$ as $d\to\infty$, we have

Theorem. *There exists an isomorphism*

$$K_2Q\cong Z/2Z\oplus \coprod_{p\neq 2} F_p^{\cdot},$$

where $F_p^{\cdot}$ denotes the multiplicative group of the prime field $F_p=Z/pZ$.

And there also exist isomorphisms

$$K_nQ \cong Z/2Z \ (n \geq 3).$$

2. 1. Let F be any field of characteristic different from 2. Let M be a quadratic module over F, that is, M is a finite dimensional vector space with a non-degenerate symmetric bilinear inner product. It is well known that M is equivalent to an orthogonal direct sum $\langle a_1, \cdots, a_r\rangle = \langle a_1\rangle \oplus \cdots \oplus \langle a_r\rangle$ of one dimensional modules. Here $\langle a\rangle$ denotes the one dimensional quadratic module such that the inner product of a suitable basis vector with itself is a. By Witt's theorem any quadratic module M can be written as $M = \underbrace{\langle 1, -1\rangle \oplus \cdots \oplus \langle 1, -1\rangle}_{i\text{-times}} \oplus M_0$ where $i \geq 0$ and M_0 is anisotropic module, that is, the inner product of any non-zero vector of M_0 with itself is not zero. The index i is uniquely determined by M and M_0 is determined up to equivalence and is called the anisotropic kernel of M. Two quadratic modules M, N are called similar if their anisotropic kernels M_0, N_0 are equivalent. Then the similarity classes of quadratic modules form a commutative group W^+F, called the *Witt group* of F. The similarity class of $\langle 1\rangle \oplus \langle -1\rangle$ is the zero element of W^+F. Together with the tensor product $\otimes$, W^+F is a commutative ring, called the *Witt ring* of F. The following theorem is the consequence of well known theorem "*Satz 7 von Witt*".

Theorem W. *The Witt group W^+F (the Witt ring WF) has a representation in terms of generators $\langle a\rangle$, where a ranges over $F^\cdot$ (the multiplicative group of F), subject only to the relations* (i) (ii) *and* (iii) ((i) (ii) (iii) *and* (iv))

(i) $\langle ab^2\rangle = \langle a\rangle$,

(ii) $\langle a\rangle + \langle b\rangle = \langle a+b\rangle + \langle ab(a+b)\rangle$,

(iii) $\langle a\rangle + \langle -a\rangle = 0$,

((iv) $\langle ab\rangle = \langle a\rangle\langle b\rangle$)

and their consequences.

Now let E be a field which is complete under a discrete valuation with residue field $\bar{E}$ of characteristic different from 2. Let π be a prime element.

Theorem S (*T. A. Springer*) *The Witt ring WE contains a subring W_0 canonically isomorphic to $W\bar{E}$ and WE is a group ring of a cyclic group $\langle \pi\rangle$ of order 2 over W_0. Hence the Witt group W^+E is a direct sum of W_0^+ and $(\langle\pi\rangle W_0)^+$ ($^+$ denotes additive group). W_0 can be defined as the subring generated by $\langle u\rangle$ where u ranges over units of E, and the isomorphism $W_0 \to W\bar{E}$ is defined by $\langle u\rangle \to \langle \bar{u}\rangle$.*

Corollary. *There exists a split exact sequence*

$$0 \to W^+\bar{E} \to W^+E \xrightarrow{\partial} W^+\bar{E} \to 0,$$

where the first homomorphism maps $\langle\bar{u}\rangle$ *to* $\langle u\rangle$ *and where* ∂ *is defined by* $\partial\langle u\rangle=0$, $\partial\langle\pi u\rangle=\langle\bar{u}\rangle$.

2. 2. Let Q be the field of rational numbers. The Witt ring WQ is generated by $\langle 1\rangle, \langle 2\rangle, \cdots, \langle p\rangle, \cdots$, where p ranges over all prime numbers.

Let $1=p_0<p_1<\cdots<p_d<\cdots$ be the increasing sequence of all rational prime numbers, and let $L_d=L(\leq p_d)$ be the subring of WQ generated by $\langle 1\rangle, \langle 2\rangle, \cdots, \langle p_d\rangle$; define $L_{d-1}=L(<p_d)$ similarly. We denote by L_d^+ the additive group of L_d. Then L_0^+ is isomorphic to Z by the "signature" of quadratic forms, and L_1^+ is isomorphic to $Z\oplus Z/2Z$. In fact, it is not difficult to show that $L_1^+=L_0^+\oplus(\langle 2\rangle-\langle 1\rangle)$, where $(\langle 2\rangle-\langle 1\rangle)$ is a cyclic group, of order 2, generated by $\langle 2\rangle-\langle 1\rangle$.

Note that the signature and the correspondence $L_1\ni\langle a\rangle\to ord_2 a$ *mod 2* map L_1 to the first and the second summand of $Z\oplus Z/2Z$.

Now, let $p_d=p\geqslant 3$.

Lemma 1. *The additive group* L_d^+ *is generated, modulo* L_{d-1}^+, *by expressions* $\langle pq_1\cdots q_s\rangle$ *where* $q_1, \cdots, q_s$ *are rational integers such that* $|q_i|<\frac{p}{2}$.

Proof. By definition, L_d^+ is generated, modulo L_{d-1}^+, by expressions $\langle pt_1\cdots t_s\rangle$ where t_i are rational integers such that $0<t_i<p$. Put $t_1\equiv q_1 \pmod p$ with $|q_1|<\frac{p}{2}$.

If $0<t_1<\frac{p}{2}$, then $t_1=q_1$. If $\frac{p}{2}<t_1<p$, then $t_1=q_1+p$. Hence the expression $\langle pt_1\cdots t_s\rangle$ is equal to a sum of terms

$$\langle q_1\rangle\langle pt_2\cdots t_s\rangle+\langle t_2\cdots t_s\rangle-\langle qt_1\cdots t_s\rangle,$$

whence

$$\langle pt_1\cdots t_s\rangle\equiv\langle q_1\rangle\langle pt_2\cdots t_s\rangle \mod L_{d-1}.$$

Therefore, if $t_1\equiv q_1$ mod p with $|q_1|<\frac{p}{2}$, we have

$$\langle pt_1\cdots t_s\rangle\equiv\langle q_1\rangle\langle pt_2\cdots t_s\rangle \mod L_{d-1}.$$

An induction on s, together with considering that L_{d-1} is a ring containing $\langle q_1\rangle$, completes the proof.

Lemma 2. *Let* $q_1, \cdots, q_s$ *be non-zero rational integers such that* $|q_1|<p$, $|q_i|<\frac{p}{2}$ $(i>1)$ *and let* q *be the rational integer defined by*

$$q_1\cdots q_s\equiv q \textit{ mod } p \textit{ and } |q|<\frac{p}{2},$$

then

$$\langle pq_1\cdots q_s\rangle\equiv\langle pq\rangle \textit{ mod } L_{d-1}^+.$$

Proof. Define $h \in Z$ by $q_1q_2 \equiv h \bmod p$ with $|h| < \frac{p}{2}$. Putting $q_1q_2 = h + pk$, we see that $|k| < p$.

If k is not zero, we have

$$\langle q_1q_2\rangle = \langle h\rangle + \langle pk\rangle - \langle pkhq_1q_2\rangle.$$

Multiplying by $\langle pq_3 \cdots q_s\rangle$, this shows that $\langle pq_1 \cdots q_s\rangle$ is equal to a sum of terms

$$\langle phq_3 \cdots q_s\rangle + \langle kq_3 \cdots q_s\rangle - \langle khq_1 \cdots q_s\rangle \equiv \langle phq_3 \cdots q_s\rangle \bmod L_{d-1}.$$

Hence, if $q_1q_2 \equiv h \bmod p$ with $|h| < \frac{p}{2}$, then

$$\langle pq_1 \cdots q_s\rangle \equiv \langle phq_3 \cdots q_s\rangle \mod L^+_{d-1}$$

holds. Therefore, an induction on s completes the proof.

Now consider the prime field $F_p = Z/pZ$ of characteristic p $(p \neq 2)$. Each residue class $\bar{a}$ modulo p can be represented by a unique rational integer a with $|a| < \frac{p}{2}$.

Lemma 3. *There exists an additive group homomorphism φ from W^+F_p to $L(\leq p)^+/L(<p)^+$ which carries $\langle \bar{a}\rangle$ to $\langle pa\rangle$ mod $L(<p)^+$.*

Proof. By Theorem W, it is enough to prove that the relations (i) (ii) and (iii) in Theorem W are valid in $L(\leq p)^+/L(<p)^+$. So if we set $(\bar{a}) = \langle pa\rangle$ mod $L(<p)^+$ for $\bar{a} \in F_p^{\cdot}$, then we must show

(i) $\quad (\bar{a}\bar{b}^2) = (\bar{a})$,

(ii) $\quad (\bar{a}) + (\bar{b}) = (\bar{a}+\bar{b}) + (\bar{a}\bar{b}\cdot\overline{a+b})$, $\quad$ if $\bar{a}+\bar{b} \neq 0$,

(iii) $\quad (\bar{a}) + (-\bar{a}) = 0$.

If $ab^2 \equiv c \bmod p$, $|c| < \frac{p}{2}$, then by Lemma 2 we have $\langle pa\rangle = \langle pab^2\rangle \equiv \langle pc\rangle$ mod $L(<p)^+$. This prove (i). Similarly, let $c, d \in Z$ be defined by $a+b \equiv c \bmod p$, $|c| < \frac{p}{2}$; $ab(a+b) \equiv d \bmod p$, $|d| < \frac{p}{2}$. Since $|a|, |b| < \frac{p}{2}$, it follows that $|a+b| < p$, whence, by Lemma 2,

$$\langle p(a+b)\rangle \equiv \langle pc\rangle \mod L(<p)^+,$$

$$\langle pab(a+b)\rangle \equiv \langle pd\rangle \mod L(<p)^+.$$

Then the relation

$$(\bar{a}) + (\bar{b}) = (\bar{a}+\bar{b}) + (\bar{a}\bar{b}\cdot\overline{a+b})$$

follows from

$$\langle pa\rangle+\langle pb\rangle=\langle p(a+b)\rangle+\langle pab(a+b)\rangle\equiv\langle pc\rangle+\langle pd\rangle \mod L(<p)^+.$$

This proves (ii). Finally $(\bar{a})+(-\bar{a})=0$ follows from $\langle pa\rangle+\langle -pa\rangle=0$.

Now let Q_p be the p-adic number field. Denote by ∂_p the composition of the natural map from WQ into WQ_p with the homomorphism ∂: $WQ\to WF_p$ of Cor. of Theorem S. Obviously $\partial_p\langle u\rangle=0$, $\partial_p\langle pu\rangle=\langle\bar{u}\rangle$ for rational p-adic units u.

Lemma 4. *The additive group homomorphism* ∂_p: $W^+Q\to W^+F_p$ $(p\neq 2)$ *gives rise to an isomorphism*

$$L(\leq p)^+/L(<p)^+\cong W^+F_p.$$

Proof. We see, by inspection, the composite

$$W^+F\xrightarrow{\varphi} L(\leq p)^+/L(<p)^+\xrightarrow{\partial_p} W^+F_p$$

is the identity. On the other hand, by Lemmas 1 and 2, $L(\leq p)^+/L(<p)^+$ is generated by $\langle pa\rangle$ mod $L(<p)^+$, where a ranges over $-\frac{p}{2}<a<\frac{p}{2}$, $a\neq 0$. Therefore, by inspection again, the composite

$$L(\leq p)^+/L(<p)^+\xrightarrow{\partial_p} W^+F_p\xrightarrow{\varphi} L(\leq p)^+/L(<p)^+$$

is the identity on the generators of $L(\leq p)^+/L(<p)^+$, whence $\varphi\circ\partial_p=id$. It follows that ∂_p is an isomorphism.

Lemma 5. *For a prime* $p_d\geqslant 2$,

$$L_d^+\cong Z\oplus Z/2Z\oplus\coprod_{2<p\leq p_d} W^+F_p,$$

where the signature and the correspondence $\langle a\rangle\to ord_2 a$ *mod 2 map to the first two summands, and homomorphisms* ∂_p *map to the third.*

Proof. Denote by M_d an additive group $Z\oplus Z/2Z\oplus\coprod_{2<p\leq p_d} W^+F_p$ and by ψ_d the correspondence

$$q\to(\text{sign } q,\ ord_2(\det q) \bmod 2,\ \cdots,\ \partial_p(q),\ \cdots)$$

from L_d^+ to M_d. Then, the following diagram is commutative.

$$\begin{array}{ccccccccl} 0 & \longrightarrow & L_{d-1}^+ & \longrightarrow & L_d^+ & \xrightarrow{\partial_{p_d}} & W^+F_{p_d} & \longrightarrow 0 & \text{(exact)}\\ & & \downarrow{\scriptstyle \psi_{d-1}} & & \downarrow{\scriptstyle \psi_d} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & M_{d-1} & \longrightarrow & M_d & \longrightarrow & W^+F_{p_d} & \longrightarrow 0 & \text{(exact)}\end{array}$$

Hence, an induction on d, together with the Five Lemma, proves the lemma.

Passing to the direct limits $W^+Q=\varinjlim L_d^+$, $\varinjlim M_d=Z\oplus Z/2Z\oplus \coprod_{p\neq 2} W^+F_p$ as $d\to\infty$, we have

Theorem. *For the field Q of rational numbers*

$$W^+Q\cong Z\oplus Z/2Z\oplus \coprod_{p\neq 2} W^+(Z/pZ),$$

where the signature and the correspondence $q\to ord_2(\det q)$ mod 2 map to the first two summands and the homomorphisms ∂_p for the third.

References

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