On an Equivalence Relation Among (0, 1)-matrices

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(Introduced by Prof. F. Niiro)

(Received February 4, 1971)

Introduction

The purpose of this note is to investigate an equivalence relation among the set $X(m, n)$ of all (0, 1)-matrices with m rows and n columns. Given two elements A, B in $X(m, n)$, we say that A is equivalent to B (in notation $A \sim B$) if there exist permutation matrices P, Q of degree m, n respectively such that $PAQ=B$. We will give a criterion for $A \sim B$ by introducing a notion of complexes with mutiplicity function (§2 for the definition) associated with A, B respectively.

Using this criterion, we will give the number of equivalence classes in $X(n, n)$ which are represented by fully indecomposable (0, 1)-matrices with $2n+1$ non-zero entries.

§1. An equivalence relation on rectangular (0, 1)-matrices

A real matrix $A=(a_{ij})$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq n)$ is called a (0, 1)-matrix if every entry a_{ij} is either 0 or 1. We denote by $X(m, n)$ the set of all (0, 1)-matrices with m rows and n columns. Thus $X(m, n)$ is a finite set of cardinality 2^{mn}.

We denote by S_n the subset of $X(n, n)$ consisting of permutation matrices, i.e., S_n is the set of $P=(p_{ij}) \in X(n, n)$ such that each row and each column of P has exactly one non-zero entry. Then S_n forms a group with respect to the matrix multiplication. The group S_n is isomorphic with the symmetric group $\mathfrak{S}_n$ consisting of all permutations of n letters $\{1, 2, \cdots, n\}$. The following mapping $f: \mathfrak{S}_n \to S_n$ gives an isomorphism of $\mathfrak{S}_n$ with S_n: let $\sigma \in \mathfrak{S}_n$, define $f(\sigma)=(p_{ij})$ by

$$p_{ij}=\begin{cases} 1 & \text{if} \quad i=\sigma(j), \\ 0 & \text{if} \quad i \neq \sigma(j). \end{cases}$$

In fact if $f(\sigma)=P=(p_{ij})$, $f(\tau)=Q=(q_{ij})$, one has for $R=PQ=(r_{ij})$

$$\begin{aligned} r_{ij} &= \sum_{k=1}^{n} p_{ik} q_{kj} \\ &= \sum_{k=1}^{n} \delta_{i,\sigma(k)} \delta_{k,\tau(j)} \\ &= \delta_{i,\sigma\tau(j)} \end{aligned}$$

Hence $R=f(\sigma\tau)$, i.e. $f(\sigma)f(\tau)=f(\sigma\tau)$. Now since f is injective and $|\mathfrak{S}_n|=n!=|S_n|$, f is bijective. We denote $f(\sigma)$ by P_σ.

Definition. Let $A\in X(m, n)$, $B\in(m, n)$. We say that A is equivalent to B (in notation $A\sim B$) if there exist $P\in S_n$, $Q\in S_n$ such that $PAQ=B$.

Obviously this relation $\sim$ is equivalence relation on the set $X(m, n)$. We denote by $[A]$ the equivalence class containing A.

§ 2. Complex with a multiplicity function

In order to find a condition for two matrices A, B in $X(m, n)$ to be equivalent, we introduce a notion of a complex with a multicity function.

Let V be a finite set. A complex over V is a subset Δ of 2^V, where 2^V is the set of all subsets of V, satisfying the following two conditions:

(C·I) if $L\in\Delta$, $K\subset L$, then $K\in\Delta$. (In particular, the empty set ϕ belongs to Δ).

(C·II) for each element v of V, the subset $\{v\}$ belongs to V.

The elements of V are called vertices of Δ. We denote by (Δ, V) a complex over V.

Definition. Let (Δ, V) be a complex. A mapping $\mu: 2^V\rightarrow\{0, 1, \cdots, m\}$ is called a multiplicity function of (Δ, V) if the following conditions are satisfied:

(M·I) $\mu(\phi)=m$; m is called the weight of μ,

(M·II) an element K of 2^V is in Δ if and only if $\mu(K)>0$,

(M·III) if $K, L\in 2^V$, $K\subset L$, then $\mu(K)\geqq\mu(L)$.

A triple $(\Delta, V; \mu)$, (consisting of a complex (Δ, V) and of a multiplicity function μ of (Δ, V)) is called a complex with multiplicity function μ.

Let (Δ, V) and (Δ', V') be complexes. A mapping $f: V\rightarrow V'$ is called a simplicial mapping of (Δ, V) into (Δ', V') if $f(\Delta)\subset\Delta'$. A simplicial mapping $f: V\rightarrow V'$ is called an isomorphism provided f is bijective and $f(\Delta)=\Delta'$. A complex (Δ, V) is said to be isomorphic with (Δ', V') if there exists an isomorphism f: $V\rightarrow V'$; then we write $(\Delta, V)\cong(\Delta' V')$.

Now let $(\Delta, V; \mu)$, $(\Delta', V'; \mu')$ be two complexes with multiplicity functions μ, μ' respectively. A bijective mapping $f: V\rightarrow V'$ is called an isomorphism of $(\Delta, V; \mu)$ with $(\Delta', V'; \mu')$ if the induced mapping f: $2^V\rightarrow 2^{V'}$ satisfies $\mu'\circ f=\mu$. Then it is immediate to see that $f(\Delta)=\Delta'$ because of the condition (M·II). Thus f is an isomorphism of complexes. Also by (M·I), μ and μ' are of the same weight.

$(\Delta, V; \mu)$ is said to be isomorphic with $(\Delta', V'; \mu')$ (in notation $(\Delta, V; \mu)\cong(\Delta', V'; \mu')$) if there exists an isomorphism $f: V\rightarrow V'$ of $(\Delta, V; \mu)$ with $(\Delta', V'; \mu')$.

§ 3. A complex with a multiplicity function associated with a (0, 1)-matrix

Let $A=(a_{ij})\in X(m, n)$. Let $V(A)$ be the subset of $\Omega=\{1, \cdots, n\}$ defined by

$$V(A)=\{j\in\Omega;\ a_{ij}=1 \text{ for some } i\in\{1, \cdots, m\}\}.$$

Now we define a mapping $\mu_A: 2^\Omega\to\{0, 1, \cdots, m\}$ as follows:

(3.1) $\mu(\phi)=m,$

(3.2) for $J\in 2^{V(A)}$, $J\neq\phi$, $\quad \mu_A(J)=\sum_{i=1}^{m}\prod_{j\in J}a_{ij}$

Then clearly one has

$$\mu_A(K)\geqq\mu_A(L)$$

for $K, L\in 2^\Omega$ such that $K\subset L$. Hence by putting

$$\Delta_A=\{K\in 2^{V(A)};\ \mu_A(K)>0\},$$

we get a complex with a multiplicity function $(\Delta_A, V(A); \mu_A)$.

Theorem 1. *Let A, B be two* (0, 1)*-matrices with m rows and n columns. Then A is equivalent to B if and only if*

$$(\Delta_A, V(A); \mu_A)\cong(\Delta_B, V(B); \mu_B).$$

Proof. Suppose $P_\sigma AP_\tau=B$ for fome $\sigma\in\mathfrak{S}_m$, $\tau\in\mathfrak{S}_n$. Then putting $A=(a_{ij})$, $B=(b_{ij})$, one has

$$b_{ij}=a_{\sigma^{-1}(i),\tau(j)}.$$

Therefore, j is in $V(B)$ if and only if $\tau(j)$ is in $V(A)$. Thus, $\tau(V(B))=V(A)$. Furthermore $\mu_A(\phi)=m=\mu_B(\phi)$. Now for $J\in 2^{V(B)}$ we have

$$\prod_{j\in J}b_{ij}=\prod_{k\in\tau(J)}a_{\sigma^{-1}(i),k}.$$

Hence $\mu_B(J)=\sum_{i=1}^{m}\prod_{j\in J}b_{ij}=\sum_{i=1}^{m}\prod_{j\in J}a_{\sigma^{-1}(i),\tau(j)}=\mu_A(\tau(J))$. Thus $\tau: V(B)\to V(A)$ is an isomorphism of $(\Delta_B, V(B); \mu_B)$ with $(\Delta_A, V(A); \mu_A)$.

Conversely suppose that

$$(\Delta_A, V(A); \mu_A)\cong(\Delta_B, V(B); \mu_B).$$

Then there exists a bijection $\tau: V(B)\to V(A)$ such that

$$\mu_B(J)=\mu_A(\tau(J))$$

for every $J\in 2^{V(B)}$. One can extend τ to a permutation (also denoted by τ) of $\Omega=\{1, \cdots, n\}$. Put $C=AP_\tau=(c_{ij})$. Then $c_{ij}=a_{i,\tau(j)}$ $(1\leqq i\leqq m, 1\leqq j\leqq n)$. Then $V(C)=\tau(V(A))=V(B)$ and for every snbset J of $V(B)$, one has $\mu_B(J)=\mu_C(J)$. Hence the proof of Theorem 1 is complete if the following lemma is established.

LEMMA 1. *Let B, $C\in X(m, n)$. Suppose that for every subset $J\in 2^\Omega$, one has $\mu_B(J)=\mu_C(J)$. Then there exists a permutation $\sigma\in\mathfrak{S}_m$ such that $P_\sigma B=C$.*

Proof. We prove by induction on m. Our assertion is immediate for $m=1$.

Suppose $B=O$. Then for every $j\in\{1, \cdots, n\}$ we have $0=\sum_{i=1}^{m} b_{ij}=\mu_B(\{j\})=\sum_{i=1}^{m} c_{ij}$. Hence $C=O$.

Thus we may assume that $B\neq O$. Then there exists a non-empty subset J such that $\mu_B(J)>0$. Let L be a maximal subset of $\Omega=\{1, \cdots, n\}$ with the property $\mu_B(L)>0$. Put M be the subset of $\Omega'=\{1, \cdots, m\}$ defined by $M=\{i\in\Omega'; b_{ij}=1$ for every $j\in L\}$. Then due to the maximality of L, we have

$$b_{ik}=0, \quad \text{for every } (i, k)\in M\times(\Omega-L).$$

Now let M' be the subset of Ω' defined by

$$M'=\{i\in\Omega'\,;\ c_{ij}=1 \text{ for every } j\in L\}.$$

Then the cardinality $|M'|$ of M' is equal to the cardinality $|M|$ of M, since

$$|M'|=\mu_C(L)=\mu_B(L)=|M|.$$

Hence there exists a permutation σ of Ω' such that $\sigma(M)=M'$. Thus by replacing C by PC with a suitable $P\in S_m$, we may assume that $M=M'$. Denote by B_0 and C_0 the submatrices of B, C respectively defined by

$$B_0=(b_{ij})_{i\in\Omega'-M,\, j\in\Omega},$$

$$C_0=(c_{ij})_{i\in\Omega'-M,\, j\in\Omega}.$$

Then B_0 and C_0 are (0, 1)-matrices with $|\Omega'|-|M|$ rows and n columns. Let us now verify that

$$\mu_{B_0}(J)=\mu_{C_0}(J)$$

for every subset J of Ω. This valid for $J=\phi$ since $\mu_{B_0}(\phi)=|\Omega'|-|M|=\mu_{C_0}(\phi)$. Suppose now that $J\neq\phi$. Put

$$S=\{i\in\Omega'-M;\ b_{ij}=1 \text{ for every } j\in J\}$$

and

$$T=\{i\in\Omega'-M;\ c_{ij}=1 \text{ for every } j\in J\}.$$

Then we have

$$\mu_{B_0}(J)=|S|, \qquad \mu_{C_0}(J)=|T|.$$

So we have to show that $|S|=|T|$. Now put

$$\tilde{S}=\{i\in\Omega';\ b_{ij}=1 \text{ for every } j\in J\}$$

and

$$\tilde{T}=\{i\in\Omega';\ c_{ij}=1 \text{ for every } j\in J\}.$$

Then $|\tilde{S}|=\mu_B(J)=\mu_C(J)=|\tilde{T}|$. We distinguish now two cases.
Case 1. $J\subset L$.

Then, we have $M\subset\tilde{S}$, $M\subset\tilde{T}$ and $S=\tilde{S}-M$, $T=\tilde{T}-M$. Hence, $|S|=|\tilde{S}|-|M|$

$=|\tilde{T}|-|M|=|T|$.

Case 2. $J \not\subset L$.

Since $b_{ik}=c_{ik}=0$ for every $(i, k)\in M\times(\Omega-L)$, we have $M\cap\tilde{S}=\phi$, $M\cap\tilde{T}=\phi$. Therefore $\tilde{S}=S$, $\tilde{T}=T$. Hence $|S|=|T|$.

Thus we may apply the induction assumption to B_0, C_0. Hence there exists a permutation ρ of $\Omega'-M$ such that $P_\rho B_0=C_0$. ρ can be extended to a permutation σ of Ω' by putting

$$\sigma(i)=\begin{cases} i & \text{for } i\in M,\\ \rho(i) & \text{for } i\in\Omega'-M.\end{cases}$$

Then it is immediate to see that $P_\rho B=C$, q.e.d.

§ 4. Fully indecomposable (0, 1)-matrices with $2n+1$ non-zero entries

Let $A=(a_{ij})$ be an $n\times n$ matrix with non-negative real entries a_{ij} $(i, j=1, \cdots, n)$. Following Marcus-Minc [1], we say that $A=(a_{ij})$ is decomposable if there exists a permutation P such that $PAP^{-1}=\begin{pmatrix} B & O\\ C & D\end{pmatrix}$ where B and D are square matrices of positive degree. Otherewise we say that A is indecomposable. We say also that A is fully indecomposable if PAQ is indecomposable for all permutation matrices P and Q. Thus, for two (0, 1)-matrices A, B in $X(n, n)$ such that $A\sim B$, A is fully indecomposable if and only if B is so.

we consider in this section the equivalence classes in $X(n, n)$ consisting of fully indecomposable matrices; we call such classes fully indecomposable. We will determine fully indecomposable classes represented by $n\times n$ (0, 1)-matrices with $2n+1$ non-zero entries.

LEMMA 2. ([2], [3]) *Let $A=(a_{ij})$ be an $n\times n$ fully indecomposable* (0, 1)-*matrix. Then for every pair (i, j) such that $a_{ij}\neq 0$, there there exists a permutation matrix $P=(p_{kl})$ such that*

i) $a_{kl}\geqq p_{kl}$ for all $k, l=1, \cdots, n$

and

ii) $p_{ij}=1$.

Proof. Put $\Omega=\{1, \cdots, n\}$. For every subset J of $\Omega_1=\Omega-\{i\}$, denote by J^* the subset of $\Omega_2=\Omega-\{j\}$ given by $J^*=\{j\in\Omega_2;\ a_{ij}\neq 0 \text{ for some } i\in J\}$. We assert then $|J^*|\geqq|J|$. In fact, if $|J^*|<|J|$ for some J, we have $a_{ij}=0$ for ever $(i, j)\in J\times(\Omega_2-J^*)$. Now $|J|+|\Omega_2-J^*|=\Omega_2|+|J|-|J^*|>|\Omega_2|=n-1$. Hence there must exist non-empty subsets K, L of Ω such that $K\subset J$, $L\subset\Omega_2-J^*$, $|K|+|L|=|\Omega|$. But this is impossible since A is fully indecomposable.

By the above assertion, one can apply the Marriage Theorem [4]. Thus, there exists an injection σ: $\Omega_1\to\Omega_2$ such that $a_{i,\sigma(i)}\neq 0$ for every $i\in\Omega_1$. Putting $\sigma(i)=j$, we can extend σ to a permutation σ of Ω. Then $P=P_\sigma$ has all the properties desired, q.e.d.

COROLLARY. *Let $A=(a_{ij})$ be an $n\times n$ fully indecomposable* (0, 1)*-matrix. Then the equivalence class* $[A]$ *contains a matrix $B=(b_{ij})$ such that $b_{11}=b_{22}\cdots=b_{nn}=1$.*

Proof. Take P as in Lemma 1. Then $B=P^{-1}A$ satisfies the above condition, q.e.d.

LEMMA 3. ([3]). *Let $A=(a_{ij})$ be an $n\times n$ (0, 1)-matrix such that $a_{11}=a_{22}=\cdots=a_{nn}=1$. Then, A is fully indecomposable if and if $A-I$ is indecomposable.*

Proof. Suppose that $A-I$ is decomposable. Then there exists a permutation matrix P such that $P(A-I)P^{-1}$ is of the form $\begin{pmatrix} B & O \\ C & D \end{pmatrix}$. Then $PAP^{-1}=I+\begin{pmatrix} B & O \\ C & D \end{pmatrix}$ is of the form $\begin{pmatrix} B' & O \\ C & D' \end{pmatrix}$, and so A is not fully indecomposable.

Coversely, suppose that A is not fully indecomposable. Then there exist non-empty subsets J, K of $\Omega=\{1, \cdots, n\}$ such that

(i) $a_{jk}=0$ for all $(j, k)\in J\times K$ and

(ii) $|J|+|K|=n$.

Since $a_{11}=\cdots=a_{nn}=1$, we have $J\cap K=\phi$. Thus $K=\Omega-J$. However, this means that A is decomposable. Hence, $A-I$ is also decomposable, q.e.d.

We note here that the indecomposability of a matrix $A=(a_{ij})\in X(n, n)$ is equivalent to the so-called strong connectivity of the oriented graph Γ_A associated with A. Namely Γ_A cansists of n vertices $1, \cdots, n$ together with oriented edges from i to j if and only if $a_{ij}\neq 0$. Γ_A is called strongly connected if for any distinct vertices i, j there exists a sequence $i_1, i_2, \cdots, i_r$ of vertices such that $i=i_1$, $j=i_r$, $a_{i_p, i_{p+1}}\neq 0$ $(p=1, \cdots, r-1)$.

We note also that if $A=(a_{ij})\in X(n, n)$ is fully indecomposable, then the number of non-zero entries of A (which we denote by $\#A$) is $\geqq 2n$. In fact, replacing A by an equivalent matrix in in $X(n, n)$, we may assume that $a_{11}=\cdots=a_{nn}=1$. Then, since $A-I$ is indecomposable, every row of $A-I$ contains at least one non-zero entry. Hence $\#(A-I)\geqq n$. Thus $\#A\geqq 2n$.

We note furthermore that if $A\in X(n, n)$ is fully indecomposable and if $\#A=2n$, then, assuming $a_{11}=\cdots=a_{nn}=1$ as above, we see that $A-I$ is indecomposable and $\#(A-I)=n$. Hence $A-I=P$ is a permutation matrix. Since P is indecomposable, there is a permutation matrix Q such that

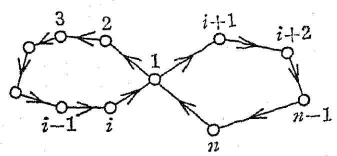
$$QPQ^{-1}=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}=J.$$

Thus, $A=I+P=I+Q^{-1}JQ=Q^{-1}(I+J)Q$ is equivalent to $I+J$. We have proved thus that two fully indecomposable (0, 1)-matrices A, B in $X(n, n)$ with $\sharp A=\sharp B=2n$ are always equivalent.

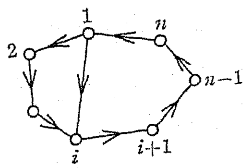
Theorem 2. *The number of equivalence classes (in the sense of* § 1) *of fully indecomposable* (0, 1)*-matrix of degree* n *satisfying* $\sharp A=2n+1$ *is equal to* $\left[\frac{n-1}{2}\right]$.

Proof. Let A be a fully indecomposable (0, 1)-matrix of degree n satisfying $\sharp A=2n+1$. We may assume that $a_{11}=\cdots=a_{nn}=1$. Then $B=A-I$ is indecomposable. Furthermore $\sharp B=n+1$. Thus, the oriented graph Γ_B associated to B must be one of the following forms. (Note that we can assign arbitrary numbers to vertices since this is realized by replacing A with $P_\sigma AP_\sigma^{-1}$ where σ is a suitable permutation of $\Omega=\{1, \cdots, n\}$.

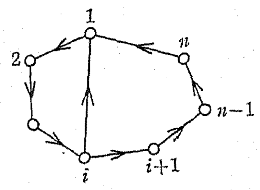
Type I_i $(i=2, \cdots, n-1)$



Type II_i $(i=3, \cdots, n)$



Type III_i $(i=2, \cdots, n-1)$

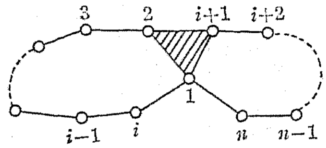


We denote also by I_i, II_i and III_i the (0, 1)-matrices corresponding to the above oriented graphs. We denote by I_i^*, II_i^*, III_i^* the (0, 1)-matrices obtained from I_i, II_i, III_i, by adding the identity matrix of degree n respectively.

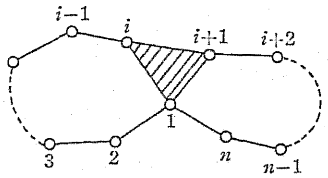
LEMMA 4. *If* $2\leqq i\leqq n-1$, *then* I_i^* *is equivalent* III_i^*.

Proof. We denote the complexes with multiplicity functions of I_i^*, III_i^*, over the same vertex-set Ω by $(\Delta, \Omega; \mu_i)$, $(\Delta_i', \Omega; \mu_i')$ respectively. Suppose now, $3\leqq i\leqq n-1$. Then these complexes are of dimension 2 and they are illustrated as follows:

Δ_i :



Δ_i' :



Furthermore the multiplicity function μ_i is given as follows:

For $J \ni 2^{\Omega}$ such that $|J| \geqq 3$, $J \neq \{1, 2, i+1\}$, $\mu_i(J)=0$.

For $J=\{1, 2, i+1\}$, $\mu_i(\{1, 2, i+1)=1$.

For $=\{1, 2\}$, $\{2, 3\}$, $\cdots$, $\{i-1, i\}$, $\{i, 1\}$, $\{2, i+1\}$, $\{1, i+1\}$, $\{i+1, i+2\}$, $\cdots$, $\{n-1, n\}$, $\{n, 1\}$, $\mu_i(J)=1$.

For all other J with $|J|=2$, $\mu_i(J)=0$.

For $J=\{1\}$, $\mu_i(\{1\})=3$.

For all other J with $|J|=1$, $\mu_i(J)=2$.

Finally $\mu_i(\phi)=n$.

On the other hand, the multiplicity function μ_i' is given as follows:

For $J \in 2^{\Omega}$ such that $|J| \geqq 3$. $J \neq \{1, i, i+1\}$, $\mu_i'(J)=0$.

For $J=\{1, i, i+1\}$, $\mu_i'(1, i, i+1\})=1$.

For $J=\{1, 2\}$, $\{2, 3\}$ $\cdots$, $\{i-1, i\}$, $\{i, 1\}$, $\{i, i+1\}$, $\{i+1, i+2\}$, $\cdots$, $\{n-1, n\}$, $\{n, 1\}$, $\mu_i'(J)=1$.

For all other J with $|J|=2$, $\mu_i'(J)=0$.

For $J=\{1\}$, $\mu_i'(1)=3$.

For all other J with $|J|=1$, $\mu_i'(J)=2$.

Finally $\mu_i'(\phi)=n$.

Thus the permutaion $\sigma: \Omega \rightarrow \Omega$ defined by

$$\sigma=\begin{cases}(2,\, i)(3,\, i-1)(4,\, i-2)\cdots\cdots(\nu,\, \nu+2) & (\text{if } i=2\nu) \\ (2,\, i)(3,\, i-1)(4,\, i-2)\cdots\cdots(\nu+1,\, \nu+) & (\text{if } i=2\nu+1)\end{cases}$$

is an isomorphism of $(\Delta_i, \Omega; \mu_i)$ with $(\Delta_i', \Omega; \mu_i')$.

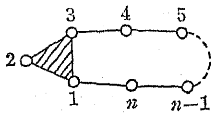
Suppose now $i=2$. Then the (0, 1)-matrix III_2^* is obtained from the (0, 1)-matrix I_2^* by interchanging the first and second row and column. Thus I_2^* is equivalent to III_2^*, q.e.d.

LEMMA 5. II_i^* *is equivalent to* III_{n-i+2}^* $(i=3, \cdots, n)$.

Proof. This is obvious since the oriented graph of II_i^* is obtained from that of III_{n-i+2}^* by rotating the vertices, q.e.d.

LEMMA 6. I_i^* *is equivalent to* I_j^* *if and only if* $i+j=n+1$.

Proof. Obviously I_i^* is equivalent to I_j^* if $i+j=n+1$. Conversely suppose now that I_i^* is equivalent to I_j^*. The complex $(\Delta_2, \Omega; \mu_2)$ is as follows:



Thus, by comparing the associated complex $\Delta_2, \cdots, \Delta_{n-1}$, we see that $i+j$ must be equal to $n+1$, q.e.d.

Thus we have seen that every II_i^* or III_i^* is equivalent to some I_i^* and that $\{\mathrm{I}_2^*, \cdots, \mathrm{I}_{n-1}^*\}$ is classified by Lemma 6. Therefore, a complete set of representatives of the equivalence classes of (0, 1)-matrices in question is given by $\{\mathrm{I}_2^*, \mathrm{I}_3^*, \cdots, \mathrm{I}_\nu^*\}$ (if $n=2\nu$) or by $\{\mathrm{I}_2^*, \mathrm{I}_3^*, \cdots, \mathrm{I}_{\nu+1}^*\}$ (if $n=2\nu+1$).

Hence the number of equivalence classes in the question is given by $\frac{n}{2}-1$ (if n is even) or by $\left[\frac{n-1}{2}\right]$, i.e. it is always equal to $\left[\frac{n-1}{2}\right]$ q.e.d.

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