On the Abstract Cauchy Problems and Semi-groups of Linear Operators in Locally Convex Spaces

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Introduction and Summary.

In this paper the author presents a unified treatment of the theory of distribution semi-groups and that of locally equicontinuous semi-groups. These theories are complementary to each other. To explain this situation, let us consider the Cauchy problem for the following evolution equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t, x)=P(x, D)u(t, x), \quad t>0, \quad x\in R^n, \\ u(0, x)=\varphi(x), \end{cases}$$

where $P(x, D)$ is a linear differential operator independent of t. The well-known Hille-Yosida theorem has attained great success in solving this problem by the method of functional analysis. In applying this theorem to this problem, we must choose a suitable Banach space X such as $L^p(R^n)$, and check the condition on the resolvent $(\lambda-A)^{-1}$ of the operator A in X which is a realization of $P(x, D)$. After these processes, the solution operator $\{T_t: t\geqslant 0\}$, called a semi-group of class C_0, will be obtained. Formally we have a representation:

$$T_t=\frac{1}{2\pi i}\int e^{t\lambda}(\lambda-A)^{-1}d\lambda, \tag{0. 1}$$

which can be considered as the inverse Laplace transform of $(\lambda-A)^{-1}$. But if one cannot check the condition on the resolvent in the Banach space chosen by him, there are, probably, two alternative ways of interpreting the formula (0. 1). The one is to understand the continuity of T_t with respect to t in a more generalized sense. The other is to change the Banach space X by the other topological vector space E to the effect that the operator $P(x, D)$ generates a semi-group of class C_0 in E. The notion of the distribution semi-groups of Lions [11] is the typical one of the former way. There have been several works concerning this subject (Foias [6], Yoshinaga [23], [24], Da Prato-Mosco [3], [4], Fujiwara [7], Barbu [1], Ushijima [20], etc.). Above all, Chazarain [2] characterized the generator of a distribution semi-group in terms of its resolvent. As for the latter way, there has been sufficient literature also. For example, the theory of equicontinuous semi-groups in locally convex spaces was written in the textbook of Yosida [25] (see also Schwartz [16], Miyadera [12], and Komatsu [9]). Kōmura [10] treated the theory of continuous semi-groups in locally convex spaces. It is to be noted that generators of such semi-groups have not necessarily resolvents because of the generality of spaces. Therefore, she introduced the generalized resolvent via the generalized Laplace transform, and gave a characterization of the generators of locally equicontinuous semi-groups in locally convex sequentially complete spaces.

Preparing the forthcoming paper [22], the author felt deeply the necessity of describing the theory of locally equicontinuous semi-groups in terms of distribution semi-groups, which will be done in this work. For the sake of self-containedness, proofs of the results will be given completely as far as possible even if some of the results are known theorems or straightforward generalizations of them. The present results will be effectively used in [22], where we will also characterize the distribution semi-groups in Banach spaces in terms of locally equicontinuous semi-groups.

Although Lions' original definition of distribution semi-groups was given in Banach spaces, this can be extended to the case in general locally convex spaces. Then we will reach the concept of the well-posedness of a linear operator for the Cauchy problem in the sense of distribution. An operator A in E is well-posed in this sense if and only if it has the generalized resolvent in the slightly modified sense of Kōmura's definition. The present discussions will be carried out mainly under the assumption of the sequential completeness only (cf. Shiraishi-Hirata [18], Fattorini [5], where the quasi-completeness was assumed). There will be also some methodological differences from the treatment of Lions. Namely, the generator A will be considered to be a closed operator in E, while it was mainly considered to be a continuous operator from the domain $D(A)$ with the graph topology into E. And we will investigate the inhomogeneous equation:

$$\frac{d}{dt}u - Au = f \tag{0. 2}$$

in $\mathcal{D}'_+(E)$, the totality of E-valued distributions on R^1 whose supports are contained in $[0, \infty)$, while $\mathcal{D}'_+(E)$ means $L(\mathcal{D}_-, E)$ in Lions' theory ($\mathcal{D}_-$ is the totality of C^∞-functions on R^1 with supports limited from the right).

The outline of this article is as follows. In §1, after some notational preparation we will give it as the definition that A is well-posed if and only if for any $f \in \mathcal{D}'_+(E)$ there exists a unique $u \in \mathcal{D}'_+(E)$ satisfying the equation (0. 2) and depending continuously on f. It is to be noted that the well-posedness of A does not necessarily imply the denseness of the domain of A. An element $\mathcal{T} \in \mathcal{D}'(L(E))$ is said to be boundedly equicontinuous if $\{\mathcal{T}(\varphi): \varphi$ belongs to a bounded set of $\mathcal{D}\}$ forms an equicontinuous family in $L(E)$. Then, in the sequentially complete space E, the well-posedness of A is equivalent to the existence of a boundedly equicontinuous $\mathcal{T}\in\mathcal{D}'_+(L(E))$ such that $\frac{d}{dt}\mathcal{T}-A\mathcal{T}=\delta$ and that $A\mathcal{T}\supset\mathcal{T}A$ (§1). In Banach spaces, A is densely defined and well-posed if and only if A is the generator of a regular distribution semi-groups. In general spaces, this correspondence will be proved under some restrictions in §3. In §4, the locally equicontinuous semi-groups and their adjoint semi-groups will be discussed in our settings. We will add some consideration on the analyticity of such semi-groups. Finally, the proofs of some technical propositions and a modified version of Kōmura's fundamental theorem will be given in Appendix.

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§1. The well-posedness of the Cauchy problem in the sense of distribution.

Let $L(E, F)$ be the totality of continuous linear mappings from E to F, where E and F are separated locally convex spaces. Throughout the present paper the set $L(E, F)$ is considered to be a separated locally convex space with the topology of uniform convergence on bounded sets. Namely a base of neighbourhoods of 0 in the space $L(E, F)$ is formed by the family of sets:

$$U(p, B)=\left\{f\colon \sup_{x\in B} p(f(x))\leqslant 1\right\},$$

where p runs on all continuous semi-norms of F, and B runs on all bounded sets in E. The set $L(E, E)$ is denoted by $L(E)$.

Let us abbreviate Schwartz space $\mathcal{D}(R^1)$ (or $\mathcal{D}_{(0,\infty)}$, or $\mathcal{D}_{(-k,k)}$) by $\mathcal{D}$ (or $\mathcal{D}^+$, or $\mathcal{D}_k$). Consider the totality of E-valued distributions, $\mathcal{D}'(E)=L(\mathcal{D}, E)$. For any $f\in\mathcal{D}'(E)$, $f(\varphi)$ denotes its value in E at $\varphi\in\mathcal{D}$. Sometimes we need the value of f at the test function φ with parameters. To clarify the independent variable we use the notations such as $f_t(\varphi(t, s))$, $\mathcal{D}_t$, $\mathcal{D}_s$ and $\mathcal{D}_{t,s}$ $(=\mathcal{D}_t\hat{\otimes}\mathcal{D}_s=\mathcal{D}(R^2))$. The notions of support, differentiation, and multiplication by C^∞-functions for E-valued distributions are defined in the same fashion as in the case of scalar valued distributions. The support of f is denoted by supp(f). We have:

$$\left(\frac{d}{dt}f\right)(\varphi)=f\left(-\frac{d}{dt}\varphi\right) \text{ for any } \varphi\in\mathcal{D},$$

$$\text{if } \alpha \text{ is } C^\infty, \text{ then } (\alpha f)(\varphi)=f(\alpha\varphi) \text{ for any } \varphi\in\mathcal{D}.$$

Let $\mathcal{D}'_+(E)$ be the totality of elements of $\mathcal{D}'(E)$ whose supports are contained in $[0, \infty)$. It is a closed subspace of $\mathcal{D}'(E)$. Hereafter $\mathcal{D}'_+(E)$ is considered to be a locally convex space with the induced topology. For $E=C$ (complex number field), $\mathcal{D}'(C)$ (or $\mathcal{D}'_+(C)$) is denoted by $\mathcal{D}'$ (or $\mathcal{D}'_+$).

For any linear operator T, we denote its domain (or range, or null space) by $D(T)$ (or $R(T)$, or $N(T)$). If $D(T)\cup R(T)\subset E$, T is said to be in E.

Definition 1. 1. A linear operator A in E is said to be well-posed for the Cauchy problem at $t=0$ in the sense of distribution (well-posed, in abbreviation) if for any $f\in\mathcal{D}'_+(E)$ there exists a unique $u\in\mathcal{D}'_+(E)$ satisfying the following conditions:

(A. 1) $u(\varphi)\in D(A)$ for any $\varphi\in\mathcal{D}$,

(A. 2) the mapping: $f\to u$ belongs to $L(\mathcal{D}'_+(E))$,

(A. 3) $\left(\frac{d}{dt}u\right)(\varphi)-Au(\varphi)=f(\varphi)$ for any $\varphi\in\mathcal{D}$.

Let O be the zero operator in E: $Ox=0$ for any $x\in E$. Then the following proposition is clear.

Proposition 1. 1. *The operator O is well-posed in E. For any $f\in\mathcal{D}'_+(E)$, u in Definition* 1. 1 *is determined as*

$$u(\varphi)=f(\alpha\psi),\quad \psi(t)=\int_t^\infty \varphi(t)dt,$$

where $\alpha(t)$ is an arbitrary C^∞-function such that $\mathrm{supp}(\alpha)\subset[a,\infty)$, $a>-\infty$, *and that $\alpha(t)=1$ for $t\geqq 0$.*

Now we recall the facts concerning the generalized Laplace transform of distributions, which were discussed by T. Kōmura in [10]. Let $\hat{\varphi}(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\lambda t}\varphi(t)dt$ for complex λ and $\varphi\in\mathcal{D}$. Let $\boldsymbol{D}=\{\hat{\varphi}:\ \varphi\in\mathcal{D}\}$. The space $\boldsymbol{D}$ can be considered to be a separated locally convex space such that the mapping: $\varphi\to\hat{\varphi}$ is a homeomorphism from $\mathcal{D}$ onto $\boldsymbol{D}$. Let $\boldsymbol{D}'(E)=L(\boldsymbol{D}, E)$. There is a homeomorphism: $f\to\hat{f}$ from $\mathcal{D}'(E)$ onto $\boldsymbol{D}'(E)$ defined by the relation: $\hat{f}(\hat{\varphi})=f(\varphi)$ for any $\varphi\in\mathcal{D}$. The functional $\hat{f}$ is called the generalized Laplace transform of f. Let $\boldsymbol{D}'_+(E)=\{\hat{f}:\ f\in\mathcal{D}'_+(E)\}$. It is a closed subspace of $\boldsymbol{D}'(E)$ since $\mathcal{D}'_+(E)$ is the closed subspace of $\mathcal{D}'(E)$, and it is homeomorphic to $\mathcal{D}'_+(E)$ by the mapping: $\hat{f}\to f$. Let $F\in\mathcal{D}'$, and let $x\in E$. Define $f=F\otimes x$ by the equality: $(F\otimes x)(\varphi)=F(\varphi)x$ for any $\varphi\in\mathcal{D}$. It is clear that $f\in\mathcal{D}'(E)$ (or $\mathcal{D}'_+(E)$) if $F\in\mathcal{D}'$ (or $\mathcal{D}'_+$), and that $\hat{f}=\hat{F}\otimes x$. For example, we have $(\delta\otimes x)^\wedge=\hat{\delta}\otimes x=1\otimes x$.

For any linear operator A in E, we define the operator $\boldsymbol{A}$ in $\boldsymbol{D}'_+(E)$ induced by A through the relations:

$$D(\boldsymbol{A})=\{\hat{f}:\ f\in\mathcal{D}'_+(E),\ f(\varphi)\in D(A) \text{ for any } \varphi\in\mathcal{D}, \text{ and the mapping } g \text{ defined by } g(\varphi)=A(f(\varphi)) \text{ belongs to } \mathcal{D}'_+(E)\},$$

$$\boldsymbol{A}\hat{f}=\hat{g} \text{ for any } \hat{f}\in D(\boldsymbol{A}).$$

It is to be noted that a linear operator A in E is closed if and only if its induced operator $\boldsymbol{A}$ is closed in $\mathcal{D}'_+(E)$ (Cf. proposition 4. 3 of [10]), and that if

$x \in D(A)$, then for any $F \in \mathcal{D}'_+$, $\hat{F} \otimes x \in D(A)$ and $A(\hat{F} \otimes x) = \hat{F} \otimes Ax$.

We also define the operator $\boldsymbol{\lambda} \in L(\mathcal{D}'_+(E))$ by the relation:

$$(\boldsymbol{\lambda}\hat{f})(\hat{\varphi}) = \hat{f}(\lambda\hat{\varphi}) = f(-\varphi') = \left(\frac{d}{dt}f\right)(\varphi).$$

Definition 1. 2. A linear operator A in E is said to have a generalized resolvent if $(\boldsymbol{\lambda} - \boldsymbol{A})$ has the inverse $(\boldsymbol{\lambda} - \boldsymbol{A})^{-1}$ belonging to $L(\boldsymbol{D}'_+(E))$.

Combining Definition 1. 1 with 1. 2, we have

Proposition 1. 3. *A linear operator A in E is well-posed if and only if it has a generalized resolvent.*

The following propositions are consequences of the above definitions which are easily proved.

Proposition 1. 4. *A well-posed operator is closed.*

Proposition 1. 5. *If $(\boldsymbol{\lambda} - \boldsymbol{A})^{-1} = (\boldsymbol{\lambda} - \boldsymbol{B})^{-1}$, then $A = B$.*

Let $\boldsymbol{O}$ be the operator induced by O. Since O is well-posed by Proposition 1. 2, it has a generalized resolvent $(\boldsymbol{\lambda} - \boldsymbol{O})^{-1}$ by Proposition 1. 3. We denote $(\boldsymbol{\lambda} - \boldsymbol{O})^{-1}$ by $\boldsymbol{\lambda}^{-1}$.

Proposition 1. 6. *The following identities hold.*

(i) $\boldsymbol{\lambda}\boldsymbol{A}\hat{f} = \boldsymbol{A}\boldsymbol{\lambda}\hat{f}$ for $\hat{f} \in D(\boldsymbol{A})$.

(ii) $\boldsymbol{\lambda}(\boldsymbol{\lambda} - \boldsymbol{A})^{-1} = (\boldsymbol{\lambda} - \boldsymbol{A})^{-1}\boldsymbol{\lambda}$.

(iii) $\boldsymbol{A}(\boldsymbol{\lambda} - \boldsymbol{A})^{-1}\hat{f} = (\boldsymbol{\lambda} - \boldsymbol{A})^{-1}\boldsymbol{A}\hat{f}$ for $\hat{f} \in D(\boldsymbol{A})$.

(iv) $(\boldsymbol{\lambda} - \boldsymbol{A})^{-1} = \boldsymbol{\lambda}^{-1} + \boldsymbol{\lambda}^{-1}\boldsymbol{A}(\boldsymbol{\lambda} - \boldsymbol{A})^{-1}$.

(v) $(\boldsymbol{\lambda} - \boldsymbol{A})^{-1}\hat{f} = \boldsymbol{\lambda}^{-1}\hat{f} + \boldsymbol{\lambda}^{-2}\boldsymbol{A}\hat{f} + \cdots + \boldsymbol{\lambda}^{-n}\boldsymbol{A}^{n-1}\hat{f} + \boldsymbol{\lambda}^{-n}(\boldsymbol{\lambda} - \boldsymbol{A})^{-1}\boldsymbol{A}^n\hat{f}$ for $\hat{f} \in D(\boldsymbol{A}^n)$.

Proof. See Lemma 5. 3 and the proof of the equality (5. 17) of [10].

Remark. Instead of $\mathcal{D}'_+(E)$, Kōmura considered the space $\mathcal{D}'_a(E)$, the totality of elements of $L(\mathcal{D}_{(-\infty,a]}, E)$ vanishing on $\mathcal{D}_{(-\infty,0]}$, with the topology $L(\mathcal{D}_{(-\infty,a]}, E)$, where $\mathcal{D}_{(-\infty,a]}$ is the space of all functions in $\mathcal{D}$ with supports contained in $(-\infty, a]$ with the topology induced by $\mathcal{D}$. For any fixed $a>0$, the induced operator $\boldsymbol{A}$ and the generalized resolvent $(\boldsymbol{\lambda} - \boldsymbol{A})^{-1}$ were considered in $\boldsymbol{D}'_a(E)$, the image of $\mathcal{D}'_a(E)$ by the generalized Laplace transform. It is quite easy to see that some statements given in [10] such as Proposition 4. 3 or Lemma 5. 3 are true for the present setting.

§ 2. A characterization of the well-posedness.

Definition 2. 1. An $L(E)$-valued distribution $\mathcal{T}$ is said to be boundedly equicontinuous, if for any continuous semi-norm p on E and any bounded set B in $\mathcal{D}$, there exists a continuous semi-norm q on E satisfying that

$$p(\mathcal{T}(\varphi)x)\leqslant q(x) \qquad \text{for any } \varphi\in B \text{ and } x\in E.$$

For any $\mathcal{T}\in\mathcal{D}'(L(E))$, the mapping $\mathcal{T}(\varphi)x$ is a separately continuous bilinear mapping from $\mathcal{D}\times E$ into E. Hence if E is barreled, then every $\mathcal{T}\in\mathcal{D}'(L(E))$ is boundedly equicontinuous, for, a fortiori, $\mathcal{T}(\varphi)x$ is hypocontinuous (see Theorem 41. 2 of Treves [19]).

Theorem 2. 1. *Let a linear operator A be well-posed in a locally convex space E. Then A is closed, and there exists a boundedly equicontinuous $\mathcal{T}\in\mathcal{D}'_+(L(E))$ satisfying the following properties.*

($\mathcal{T}$. 1) *For any $x\in E$ and $\varphi\in\mathcal{D}$, $\mathcal{T}(\varphi)x\in D(A)$ and $\left(\frac{d}{dt}\mathcal{T}\right)(\varphi)x-A\mathcal{T}(\varphi)x=\delta(\varphi)x$.*

($\mathcal{T}$. 2) *For any $x\in D(A)$ and $\varphi\in\mathcal{D}$, $\mathcal{T}(\varphi)Ax=A\mathcal{T}(\varphi)x$.*

Proof. Let us define

$$\mathcal{T}(\varphi)x=(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}(1\otimes x)(\check{\varphi}) \tag{2. 1}$$

for any $x\in E$ and $\varphi\in\mathcal{D}$. Then $\mathcal{T}x$ $((\mathcal{T}x)(\varphi)=\mathcal{T}(\varphi)x)$ can be considered as an element of $\mathcal{D}'_+(E)$, since $u=\mathcal{T}x$ for $f=\delta\otimes x$ in Definition 1. 1. The condition (A. 2) is equal to the fact that for any continuous semi-norm p on E and any bounded set A of $\mathcal{D}$, there exist a continuous semi-norm q and a bounded set B satisfying

$$\sup_{\varphi\in A} p(u(\varphi))\leqslant\sup_{\varphi\in B} q(f(\varphi)).$$

Substituting $u=\mathcal{T}x$ and $f=\delta\otimes x$ into this inequality, we have

$$\sup_{\varphi\in A} p(\mathcal{T}(\varphi)x)\leqslant\sup_{\varphi\in B}|\varphi(0)|q(x). \tag{2. 2}$$

This inequality implies $\mathcal{T}(\varphi)\in L(E)$. Next we prove that $\mathcal{T}\in\mathcal{D}'(L(E))$. For an arbitrary continuous semi-norm p on E and bounded set B of E, let $V=\{T\in L(E)\colon \sup_{x\in B} p(Tx)\leqslant 1\}$, and let $U=\{\varphi\in\mathcal{D}\colon \sup_{x\in B} p(\mathcal{T}(\varphi)x)\leqslant 1\}$. Then clearly $\mathcal{T}U\subset V$. On the other hand the closed absolutely convex set U is a barrel, for it absorbs any element $\varphi\in\mathcal{D}$, namely we have

$$\sup_{x\in B} p(\mathcal{T}(\varphi)x)=C_\varphi<\infty,$$

since the set $\{\mathcal{T}x\colon x\in B\}$ is bounded in $\mathcal{D}'(E)$. Therefore U is a neighbourhood of zero in the barreled space $\mathcal{D}$. Hence we have that $\mathcal{T}\in\mathcal{D}'(L(E))$. Moreover the estimate (2. 2) implies that $\mathcal{T}$ is boundedly equicontinuous. Since $\mathcal{T}x \in \mathcal{D}'_+(E)$ for any $x \in E$, $\mathrm{supp}(\mathcal{T}) \subset [0, \infty)$. The condition ($\mathcal{T}$. 1) follows from the conditions (A. 1) and (A. 3). The condition ($\mathcal{T}$. 2) follows from (iii) of Proposition 1. 6.

The converse assertion holds if E is sequentially complete. Namely we have

Theorem 2. 2. *Let E be a locally convex sequentially complete space. Then a linear operator A in E is well-posed if and only if A is closed and there exists a boundedly equicontinuous $\mathcal{T} \in \mathcal{D}'_+(L(E))$ satisfying ($\mathcal{T}$. 1) and ($\mathcal{T}$. 2).*

The proof of 'if' part is based upon the following two Propositions in which the sequentially completeness of E is a priori assumed. Although the proofs of these Propositions are standard arguments, these facts consist of the crucial points of our theory. So we will give them in Appendix of this article.

Proposition 2. 1. *The algebraic tensor product $\boldsymbol{D}'_+ \otimes E$ is dense in $\boldsymbol{D}'_+(E)$. If A is a closed operator, then $\boldsymbol{D}'_+ \otimes D(A)$ is dense in $D(\boldsymbol{A})$ which is topologized by the graph topology of $\boldsymbol{A}$.*

Proposition 2. 2. *For any boundedly equicontinuous $\mathcal{T} \in \mathcal{D}'_+(L(E))$, there exists a unique convolution operator $\mathcal{T}* \in L(\mathcal{D}'_+(E))$ satisfying that for $f = F \otimes x$ with $F \in \mathcal{D}'_+$ and $x \in E$*

$$(\mathcal{T}*f)(\varphi) = \mathcal{T}_t(\alpha(t)F_s(\varphi(t+s)))x, \tag{2. 3}$$

where $\alpha(t)$ is an arbitrary C^∞-function such that $\mathrm{supp}(\alpha) \subset [a, \infty)$, $a > -\infty$ and that $\alpha(t) = 1$ for $t \geqslant 0$.

Proof of 'if' part of Theorem 2. 2. Let us define a linear operator $\boldsymbol{R}$ in $\boldsymbol{D}'_+(E)$ as

$$\boldsymbol{R}\hat{f} = (\mathcal{T}*f)^\wedge, \qquad \hat{f} \in \boldsymbol{D}'_+(E).$$

Then by Proposition 2. 2, $\boldsymbol{R} \in L(\boldsymbol{D}'_+(E))$. By (2. 3) and ($\mathcal{T}$. 1) we have

$$(\lambda - \boldsymbol{A})\boldsymbol{R}\hat{f} = \hat{f} \tag{2. 4}$$

for any $\hat{f} \in \boldsymbol{D}'_+ \otimes E$. And also by (2. 3), ($\mathcal{T}$. 1) and ($\mathcal{T}$. 2) we have

$$\boldsymbol{R}(\lambda - \boldsymbol{A})\hat{f} = \hat{f} \tag{2. 5}$$

for any $\hat{f} \in \boldsymbol{D}'_+ \otimes D(A)$. The closedness of $\boldsymbol{A}$ and the first statement of Proposition 2. 1 imply that (2. 4) holds for any $\hat{f} \in \boldsymbol{D}'_+(E)$. The second statement of Proposition 2. 1 assures that (2. 5) holds for any $\hat{f} \in D(\boldsymbol{A})$. Therefore we have $\boldsymbol{R} = (\lambda - \boldsymbol{A})^{-1}$, namely A has a generalized resolvent. By Proposition 1. 3, A is well-posed.

Now we examine the well-posedness of the dual of a well-posed operator.

Let $E'=L(E, C)$. We denote the elements of E' by $x', y', \cdots$, and write $\langle x, x'\rangle$ for the value of the linear form x' at the point x of E. For a densely defined linear operator A in E, define its dual operator A' through the relations:

$$D(A')=\{x': \text{there exists } y'\in E' \text{ such that } \langle Ax, x'\rangle=\langle x, y'\rangle \text{ for any } x\in D(A)\},$$

$$A'x'=y'.$$

Since $D(A)$ is dense in E, A' is uniquely defined and closed in E'.

Theorem 2. 3. *Let E be a barreled sequentially complete space. If a linear operator A in E is well-posed and densely defined, then A' is well-posed in E'.*

Proof. The space E' is sequentially complete since E is barreled. Let $T=\mathcal{T}(\varphi)$ for an arbitrarily fixed $\varphi\in\mathcal{D}$. It is clear that $T'\in L(E')$. For any $x'\in E'$, we have that $T'x'\in D(A')$ and $A'T'x'=(AT)'x'$. In fact, for any $x\in D(A)$ it holds that

$$\langle Ax, T'x'\rangle=\langle TAx, x'\rangle=\langle ATx, x'\rangle,$$

where the last equality follows from the condition ($\mathcal{T}$. 2) in Theorem 2. 1. If $x'\in D(A')$ we have

$$\langle Ax, T'x'\rangle=\langle ATx, x'\rangle=\langle Tx, A'x'\rangle=\langle x, T'A'x'\rangle.$$

This implies that $A'T'x'=T'A'x'$ if $x'\in D(A')$. Thus we have established the conditions ($\mathcal{T}$. 1) and ($\mathcal{T}$. 2) for A' and $\mathcal{T}'$ where $\mathcal{T}'$ is defined by $\mathcal{T}'(\varphi)=(\mathcal{T}(\varphi))'$ for $\varphi\in\mathcal{D}$. Now we show that $\mathcal{T}'$ can be considered as an element of $\mathcal{D}'(L(E'))$. The topology of $L(E')$ is determined by the semi-norm system $\{P_{B,B'}\}$:

$$P_{B,B'}(T')=\sup_{x\in B,\ x'\in B'}|\langle x, T'x'\rangle| \quad \text{for} \quad T'\in L(E'),$$

where B (and B') runs on all bonded sets in E (and E'). Hence we have

$$P_{B,B'}(\mathcal{T}(\varphi)')=\sup_{x\in B,\ x'\in B'}|\langle\mathcal{T}(\varphi)x, x'\rangle|=\sup_{x\in B} P_{B'}(\mathcal{T}(\varphi)x),$$

where $P_{B'}(x)=\sup_{x'\in B'}|\langle x, x'\rangle|$. Since E is barreled, $P_{B'}(x)$ is a continuous semi-norm on E. Since $\mathcal{T}\in\mathcal{D}'(L(E))$, for any $k>0$, there exist a constant C and an integer $n\geqslant 0$ such that

$$P_{B,B'}(\mathcal{T}(\varphi)')\leqslant C\sup_{0\leqslant j\leqslant n,\ t\in R^1}|\varphi^{(j)}(t)|$$

holds for any $\varphi\in\mathcal{D}_k$. Thus we have shown that $\mathcal{T}'\in\mathcal{D}'(L(E'))$. It is clear that $\text{supp}(\mathcal{T}')\subset[0, \infty)$. The bounded equicontinuity of $\mathcal{T}$ implies that the set $C=\{\mathcal{T}(\varphi)x: \varphi\in A, x\in B\}$ is bounded in E where A (and B) is a bounded set in $\mathcal{D}$ (and E). Then we have for any $\varphi\in A$

$$P_B(\mathcal{T}'(\varphi)x') = \sup_{x\in B} |\langle \mathcal{T}(\varphi)x,\ x'\rangle| \leqslant \sup_{y\in C} |\langle y,\ x'\rangle| = P_C(x').$$

This implies the bounded equicontinuity of $\mathcal{T}'$.

Now applying 'if' part of Theorem 2. 2, we have the assertion.

Remark. The topological properties, which were used in the above proof, are 1) the sequential completeness of E', and 2) the equicontinuity of the strongly bounded set in E'. Since E is assumed to be sequentially complete, every weakly bounded set in E' is strongly bounded. Hence every barrel in E is the polar of a strongly bounded set in E'. Therefore 2) implies that E is a barreled space. Then the property 1) is automatically satisfied.

In this direction, we have the following

Corollary. *Let E be a reflexive locally convex space* (i.e. *it is barrelled and every bounded set is relatively weakly compact*). *An operator A in E is well-posed with dense domain if and only if its dual operator A' in E' is well-posed with dense domain.*

Proof. Since a reflexive space is quasi-complete, E and E' are sequentially complete. Note that $D(A')$ is $\sigma(E', E)$ dense, which implies $\beta(E', E)$ dense by the reflexivity of E. Hence the assertion follows from Theorem 2. 3.

§ 3. Distribution semi-groups.

Throughout this paragraph, the sequential completeness is always assumed on a locally convex space E. Now we give the definition of distribution semi-groups under some modifications of the original one of Lions [11]. Almost all the results of this paragraph are essentially due to him.

Definition 3. 1. An $L(E)$-valued distribution $\mathcal{T}$ is said to be a distribution semi-group (D.S.G., in short), if $\mathcal{T}$ satisfies the following conditions.

($\mathcal{D}$. 1) $\mathcal{T}$ is boundedly equicontinuous and belongs to $\mathcal{D}'_+(L(E))$.

($\mathcal{D}$. 2) $\mathcal{T}(\varphi*\phi) = \mathcal{T}(\varphi)\mathcal{T}(\phi)$ if $\varphi,\ \phi\in\mathcal{D}^+$.

($\mathcal{D}$. 3) For any $x = \mathcal{T}(\phi)y$, where $\phi\in\mathcal{D}^+$ and $y\in E$, there exists an E-valued function $x(t)$ such that:

(i) $x(t) = 0$ for $t < 0$,

(ii) $x(0) = x$,

(iii) $x(t)$ is continuous for $t \geqslant 0$,

(iv) $\mathcal{T}(\varphi)x = \int_0^\infty \varphi(t)x(t)dt$ for any $\varphi\in\mathcal{D}$.

($\mathcal{D}$. 4) Let $\mathfrak{N}(\mathcal{T})=\bigcap_{\varphi\in\mathcal{D}^+} N(\mathcal{T}(\varphi))$, then $\mathfrak{N}(\mathcal{T})=\{0\}$.

Let $\mathfrak{R}(\mathcal{T})$ be the totality of finite linear combinations of elements of the set $\bigcup_{\varphi\in\mathcal{D}^+} R(\mathcal{T}(\varphi))$. Then a D.S.G. $\mathcal{T}$ is said to be a regular D.S.G. (R.D.S.G., in short) if it satisfies

($\mathcal{D}$. 5) $\mathfrak{R}(\mathcal{T})$ is dense in E.

Let $\mathcal{T}$ be a D.S.G. in E. For any $F\in\mathcal{D}'_+$ with compact support, we can define an operator $\mathcal{T}(F)$ in E by the relation

$$\mathcal{T}(F)x=\sum_{j=1}^{n}\mathcal{T}(F*\varphi_j)y_j \qquad \text{for} \quad x=\sum_{j=1}^{n}\mathcal{T}(\varphi_j)y_j\in\mathfrak{R}(\mathcal{T}).$$

By the argument due to Peetre [13], $\mathcal{T}(F)$ is uniquely determined and closable in E. In fact, for any $\varphi\in\mathcal{D}^+$ we have

$$\begin{aligned}\mathcal{T}(\varphi)\mathcal{T}(F)x&=\sum_{j=1}^{n}\mathcal{T}(\varphi)\mathcal{T}(F*\varphi_j)y_j\\&=\sum_{j=1}^{n}\mathcal{T}(F*\varphi_j*\varphi)y_j\\&=\sum_{j=1}^{n}\mathcal{T}(F*\varphi)\mathcal{T}(\varphi_j)y_j\\&=\mathcal{T}(F*\varphi)x,\end{aligned}$$

where we have used the condition ($\mathcal{D}$. 2) and the fact $F*\varphi\in\mathcal{D}^+$ for any $\varphi\in\mathcal{D}^+$. Therefore $\mathcal{T}(F)x$ is a uniquely defined linear operator by ($\mathcal{D}$. 4). Let $\{x_\alpha\}$ be a net in $\mathfrak{R}(\mathcal{T})$ satisfying that $\lim_\alpha x_\alpha=0$ and $\lim_\alpha \mathcal{T}(F)x_\alpha=z$. Then we have for any $\varphi\in\mathcal{D}^+$, $\lim_\alpha \mathcal{T}(\varphi)x_\alpha=0$, and $\mathcal{T}(\varphi)z=\lim_\alpha \mathcal{T}(\varphi)\mathcal{T}(F)x_\alpha=\lim_\alpha \mathcal{T}(\varphi*F)x_\alpha=0$. By ($\mathcal{D}$. 4) $z=0$, which implies that $\mathcal{T}(F)$ is closable.

Definition 3. 2. The closure A of $\mathcal{T}(-\delta')$ is called the generator of a D.S.G. $\mathcal{T}$, where δ' is the derivative of the Dirac measure δ.

Proposition 3. 1. *Let $\mathcal{T}$ be a D.S.G. in E with the generator A, and let $\bar{\mathfrak{R}}$ be the closure of $\mathfrak{R}(\mathcal{T})$ in E. Then A is an operator in $\bar{\mathfrak{R}}$, and $D(A)$ is dense in $\bar{\mathfrak{R}}$. Consequently $\mathcal{T}$ is an R.D.S.G. if and only if $D(A)$ is dense in E.*

Proof. Note that $\mathfrak{R}(\mathcal{T})$ is a core of A, and that $A\mathfrak{R}(\mathcal{T})\subset\mathfrak{R}(\mathcal{T})$.

Theorem 3. 1. *Let $\mathcal{T}$, E, A, $\bar{\mathfrak{R}}$ be as in Proposition 3. 1. Then A is densely defined and well-posed in $\bar{\mathfrak{R}}$. Let $\mathcal{T}^+$ be the restriction of $\mathcal{T}$ on $\bar{\mathfrak{R}}$. Namely $\mathcal{T}^+(\varphi)x=\mathcal{T}(\varphi)x$ if $x\in\bar{\mathfrak{R}}$ and $\mathcal{T}(\varphi)x\in\bar{\mathfrak{R}}$ for any $\varphi\in\mathcal{D}$. Then $\mathcal{T}^+$ is an R.D.S.G. in $\bar{\mathfrak{R}}$ with the generator A. The generator of an R.D.S.G. in E is densely defined and well-posed in E.*

Proof. The conditions $(\mathcal{D}.1)$, $(\mathcal{D}.2)$ and the sequential completeness of E imply that the function $x(t)$ in $(\mathcal{D}.3)$ is represented as

$$x(t)=Y(t)\mathcal{T}_s(\psi(s-t))y \tag{3.1}$$

where $Y(t)$ is the Heaviside function. In fact, the right hand side of (3.1) clearly satisfies the requirements (i) to (iii) in $(\mathcal{D}.3)$. Moreover we have for any $\varphi\in\mathcal{D}^+$

$$\mathcal{T}(\varphi)\mathcal{T}(\psi)y=\mathcal{T}(\varphi*\psi)y=\mathcal{T}_s\Big(\int_0^\infty\varphi(t)\psi(s-t)dt\Big)y$$

$$=\int_0^\infty\varphi(t)\mathcal{T}_s(\psi(s-t))ydt.$$

On the other hand we have by $(\mathcal{D}.3)$

$$\mathcal{T}(\varphi)\mathcal{T}(\psi)y=\int_0^\infty\varphi(t)x(t)dt.$$

Hence (3.1) holds. Using this representation, we have

$$\mathcal{T}(\varphi)\mathcal{T}(\psi)=\mathcal{T}((Y\varphi)*\psi)\qquad\text{for any } \varphi\in\mathcal{D} \text{ and } \psi\in\mathcal{D}^+, \tag{3.2}$$

for it holds that for any $y\in E$,

$$\mathcal{T}(\varphi)\mathcal{T}(\psi)y=\int_0^\infty\varphi(t)x(t)dt=\int_0^\infty\varphi(t)\mathcal{T}_s(\psi(s-t))ydt$$

$$=\mathcal{T}_s\Big(\int_0^\infty\varphi(t)\psi(s-t)dt\Big)y=\mathcal{T}((Y\varphi)*\psi)y.$$

Since $(Y\varphi)*\psi\in\mathcal{D}^+$ for $\varphi\in\mathcal{D}$ and $\psi\in\mathcal{D}^+$, the equality (3.2) implies that $\mathcal{T}(\varphi)$ transforms $\mathcal{R}(\mathcal{T})$ into $\mathcal{R}(\mathcal{T})$, hence $\bar{\mathcal{R}}$ into $\bar{\mathcal{R}}$. Therefore $\mathcal{T}^+$ is a D.S.G. in $\bar{\mathcal{R}}$. It is clear that the generator of $\mathcal{T}^+$ in $\bar{\mathcal{R}}$ coincides with A. Since $\overline{D(A)}=\bar{\mathcal{R}}$, by Proposition 3.1, $\mathcal{T}^+$ is an R.D.S.G. with the generator A.

Now we check the conditions $(\mathcal{T}.1)$ and $(\mathcal{T}.2)$ in Theorem 2.1 to the present A in $\bar{\mathcal{R}}$. First we show that for any $x\in\mathcal{R}(\mathcal{T})$ and $\varphi\in\mathcal{D}$,

$$\begin{cases}\mathcal{T}(\varphi)x\in D(A) \text{ and}\\ A\mathcal{T}(\varphi)x=-\mathcal{T}(\varphi')x-\varphi(0)x.\end{cases} \tag{3.3}$$

We may assume without loss of generality that $x=\mathcal{T}(\psi)y$ for some $\psi\in\mathcal{D}^+$ and $y\in E$. Then (3.2) implies that $\mathcal{T}(\varphi)x\in\mathcal{R}(\mathcal{T})\subset D(A)$, and that

$$A\mathcal{T}(\varphi)x=A\mathcal{T}((Y\varphi)*\psi)y$$

$$=-\mathcal{T}(((Y\varphi)*\psi)')y$$

$$=-\mathcal{T}((Y\varphi)'*\psi)y.$$

Since $(Y\varphi)'=Y\varphi'+\varphi(0)\delta$, it holds that

$$A\mathcal{T}(\varphi)x=-\mathcal{T}((Y\varphi')*\psi)y-\varphi(0)\mathcal{T}(\delta*\psi)y$$
$$=-\mathcal{T}(\varphi')\mathcal{T}(\psi)y-\varphi(0)\mathcal{T}(\psi)y.$$

Thus (3. 3) is valid for any $x\in\mathcal{R}(\mathcal{T})$ and $\varphi\in\mathcal{D}$. Since the operators in the right hand of (3. 3) are continuous on $\bar{\mathcal{R}}$ and A is closed, (3. 3) is valid for any $x\in\bar{\mathcal{R}}$ and $\varphi\in\mathcal{D}$. Namely we have $(\mathcal{T}.1)$. Also we have for $x=\mathcal{T}(\psi)y$ with $\psi\in\mathcal{D}^+$ and $y\in E$,

$$A\mathcal{T}(\varphi)x=-\mathcal{T}(((Y\varphi)*\psi)')y$$
$$=-\mathcal{T}((Y\varphi)*\psi')y$$
$$=-\mathcal{T}(\varphi)\mathcal{T}(\psi')y$$
$$=\mathcal{T}(\varphi)Ax.$$

Again by the closedness of A, $(\mathcal{T}.2)$ follows from this equality.

It is quite natural to imagine that the $L(E)$-valued distribution $\mathcal{T}$ in Theorem 2. 1 must be a D.S.G. with the generator A if A is well-posed in E. The author, however, can not prove the validity of this statement. In this direction, we can say

Proposition 3. 2. *If A is well-posed in E, then $\mathcal{T}$ in Theorem 2. 1 satisfies the conditions $(\mathcal{D}.1)$ to $(\mathcal{D}.3)$.*

Proof. The condition $(\mathcal{D}.1)$ is already proved in Theorem 2. 1. For any $\varphi\in\mathcal{D}^+$ and $y\in E$, let $x(t)=Y(t)\mathcal{T}_s(\varphi(s-t))y$. Then $x(t)$ satisfies the requirements (i) to (iii) in the condition $(\mathcal{D}.3)$. By the condition $(\mathcal{T}.1)$, we have that $x(t)\in D(A)$, and that

$$\frac{d}{dt}x(t)-Ax(t)=\delta\otimes y$$

as an identify in $\mathcal{D}'(E)$. In other word the generalized Laplace transform $\hat{x}$ of $x(t)$ belongs to $\boldsymbol{D}(\boldsymbol{A})$, and satisfies $\hat{x}=(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}(1\otimes y)$. Therefore we have $x(t)=\mathcal{T}y$. This implies the condition $(\mathcal{D}.3)$. Noticing the sequential completeness of E, we have for any $\psi\in\mathcal{D}$,

$$\mathcal{T}(\psi)\mathcal{T}(\varphi)y=\int_0^\infty\psi(t)x(t)dt$$
$$=\int_0^\infty\psi(t)\mathcal{T}_s(\varphi(s-t))ydt$$
$$=\mathcal{T}((Y\psi)*\varphi)y.$$

As a special case of this equality, we have the condition ($\mathcal{D}$. 2) for $\phi\in\mathcal{D}^+$.

Consider the following condition:

(3. 4) Any $f\in\mathcal{D}'(E)$ whose support is the origin can be represented as a finite sum of Dirac measure and its derivatives:

$$f=\sum_{j=0}^{n}\delta^{(j)}\otimes x_j, \qquad x_j\in E.$$

From the treatment of Lions, we can understand that the condition (3. 4) is one of sufficient conditions for ($\mathcal{D}$. 4) (see the proof of Proposition 3. 4). Of course the Banach space E satisfies (3. 4). The following Proposition is a corollary of this fact.

Proposition 3. 3. *If there exists a continuous norm p on E, then E has the property* (3. 4).

Proof. Let E_p be the completion of E by the norm p. Let $f\in\mathcal{D}'(E)$ with supp $(f)=\{0\}$. We may consider f as an element of $\mathcal{D}'(E_p)$. Since E_p is a Banach space, there exist an integer $n<\infty$ and $x_j\in E_p$ such that $f=\sum_{j=0}^{n}\delta^{(j)}\otimes x_j$ holds as an element of $\mathcal{D}'(E_p)$. The value $f(\varphi)$, however, belongs to E for any $\varphi\in\mathcal{D}$. This implies that $x_j\in E$ for $0\leqslant j\leqslant n$. Hence $f=\sum_{j=0}^{n}\delta^{(j)}\otimes x_j$ holds as an element of $\mathcal{D}'(E)$.

From this proposition, every countably normed space has the property (3. 4). Other examples of spaces E satisfying (3. 4) were investigated by Shiraishi-Hirata [18]. Here we give another formal extension of the result of Lions.

Definition 3. 3. An E-valued distribution $f\in\mathcal{D}'(E)$ is said to be of finite order on the interval $(-k, k)$, if there exist an integer $n\geqslant 0$ and an E-valued continuous function $g(t)$ on $[-k, k]$ satisfying that for any $\varphi\in\mathcal{D}_k$

$$f(\varphi)=(-1)^n\int_{-\infty}^{\infty}\varphi^{(n)}(t)g(t)dt.$$

The integer n is called the order of f on $(-k, k)$.

An $L(E)$-valued distribution $\mathcal{T}\in\mathcal{D}'(L(E))$ is said to be normal if for any $x\in E$, $\mathcal{T}x\in\mathcal{D}'(E)$ is of finite order on some interval $(-k, k)$.

A well-posed operator A is said to be normally well-posed, if, for any $f=\delta\otimes x$, $x\in E$, u in Definition 1. 1 is of finite order on some interval $(-k, k)$.

Remark. In the above definition, the interval $(-k, k)$ and the order of $\mathcal{T}x$ (or u) on it may depend on x. If E is a Banach space or a complete (DF) space, then every E-valued distribution is of finite order on any bounded interval $(-k, k)$ (see Schwartz [14], Propositions 23, 24).

Proposition 3. 4. *If a linear operator A in E is normally well-posed, then the relation*:

$$\mathcal{T}(\varphi)x=(\boldsymbol{\lambda}-A)^{-1}(1\otimes x)(\hat{\varphi})$$

determines a normal D.S.G.

Proof. By Definition 3. 3, $\mathcal{T}$ is normal. By Proposition 3. 2, $\mathcal{T}$ satisfies ($\mathcal{D}$. 1) to ($\mathcal{D}$. 3).

Let $x\in\mathcal{N}(\mathcal{T})$. Then $\operatorname{supp}(\mathcal{T}x)=\{0\}$. Since $\mathcal{T}x$ is of finite order on some interval $(-k, k)$, there exist an integer $n<\infty$ and $x_j\in E(0\leqslant j\leqslant n)$ satisfying that

$$\mathcal{T}x=\sum_{j=0}^{n}\delta^{(j)}\otimes x_j.$$

This can be proved just as the same as in the scalar case (see Schwartz [21], p. 100). Moreover $x_j\in D(A)$ for any j since $\mathcal{T}(\varphi)x\in D(A)$ for any $\varphi\in\mathcal{D}$. By the condition ($\mathcal{T}$. 1), we have the identity

$$\sum_{j=0}^{n}\delta^{(j)}\otimes Ax_j=\sum_{j=0}^{n}\delta^{(i+1)}\otimes x_j-\delta\otimes x.$$

From this, we have

$$0=x_n=x_{n-1}=\cdots\cdots\cdots=x$$

Thus ($\mathcal{D}$. 4) is proved.

We summarize the result concerning the normal R.D.S.G.

Theorem 3. 2. *A linear operator A in E is the generator of a normal R.D.S.G. if and only if A is normally well-posed with dense domain.*

Proof. The 'if' part follows from Propositions 3. 3 and 3. 1. The 'only if' part follows from Theorem 3. 1 and Definition 3. 3.

As for the dual of the generator of an R.D.S.G., we have

Theorem 3. 3. *Let E be a sequentially complete barreled space. Let $\mathcal{T}$ be an R.D.S.G. in E with the generator A. Then A is densely defined, A′ is well-posed in E′, and $\mathcal{T}'$ is a D.S.G. in E′. Let E^+ be the closure of $\mathcal{R}(\mathcal{T}')$ in E′, and let A^+ (and $\mathcal{T}^+$) be the restriction $A'|_{E^+}$ (and $\mathcal{T}'|_{E^+}$). Namely $D(A^+)=\{x'\in D(A'): x', A'x'\in E^+\}$, and $A^+x'=A'x'$ for $x'\in D(A^+)$. Then A^+ is the generator of $\mathcal{T}'$. And $\mathcal{T}^+$ is an R.D.S.G. in E^+ with the densely defined generator A^+. If E is reflexive, then an operator A in E is the generator of an R.D.S.G. if and only if A′ in E′ is the generator of an R.D.S.G.*

Proof. By Theorem 3. 1, A is well-posed and densely defined in E. So by Theorem 2. 3, A' is well-posed in E'. By Proposition 3. 2, $\mathcal{T}'$ has the properties ($\mathcal{D}$. 1) to ($\mathcal{D}$. 3). Moreover the property ($\mathcal{D}$. 5) for $\mathcal{T}$ implies the property ($\mathcal{D}$. 4) for $\mathcal{T}'$. Hence $\mathcal{T}'$ is a D.S.G. in E'.

Let A_+ be the generator of $\mathcal{T}'$, and let $\mathcal{T}^+$ be the restriction of $\mathcal{T}'$ on E^+. Then by Theorem 3. 1 again, A_+ is the generator of an R.D.S.G. $\mathcal{T}^+$ in E^+. And it has the generalized resolvent $\boldsymbol{R}^+$ such that $\boldsymbol{R}^+\hat{f}=\widehat{(\mathcal{T}^+*f)}$ for any $f\in\mathcal{D}'_+(E^+)$. On the other hand, the closedness of A^+ follows from the closedness of A'. Since A_+ coincides with A' on $\mathcal{R}(\mathcal{T}')$, we have $A_+\subset A^+$. Hence ($\mathcal{T}$. 1) for A_+ and $\mathcal{T}^+$ implies ($\mathcal{T}$. 1) for A^+ and $\mathcal{T}^+$. Moreover we have for any $\varphi\in\mathcal{D}$ and $x'\in D(A^+)$

$$\begin{aligned}\mathcal{T}^+(\varphi)A^+x' &= \mathcal{T}'(\varphi)A'x' = A'\mathcal{T}'(\varphi)x' \\ &= A'\mathcal{T}^+(\varphi)x' = A^+\mathcal{T}^+(\varphi)x'.\end{aligned}$$

Thus ($\mathcal{T}$. 2) also holds for A^+ and $\mathcal{T}^+$. By Theorem 2. 1 and its proof, the generalized resolvent of A^+ exists and coincides with $\boldsymbol{R}^+$. Hence by Proposition 1. 5, we have $A_+=A^+$.

The last statement follows from the fact that if E is reflexive, then the property ($\mathcal{D}$. 4) for $\mathcal{T}$ implies the property ($\mathcal{D}$. 5) for $\mathcal{T}'$.

§4. The locally equicontinuous semi-group and its analyticity.

Definition 4. 1. Let E be a separated locally convex space. A family of operators $\{T_\alpha\in L(E)\colon \alpha\in A\}$ is said to be equicontinuous, if for any continuous semi-norm p on E there exists a continuous semi-norm q on E such that

$$p(T_\alpha x)\leqslant q(x)$$

holds for any $x\in E$ and $\alpha\in A$.

When the index set A is a topological space, the family $\{T_\alpha\colon \alpha\in A\}$ is said to be locally equicontinuous, if for any compact subset K of A the subfamily $\{T_\alpha\colon \alpha\in K\}$ is equicontinuous.

Definition 4. 2. A family $\{T_t\colon t\geqslant 0\}$ in $L(E)$ is called a semi-group, if it satisfies the following conditions:

(T_0. 1) $T_tT_s=T_{t+s}$ for any t, $s>0$,

(T_0. 2) $T_0=1$ (the identity operator),

(T_0. 3) $\lim_{t\to s} T_tx=T_sx$ for any $s\geqslant 0$ and $x\in E$.

It is noted that every semi-group $\{T_t\}$ is locally equicontinuous if the space E is barrelled (see Proposition 1. 1 of Kōmura [10]).

The generator A of a semi-group $\{T_t\}$ is defined by

$$Ax=\lim_{h\downarrow 0} h^{-1}(T_h-1)x$$

whenever the limit exists in E.

Now we state a characterization for the generator of a locally equicontinuous semi-group, which corresponds to Theorem 2. 2. Throughout the rest of this paragraph the sequentially completeness of the space E will be always assumed.

Proposition 4. 1. *A linear operator A in E is the generator of a locally equicontinuous semi-group if and only if A is a densely defined closed operator and there exists a locally equicontinuous family $\{T_t: t\geqslant 0\}$ in $L(E)$ satisfying the conditions $(T_0. 2)$, $(T_0. 3)$ and the following:*

(A_0) *For any $x\in D(A)$ and $t\geqslant 0$,*

$$T_t x\in D(A) \text{ and } \frac{d}{dt} T_t x = A T_t x = T_t A x.$$

In this case, $\{T_t: t\geqslant 0\}$ becomes the locally equicontinuous semi-group with the generator A. The semi-group $\{T_t: t\geqslant 0\}$ can be considered as a normal R.D.S.G. $\mathcal{T}$ with the generator A, where $\mathcal{T}$ is defined by

$$\mathcal{T}(\varphi)x=\int_0^\infty \varphi(t) T_t x dt \quad \text{for any } \varphi\in\mathcal{D} \text{ and } x\in E. \tag{4. 1}$$

Proof. **'Only if' part.** Let $\{T_t\}$ be the locally equicontinuous semi-group whose generator is A. So as to have the statement, it suffices to prove that A is densely defined and closed satisfying the condition (A_0) for $\{T_t\}$. Although this has been shown in Kōmura [10], we prove here in terms of E-valued distributions. By definition, we have easily

$$\left(\frac{d}{dt}\right)^+ T_t x = A T_t x = T_t A x \tag{4. 2}$$

for any $x\in D(A)$ and $t\geqslant 0$, where $\left(\frac{d}{dt}\right)^+$ means the right derivative. Let $\varphi\in\mathcal{D}$. Then we have

$$\int_0^\infty \varphi(t)\left(\frac{d}{dt}\right)^+ T_t x\, dt=\int_0^\infty \varphi(t) T_t A x\, dt.$$

After some elementary calculation, we have

$$\text{the left hand side } = -\int_0^\infty \varphi'(t) T_t x\, dt-\varphi(0)x,$$

and

$$\text{the right hand side } =\int_0^\infty \varphi'(t)\int_0^t T_s A x ds\, dt.$$

Therefore, we have

$$-\int_0^\infty \varphi'(t)\Big(T_t x - \int_0^t T_s Axds\Big)\ dt = \varphi(0)x.$$

Let f be an element of $\mathcal{D}'_+(E)$ such that

$$f(t) = \begin{cases} T_t x - \int_0^t T_s Axds & \text{for} \quad t \geqslant 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

Then we have

$$\frac{d}{dt} f = \delta \otimes x$$

as an identity of an E-valued distribution. By Proposition 1. 1, $f = Y \otimes x$. Hence

$$T_t x - x = \int_0^t T_s Axds \tag{4. 3}$$

holds for any $t \geqslant 0$ and $x \in E$. Since $\{T_t : t \geqslant 0\}$ is locally equicontinuous, (4. 3) implies the closedness of A. Consider $\mathcal{T}(\varphi)x$ defined by (4. 1). Then for any $x \in E$, we have that $\mathcal{T}(\varphi)x \in D(A)$, for $\lim_{h \downarrow 0} h^{-1}(T_h - I)\mathcal{T}(\varphi)x = -\mathcal{T}(\varphi')x$, and that $\mathcal{T}(\varphi_j)x \to x$ where $\varphi_j \to \delta$ in measure. Therefore $D(A)$ is dense in E. Finally the condition (A_0) follows from (4. 3) and (4. 2).

'If' part. Consider $\mathcal{T}$ defined by (4. 1). By the local equicontinuity of $\{T_t\}$ and the condition $(T_0. 3)$, $\mathcal{T}$ is a boundedly equicontinuous element of $\mathcal{D}'_+(L(E))$. By the conditions (A_0) and $(\mathcal{T}_0. 2)$, the conditions $(\mathcal{T}. 1)$ and $(\mathcal{T}. 2)$ are satisfied for the present A and $\mathcal{T}$. Hence by Theorem 2. 2, A is well-posed. Moreover it is normally well-posed, for the Cauchy problem: $\frac{d}{dt}u - Au = \delta \otimes x$ in $\mathcal{D}'_+(E)$ has the solution $u = \mathcal{T}x$. Since A is densely defined by condition, we have, by Theorem 3. 2, that A is the generator of the normal R.D.S.G. $\mathcal{T}$. The condition $(\mathcal{D}. 2)$ for $\mathcal{T}$ implies the condition $(T_0. 1)$ for $\{T_t : t \geqslant 0\}$.

To make sure the discussion, we prepare the definition of an E-valued holomorphic function. Under this definition, almost all good properties on holomorphic functions, such as the Cauchy's integral theorem and the power series expansion theorem, are valid on E-valued holomorphic functions. It holds also that if a sequence of E-valued holomorphic functions on a domain converges uniformly on every compact subdomain, the limit function is holomorphic. For the proofs of these facts, see e.g. Komatsu [9].

Definition 4. 3. Let $f(z)=f(t+is)$ is an E-valued function defined on a domain in the complex plane. We call $f(z)$ holomorphic if it satisfies the conditions:

$(H.\,1)$ $f(z)$ is infinitely differentiable as an E-valued function of real variables t and s.

$(H.\,2)$ It satisfies the Cauchy-Riemann differential equation:

$$\frac{\partial}{\partial \bar{z}}f(z)=\frac{1}{2}\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial s}\right)f(z)=0.$$

The following Theorem is a slight variant of the fundamental result due to Kōmura (see Theorem 3′ of Kōmura [10]). We will give its proof in Appendix 3.

Theorem 4. 1. *A linear operator A in E is the generator of a locally equicontinuous semi-group if and only if A is densely defined and well-posed such that there exists an $L(E)$-valued entire function $R_1(\lambda)$ satisfying the following two conditions*:

$(R_0.\,1)$
$$\left\{\frac{\mu^{n+1}}{n!}\frac{d^n}{d\mu^n}R_1(\mu):\quad \mu>0,\quad n=0, 1, 2, \cdots\right\}$$

is equicontinuous.

$(R_0.\,2)$
$$(\lambda-A)^{-1}(1\otimes x)(\varphi)=\frac{1}{i}\int_{\omega-i\infty}^{\omega+i\infty}\hat{\varphi}(\lambda)R_1(\lambda)x d\lambda$$

holds for any $x\in E$ and $\varphi\in\mathcal{D}_1$ and $\omega>0$.

Now we rewrite Theorem 3. 3 to the case of the dual semi-group of a semi-group.

Proposition 4. 2. *Let E be a sequentially complete barreled space. Let $\{T_t\colon t\geqslant 0\}$ be a semi-group with the generator A. Let A^+ be the generator of the D.S.G. $\mathcal{T}'$ in E', where $\mathcal{T}$ is defined by* (4. 1). *Let E^+ be the closure of $\mathcal{R}(\mathcal{T}')$ in E'. Then $\{T_t^+\colon t\geqslant 0\}$ forms a locally equicontinuous semi-group in E^+ with the generator A^+, which coincides with the restriction A' to E^+.*

Proof. First we note that $\mathcal{T}$ is an R.D.S.G. with the generator A by Proposition 4. 1. Hence by Theorem 3. 3 $\mathcal{T}'$ is an R.D.S.G. Let $\mathcal{T}^+$ be the restriction of $\mathcal{T}'$ to E^+. Then $\mathcal{T}^+$ is an R.D.S.G. with the generator $A^+=A'|_{E^+}$.

The family of linear operators $\{T_t'\colon t\geqslant 0\}$ in $L(E')$ is locally equicontinuous, for $\{T_t\colon t\geqslant 0\}$ is locally equicontinuous since E is barreled. The family $\{T_t'\}$ satisfies the conditions $(T_0.\,1)$ and $(T_0.\,2)$ and the condition:

(4. 4) $$T_t'\mathcal{T}(\varphi)'=\mathcal{T}(\varphi)'T_t' \qquad \text{for any } \varphi\in\mathcal{D} \text{ and } t\geqslant 0,$$

for these conditions are satisfied by $\{T_t\}$ and $\mathcal{T}$. From (4.4), we have $T_t^+ \in L(E^+)$. The family $\{T_t^+: t \geqslant 0\}$ in $L(E^+)$ satisfies $(T_0.1)$ and $(T_0.2)$. It is quite easy to show that for any $t \geqslant 0$, $\varphi \in \mathcal{D}^+$ and $x' \in E'$, it holds

$$\langle x, T_t' \mathcal{T}'(\varphi) x' \rangle = \int_0^\infty \varphi(s-t) \langle T_s x, x' \rangle ds.$$

This implies that $T_t^+ x'$ is an E'-valued continuous function if $x' \in \mathcal{R}(\mathcal{T}')$. Since the family $\{T_t^+: t \geqslant 0\}$ is locally equicontinuous, $T_t^+ x'$ is continuous for $t \geqslant 0$ for any $x' \in E^+$. Namely $\{T_t^+\}$ be a locally equicontinuous semi-group in E^+. Moreover we have the representation:

$$\mathcal{T}^+(\varphi) x' = \int_0^\infty \varphi(t) T_t^+ x' dt \qquad \text{for any } \varphi \in \mathcal{D} \text{ and } x' \in E^+.$$

In fact, the right hand converges in E^+ and coincides with $\mathcal{T}'(\varphi)x'$. Therefore, the generator of $\{T_t^+: t \geqslant 0\}$ coincides with the generator A^+ of $\mathcal{T}^+$ by Proposition 4.1.

Now we can reprove the following Theorem, which has been already obtained by Kōmura [10] in a little more general form.

Theorem 4.2. *Let E be a sequentially complete barreled space. Let $\{T_t: t \geqslant 0\}$ be a semi-group with the generator A. Let E^+ be the totality of elements $x' \in E'$ such that $T_t' x'$ is continuous in E' for $t \geqslant 0$. Then E^+ coincides with the closure $\overline{D(A')}$. Let T_t^+ be the restriction T_t' to E^+. Then we have $\{T_t^+: t \geqslant 0\}$ forms a locally equicontinuous semi-group in E^+ such that its generator A^+ is the largest restriction of A' with range in E^+.*

If E is reflexive, then A' generates a locally equicontinuous semi-group $\{T_t': t \geqslant 0\}$ in E'.

Proof. The space E^+ defined in Proposition 4.2 is denoted by E_0^+. Clearly the present E^+ contains E_0^+. Conversely if $x' \in E^+$, then for any $\varphi \in \mathcal{D}$ the integral $\int_0^\infty \varphi(t) T_t' x' dt$ exists in E_0^+, and equals to $\mathcal{T}'(\varphi)x'$. Choose a sequence $\{\varphi_j\} \subset \mathcal{D}$ converging to δ in measure. Then $\mathcal{T}'(\varphi_j)x'$ converges to x'. Namely we have $E^+ = E_0^+$.

Let $x \in D(A)$ and $x' \in D(A')$. Then we have

$$\begin{aligned} \langle x, T_{t+h}' x' - T_t' x' \rangle &= \langle T_{t+h} x - T_t x, x' \rangle \\ &= \left\langle \int_t^{t+h} T_\sigma A x d\sigma, x' \right\rangle \\ &= \left\langle A \int_t^{t+h} T_\sigma x d\sigma, x' \right\rangle \\ &= \left\langle \int_t^{t+h} T_\sigma x d\sigma, A' x' \right\rangle \end{aligned}$$

Since $D(A)$ is dense in E, we have for any $x\in E$ and $x'\in D(A')$,

(4. 5) $$\langle x, T'_{t+h}x'-T'_t x'\rangle=\left\langle\int_t^{t+h} T_\sigma x d\sigma,\ A'x'\right\rangle.$$

Since the set $\{T_\sigma x:\ t\leqslant\sigma\leqslant t+1,\ x\in B\}$ is bounded for any bounded set B, (4. 5) implies that $x'\in E^+$. Therefore we have that $\overline{D(A')}=E^+$. The remaining statements follow from Proposition 4. 2.

Remark. *A well-posed operator is not necessarily densely defined.* Consider the left translation in the space $L^p(R^1)$. Let A_p be the operator $\frac{d}{ds}$ in $L^p(R^1)$ $(1\leqslant p\leqslant\infty)$. Namely we have

$$D(A_p)=\left\{f\in L^p(R^1):\ f \text{ is absolutely continuous on any compact interval,}\right.$$
$$\left.\text{and } \frac{d}{ds}f\in L^p(R^1)\right\},$$
$$A_p f=\frac{d}{ds}f \text{ for } f\in D(A_p).$$

The operator A_∞ is closed, but not densely defined. Let us define

$$\mathcal{T}(\varphi)f(s)=\int_0^\infty \varphi(t)f(t+s)dt$$

for any $\varphi\in\mathcal{D}$ and $f\in L^\infty(R^1)$. Then $\mathcal{T}\in\mathcal{D}'_+(L^\infty(R^1))$ and satisfies ($\mathcal{T}$. 1) and ($\mathcal{T}$. 2) for A_∞. $\mathcal{T}$ is a D.S.G. by Theorem 3. 1. It is an R.D.S.G. in $E_+=\overline{\mathcal{R}(\mathcal{T})}$ with the generator A^+. What are A_+ and E_+? Incidentally we have that $A_\infty=(-A_1)'$, and that $-A_1$ is the generator of the semi-group of right translation in $L^1(R^1)$. Hence, by the proof of Theorem 4. 2, $E_+=E^+=$the totality of bounded functions which are uniformly continuous on R^1. Also we have $A_+=(-A_1)^+$, i.e.

$$D(A_+)=\{f(s)\in C^1(R^1), f, f'\in E^+\},$$
$$(A_+f)(s)=\frac{d}{ds}f(s).$$

Now we give some consideration on the analyticity of locally equicontinuous semi-groups. Let $\Sigma_\theta=\{z:\ |\arg z|<\theta\}$ for real θ, and let $\bar{\Sigma}_\theta$ be its closure.

Definition 4. 4. A family $\{T(z):\ z\in\bar{\Sigma}_\theta\}$ in $L(E)$ is called a holomorphic semi-group, if it satisfies the following conditions:

(T_h. 1) $T(z)T(z')=T(z+z')$ for any $z, z'\in\bar{\Sigma}_\theta$,

(T_h. 2) $T(0)=1$,

$(T_h. 3)$ $\lim_{z'\to z} T(z')x = T(z)x$ for any $z\in\bar{\Sigma}_\theta$ and $x\in E$,

$(T_h. 4)$ $T(z)x$ is an E-valued holomorphic function in Σ_θ for any $x\in E$.

Theorem 4. 3. *Let θ be such that $0<\theta<\pi/2$. Then the following three conditions are equivalent for a linear operator A in E.*

(I) *A is the generator of a locally equicontinuous semi-group $\{T_t: t\geqq 0\}$ having a holomorphic extension $\{T(z): z\in\bar{\Sigma}_\theta\}$, which is a locally equicontinuous family in $L(E)$ and satisfies that for any $x\in E$ $T(z)x$ is holomorphic in Σ_θ, and continuous on $\bar{\Sigma}_\theta$.*

(II) *A is a densely defined closed operator. And there exists a locally equicontinuous holomorphic semi-group $\{T(z): z\in\bar{\Sigma}_\theta\}$ satisfying that*

$(A_h. 1)$ *$T(z)x\in D(A)$ and $\frac{d}{dz}T(z)x = AT(z)x$ for any $x\in E$ and $z\in\Sigma_\theta-\{0\}$.*

$(A_h. 2)$ *$T(z)Ax = AT(z)x$ for any $x\in D(A)$ and $z\in\bar{\Sigma}_\theta$.*

(III) *Both $e^{i\theta}A$ and $e^{-i\theta}A$ generate locally equicontinuous semi-groups.*

Proof. (I)$\Longrightarrow$(II). By Proposition 4. 1, A is densely defined and closed. It suffices to show $(A_h. 1)$, $(A_h. 2)$ and $(T_h. 1)$ for the given holomorphic extension $\{T(z): z\in\bar{\Sigma}_\theta\}$. Let $z\in\Sigma_\theta-\{0\}$. Then we have

$$\frac{d}{dz}T(z)x = \frac{-1}{2\pi i}\int_C \frac{T(\zeta)x}{(\zeta-z)^2}\,d\zeta \qquad \text{for } x\in E,$$

where C is a sufficiently small circle enclosing z. By this representation, $\frac{d}{dz}T(z)$ can be considered as an element of $L(E)$ for $z\in\Sigma_\theta-\{0\}$. For $x\in D(A)$, two functions $T(z)Ax$ and $\frac{d}{dz}T(z)x$ are both holomorphic in z, and coincide with each other on the real positive line. Therefore, we have

$$(4.6)\qquad \frac{d}{dz}T(z)x = T(z)Ax \qquad \text{for any } x\in D(A) \text{ and } z\in\Sigma_\theta-\{0\}.$$

Next let $D(A^\infty) = \bigcap_{n=0}^{\infty} D(A^n)$. Since $R(\mathcal{T}(\varphi))\subset D(A)$ and $A\mathcal{T}(\varphi) = \mathcal{T}(-\varphi')$ for any $\varphi\in\mathcal{D}^+$ where $\mathcal{T}$ is the R.D.S.G. defined in Proposition 4. 1, $D(A^\infty)$ is dense in E. For any $x\in D(A^\infty)$, $T_t x$ is infinitely differentiable:

$$(4.7)\qquad \left(\frac{d}{dt}\right)^n T_t x = A^n T_t x = T_t A^n x \qquad \text{for } t\geqq 0.$$

For any $z\in\Sigma_\theta-\{0\}$, there exists a positive number t such that the following expansion is valid in the topology of E,

$$(4.8)\qquad T(z)x = \sum_{n=0}^{\infty}\frac{(z-t)^n}{n!}\left(\frac{d}{dt}\right)^n T_t x \qquad \text{for any } x\in E.$$

From the above two identities (4. 7) and (4. 8), we have that

(4. 9) $$T(z)Ax=AT(z)x \quad \text{for any } x\in D(A^\infty) \text{ and } z\in\Sigma_\theta-\{0\}.$$

By (4. 6) for any $x\in D(A^\infty)$, it holds that

(4. 10) $$AT(z)x=\frac{d}{dz}T(z)x \quad \text{for any } z\in\Sigma_\theta-\{0\}.$$

Since $\frac{d}{dz}T(z)\in L(E)$, the closedness of A and the denseness of $D(A^\infty)$ guarantee the validity of (4. 10) for any $x\in E$. Moreover, since $D(A^\infty)$ is a core of A, we have from (4. 9) for any $z\in\Sigma_\theta-\{0\}$

(4. 11) $$AT(z)x=T(z)Ax \quad \text{for any } x\in D(A).$$

By the closedness of A and the local equicontinuity of $\{T(z): z\in\bar{\Sigma}_\theta\}$, (4. 11) holds for any $z\in\bar{\Sigma}_\theta$. Namely we have $(A_h. 1)$ and $(A_h. 2)$. Hence we have for $z, z'\in\Sigma_\theta-\{0\}$ and $x\in E$

$$\begin{aligned}&T(z+z')x-T(z)T(z')x\\ &=\int_{z'}^{z+z'}\frac{d}{d\zeta}T(z+z'-\zeta)T(\zeta)xd\zeta\\ &=\int_{z'}^{z+z'}T(z+z'-\zeta)(A-A)T(\zeta)xd\zeta\\ &=0\end{aligned}$$

Thus the identity

$$T(z)T(z')=T(z\pm z')$$

holds for $z, z'\in\Sigma_\theta-\{0\}$, and for $z, z'\in\bar{\Sigma}_\theta$ by continuity.

(II)$\Longrightarrow$(III). Let $T_{\theta'}(t)=T(te^{i\theta'})$ for $t\geqslant 0$ and $|\theta'|\leqslant\theta$. Then $\{T_{\theta'}(t): t\geqslant 0\}$ is a locally equicontinuous semi-group. If $|\theta'|<\theta$, conditions $(A_h. 1)$ and $(A_h. 2)$ imply that

$(A_0. \theta')$ $$\begin{cases}\text{for any } x\in D(A) \text{ and } t\geqslant 0,\ T_{\theta'}(t)x\in D(A)=D(e^{i\theta'}A) \text{ and}\\ \dfrac{d}{dt}T_{\theta'}(t)x=e^{i\theta'}AT_{\theta'}(t)x=T_{\theta'}(t)(e^{i\theta'}A)x.\end{cases}$$

Tending θ' to $\pm\theta$, we have the relation $(A_0. \theta')$ for $\theta'=\pm\theta$. Applying Proposition 4. 1 to $e^{\pm i\theta}A$, we have the conclusion (III).

(III)$\Longrightarrow$(I). Let $T_+(t)$ (and $T_-(t)$) be a locally equicontinuous semi-group generated by $e^{i\theta}A$ (and $e^{-i\theta}A$). First we show that

(4. 12) $$T_+(t)T_-(s)x=T_-(s)T_+(t)x \quad \text{for any } x\in E \text{ and } t, s\geqslant 0.$$

If $x\in D(A)$, as functions of t, both sides of (4.12) represent solutions of the Cauchy problem:

$$\frac{d}{dt}x(t)=e^{i\theta}Ax(t),\ x(0)=T_-(s)x.$$

Since $e^{i\theta}A$ is well-posed, the solution is unique. Hence (4.12) holds for $x\in D(A)$. By continuity it holds for any $x\in E$. Now noticing that $z\in\bar{\Sigma}_\theta$ can be represented uniquely as $z=te^{i\theta}+se^{-i\theta}$ with $t,\ s\geqslant 0$, we define

$$T(z)=T(te^{i\theta}+se^{-i\theta})=T_+(t)T_-(s).$$

By (4.12) and the local equicontinuity of $\{T_+(t)\}$ and $\{T_-(t)\}$, the family of operators $\{T(z):\ z\in\bar{\Sigma}_\theta\}$ is locally equicontinuous satisfying (T_h. 1) to (T_h. 3). To prove (T_h. 4), it suffices to show that $T(z)x$ is holomorphic for $x\in D(A)$. By an elementary calculation, the Cauchy-Riemann operator takes the following form:

$$\frac{\partial}{\partial\bar{z}}=\frac{i}{\sin 2\theta}\left\{e^{-i\theta}\frac{\partial}{\partial t}-e^{i\theta}\frac{\partial}{\partial s}\right\}.$$

Since $\frac{\partial}{\partial t}T_+(t)T_-(s)x=e^{i\theta}AT_+(t)T_-(s)x$ and $\frac{\partial}{\partial s}T_+(t)T_-(s)x=T_+(t)e^{-i\theta}AT_-(s)x$ $=e^{-i\theta}AT_+(t)T_-(s)x$ if $x\in D(A)$, we have that $\frac{\partial}{\partial\bar{z}}T(z)x=0$. Therefore $T(z)x$ is holomorphic, and satisfies that for any $x\in D(A)$ and $z\in\bar{\Sigma}_\theta$, $T(z)x\in D(A)$ and

$$\frac{d}{dz}T(z)x=AT(z)x=T(z)Ax.$$

Applying Proposition 4.1 to the above situation again, we have that A is the generator of a locally equicontinuous semi-group $\{T(t):\ t\geqslant 0\}$.

Appendix.

1. Proof of Proposition 2.1.

First we show that $\mathcal{D}'_+\otimes E$ (and $\boldsymbol{D}'_+\otimes E$) is dense in $\mathcal{D}'_+(E)$ (and $\boldsymbol{D}'_+(E)$). This can be executed through two steps. The 1st step is to show that the totality of E-valued C^∞-functions with supports contained in $[0,\infty)$, which is denoted by $C^\infty_+(E)$, is dense in $\mathcal{D}'_+(E)$. The 2nd step is to show that the closure of $\mathcal{D}'_+\otimes E$ in $\mathcal{D}'_+(E)$ contains $C^\infty_+(E)$.

It must be noted that an E-valued piece-wise continuous function f is considered to be an element of $\mathcal{D}'(E)$ by the formula:

$$f(\varphi)=\int_{-\infty}^{\infty}\varphi(t)f(t)dt\qquad\text{for any }\varphi\in\mathcal{D},$$

where the integral in the right hand is convergent in E in the sense of

Riemann since E is sequentially complete. The 2nd step is easily obtained by the approximation of $f\in C_+^\infty(E)$ by a sequence of step functions $\{f_n\}$. For example we choose f_n as

$$f_n(t)=\sum_{j=0}^{n2^n}\chi_{n,j}(t)f(j2^{-n}),$$

where $\chi_{n,j}(t)$ is the characteristic function of the interval $[j2^{-n}, (j+1)2^{-n})$. Then we have

$$f_n(\varphi)-f(\varphi)=\int_{-\infty}^{\infty}\varphi(t)(f_n(t)-f(t))dt$$

converges 0 in E uniformly with respect to φ belonging to a bounded set of $\mathcal{D}$. This implies that $f_n\to f$ in $\mathcal{D}'_+(E)$.

Now we proceed to the 1st step. Choose an even function $\rho(t)\in\mathcal{D}$ satisfying that supp $(\rho)\subset(-1, 1)$, $\rho\geqslant 0$ and $\int_{-1}^{1}\rho(t)dt=1$. Let $\rho_\varepsilon(t)=\varepsilon^{-1}\rho(\varepsilon^{-1}t)$ for $\varepsilon>0$. For any $f\in\mathcal{D}'(E)$, define $\rho_\varepsilon*f\in\mathcal{D}'(E)$ as

$$(\rho_\varepsilon*f)(\varphi)=f(\rho_\varepsilon*\varphi) \qquad \text{for any } \varphi\in\mathcal{D}.$$

Then $\rho_\varepsilon*f$ converges to f in $\mathcal{D}'(E)$ as ε tends to 0 by definition. On the other hand $f_t(\rho_\varepsilon(t-s))$ is an E-valued C^∞-function. Moreover it is equal to $\rho_\varepsilon*f$. In fact, we have by the sequential completeness of E

$$\begin{aligned}
&\int_{-\infty}^{\infty}\varphi(s)f_t(\rho_\varepsilon(t-s))ds\\
&=\lim\sum_j\varphi(s_j)\Delta s_j f_t(\rho_\varepsilon(t-s_j))\\
&=\lim f_t\Big(\sum_j\varphi(s_j)\rho_\varepsilon(t-s_j)\Delta s_j\Big)\\
&=f_t\Big(\lim\sum_j\rho_\varepsilon(t-s_j)\varphi(s_j)\Delta s_j\Big)\\
&=f_t\Big(\int_{-\infty}^{\infty}\rho_\varepsilon(t-s)\varphi(s)ds\Big)\\
&=(\rho_\varepsilon*f)(\varphi),
\end{aligned}$$

where we have used, for the validity of the 3rd equality, the fact that the finite Rieman sum $\sum_j\rho_\varepsilon(t-s_j)\varphi(s_j)\Delta s_j$ converges to $(\rho_\varepsilon*\varphi)(t)$ in $\mathcal{D}_t$. Next we define a shift operator τ_t in $\mathcal{D}'(E)$:

$$(\tau_t f)(\varphi)=f_s(\varphi(s+t)) \qquad \text{for any } \varphi\in\mathcal{D}.$$

Let us define $f_\varepsilon=\tau_\varepsilon(\rho_\varepsilon * f)$ for $\varepsilon>0$. Then $f_\varepsilon\in\mathcal{D}'_+(E)$ if $f\in\mathcal{D}'_+(E)$. In order to complete the step 1, it suffices to show that f_ε converges to f in $\mathcal{D}'(E)$ as $\varepsilon\downarrow 0$. But this follows from the facts that $\{\rho_\varepsilon * f: 0<\varepsilon\leqslant 1\}$ forms an equicontinuous set in $\mathcal{D}'(E)$, and that $\tau_t f$ converges to f uniformly on an equicontinuous set of $\mathcal{D}'(E)$.

The fact that $\boldsymbol{D}'_+\otimes D(A)$ is dense in $D(\boldsymbol{A})$ follows also from the above proof, for it holds that $D(\boldsymbol{A})=\boldsymbol{D}'_+(D(A))$ where $D(A)$ is considered to be a sequentially complete space topologized by the graph topology of A.

2. Proof of Proposition 2. 2.

If E is a quasi-complete barreled space, then Proposition 2. 2 is a corollary of the general result concerning the convolution of vector-valued distributions (see Proposition 39 of Schwartz [17]). It seems to need some words for the extension of the result to a sequentially complete space. This is based upon the following Proposition (cf. Proposition 33 of [17]).

Proposition A. 1. *Let E be a locally convex sequentially complete space. Then for any boundedly equicontinuous $\mathcal{T}\in\mathcal{D}'(L(E))$, there exists a unique product operator $\mathcal{T}\otimes\in L(\mathcal{D}'(E), L(\mathcal{D}(R^2), E))$ satisfying the following conditions*:

(1) $(\mathcal{T}\otimes f)_{s,t}(\varphi(s)\otimes\psi(t))=\mathcal{T}(\varphi)f(\psi)$ *for* $\varphi\in\mathcal{D}_s$ *and* $\psi\in\mathcal{D}_t$.

(2) $(\mathcal{T}\otimes f)(\theta)=\mathcal{T}_s(F_t(\theta(s, t))x$ *for* $f=F\otimes x$ *with* $F\in\mathcal{D}'$, $x\in E$ *and* $\theta\in\mathcal{D}(R^2)$.

(3) $\mathrm{supp}\,(\mathcal{T}\otimes f)\subset\mathrm{supp}\,(\mathcal{T})\times\mathrm{supp}\,(f)$.

Proof. Define $\mathcal{T}\otimes f$ on $\mathcal{D}_s\otimes\mathcal{D}_t$ by the relation (1). Since $\mathcal{T}\otimes f$ can be considered as a continuous bilinear mapping from $\mathcal{D}_s\times\mathcal{D}_t$ to E, it has a unique extension which is denoted by the same symbol,

$$\mathcal{T}\otimes f\in L(\mathcal{D}\,\widehat{\otimes}_\pi\mathcal{D},\ \hat{E})=L(\mathcal{D}(R^2),\ \hat{E})$$

where the symbol $\wedge$ means the completion (see e.g. Proposition 43. 4 of [19]). Let K be a bounded set in $\mathcal{D}(R^2)$. Then there exist bounded sets A and B in $\mathcal{D}$ such that any $\theta\in K$ can be represented as

$$\theta(s, t)=\sum_{n=1}^{\infty}\lambda_n\varphi_n(s)\psi_n(t)$$

where $\sum_{n=1}^{\infty}\lambda_n\leqslant 1$ with $\lambda_n\geqslant 0$, $\varphi_n\in A$, $\psi_n\in B$, and $\sum_{n=1}^{\infty}$ is taken in the topology of $\mathcal{D}(R^2)$ (see Corollary 2 of Theorem 45. 2 of [19]). Hence we have for such θ

$$(\mathcal{T}\otimes f)(\theta)=\sum_{n=1}^{\infty}\lambda_n\mathcal{T}(\varphi_n)f(\psi_n).$$

The right hand is in E because of the sequential completeness of E. Therefore

$\mathcal{T}\otimes f\in L(\mathcal{D}(R^2), E)$. The properties (2) and (3) are easy consequences of the above mentioned facts. Let p be a continuous semi-norm on E. Since $\mathcal{T}$ is boundedly equicontinuous, there exists a continuous semi-norm q such that $p(\mathcal{T}(\varphi)x)\leqslant q(x)$ for any $\varphi\in A$ and $x\in E$. Then we have for $\theta\in K$

$$p((\mathcal{T}\otimes f)(\theta))\leqslant \sum_{n=1}^{\infty}\lambda_n p(\mathcal{T}(\varphi_n)f(\psi_n))$$

$$\leqslant \sum_{n=1}^{\infty}\lambda_n q(f(\psi_n))$$

$$\leqslant \sup_{\psi\in B} q(f(\psi)).$$

This estimate implies that $\mathcal{T}\otimes$ is a continuous linear mapping from $\mathcal{D}'(E)$ into $L(\mathcal{D}(R^2), E)$.

To prove Proposition 2. 2, it suffices to define

$$(\mathcal{T}*f)(\varphi)=(\mathcal{T}\otimes f)(\alpha(t)\varphi(t+s)\alpha(s)) \qquad \text{for any } \varphi\in\mathcal{D},$$

for the mapping:

$$\varphi(t)\rightarrow\alpha(t)\varphi(t+s)\alpha(s)$$

is continuous from $\mathcal{D}$ into $\mathcal{D}(R^2)$.

3. Proof of Theorem 4. 1.

'Only if' part. Proposition 4. 1 implies that A is densely defined and well-posed. Let

$$R_1(\lambda)x=\int_0^1 e^{-\lambda t}T_t x dt.$$

Then $R_1(\lambda)$ is an $L(E)$-valued entire function satisfying the condition (R_0. 1). Moreover for $\omega>0$ and $\varphi\in\mathcal{D}_1$, we have

$$\frac{1}{i}\int_{\omega-i\infty}^{\omega+i\infty}\hat{\varphi}(\lambda)R_1(\lambda)xd\lambda$$

$$=\frac{1}{i}\int_{\omega-i\infty}^{\omega+i\infty}\hat{\varphi}(\lambda)\int_0^1 e^{-\lambda t}T_t x dt d\lambda$$

$$=\int_0^1\left(\frac{1}{i}\int_{\omega-i\infty}^{\omega+i\infty}\hat{\varphi}(\lambda)e^{-\lambda t}d\lambda\right)T_t x dt$$

$$=\int_0^1\varphi(t)T_t x dt$$

$$= \mathcal{T}(\varphi)x$$
$$= (\boldsymbol{\lambda}-\boldsymbol{A})^{-1}(1\otimes x)(\hat{\varphi}).$$

'If' part. Noticing the following expansion

$$R_1(\lambda)x = \sum_{n=0}^{\infty} \frac{1}{n!}(\lambda-\mu)^n \left[\left(\frac{d}{d\lambda}\right)^n R_1(\lambda)\right]_{\lambda=\mu} x$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda-\mu)^n}{\mu^{n+1}} \left[\frac{\mu^{n+1}}{n!}\left(\frac{d}{d\mu}\right)^n R_1(\mu)\right] x,$$

we can conclude, by the condition (R_0. 1), that for any $\omega>0$, the family $\{R_1(\lambda)$: $\mathrm{Re}\,\lambda\geqslant\omega\}$ in $L(E)$ is equicontinuous. Therefore we can define an $L(E)$-valued continuous function $G(t)$ by the formula:

$$G(t) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{e^{t\lambda}}{\lambda^2} R_1(\lambda)d\lambda,$$

which vanishes for $t<0$. Consider $\mathcal{G} = D^2G\in\mathcal{D}'_+(L(E))$. Namely we have

$$\mathcal{G}(\varphi)x = \int_0^{\infty} \varphi''(t)G(t)x\,dt \qquad \text{for } \varphi\in\mathcal{D} \text{ and } x\in E. \tag{A. 1}$$

The condition (R_0. 2) implies that

$$\mathcal{G}(\varphi) = \int_{\omega-i\infty}^{\omega+i\infty} \hat{\varphi}(\lambda)R_1(\lambda)d\lambda,$$

and that

$$\mathcal{G}(\varphi) = \mathcal{T}(\varphi) \qquad \text{for} \quad \varphi\in\mathcal{D}_1. \tag{A. 2}$$

Here $\mathcal{T}$ is the $L(E)$-valued distribution determined by the generalized resolvent $(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}$. By (A. 1) and (A. 2), $\mathcal{T}$ is of finite order on the interval $(-1, 1)$. Hence $\mathcal{T}$ is an R.D.S.G. by Propositions 3. 1, 3. 2 and 3. 3.

Let δ_t be the dirac measure at $t\in R^1$. For any $t\geqslant0$, a linear operator $\mathcal{T}(\delta_t)$ in $\mathcal{R}(\mathcal{T})$ is uniquely determined (see the discussion after Definition 3. 1). Let us denote $\mathcal{T}(\delta_t)$ by S_t. Namely we have

$$S_t x = \sum_{j=1}^{n} \mathcal{T}(\delta_t * \varphi_j)y_j = \sum_{j=1}^{n} \mathcal{T}(\varphi_j(s-t))y_j$$
$$\text{for } t\geqslant0 \text{ and } x = \sum_{j=1}^{n} \mathcal{T}(\varphi_j)y_j\in\mathcal{R}(\mathcal{T}).$$

It holds from ($\mathcal{D}$. 3)

$$\mathcal{T}(\varphi)x = \int_0^{\infty} \varphi(t)S_t x\,dt \qquad \text{for } x\in\mathcal{R}(\mathcal{T}) \text{ and } \varphi\in\mathcal{D}. \tag{A. 3}$$

Choose a function $\rho\in\mathcal{D}^+$ such that $\int_0^\infty \rho(t)dt=1$, $\rho\geqslant 0$. Consider the convolution $\rho*D^2G$:

$$(\rho*D^2G)(t)=\int_0^\infty \rho''(t-s)G(s)ds\in L(E).$$

Then the family in $L(E)$

$$\left\{(\rho*D^2G)(t):\ t>0,\ \rho\in\mathcal{D}^+,\ \int_0^\infty \rho(t)dt=1,\ \rho\geqslant 0\right\}$$

is equicontinuous. In fact, since $(\rho*D^2G)(t)$ vanishes for $t<0$, being estimated by the function $e^{t\omega}$, we have an approximate formula:

$$\begin{aligned}&(\rho*D^2G)(t)\\ &=\lim_{\mu\to\infty} e^{-\mu t}\sum_{n=0}^{\infty}\frac{(-1)^n(\mu^2t)^{n+1}}{n!\,(n+1)!}\frac{d^n}{d\mu^n}(\widetilde{\rho*D^2G})(\mu)\end{aligned}$$

for $t>0$, where $\tilde{F}(\lambda)=\int_0^\infty e^{-\lambda t}F(t)dt$ (see Hille-Phillips [8], Theorem 6.3.3). On the other hand, we can easily justify the following calculus.

$$\begin{aligned}&(\widetilde{\rho*D^2G})(\lambda)=(\widetilde{D^2\rho*G})(\lambda)=\widetilde{D^2\rho}(\lambda)\tilde{G}(\lambda)\\ &=\lambda^2\tilde{\rho}(\lambda)\lambda^{-2}R_1(\lambda)=\tilde{\rho}(\lambda)R_1(\lambda).\end{aligned}$$

Since the condition $(R_0.1)$ implies the equicontinuity of the family

$$\left\{\frac{\mu^{n+1}}{n!}\left(\frac{d}{d\mu}\right)^n(\tilde{\rho}(\mu)R_1(\mu)):\ \mu>0,\ \rho\geqslant 0,\ \int_0^\infty \rho(t)dt=1\right\},$$

we have

$$e^{-\mu t}\sum_{\mu=0}^{\infty}\frac{(-1)^n(\mu^2t)^{n+1}}{n!\,(n+1)!}\left(\frac{d}{d\mu}\right)^n(\tilde{\rho}(\mu)R_1(\mu))$$

is equicontinuous with respect to $\mu>0$, $t>0$ and $\rho\in\mathcal{D}^+$ satisfying $\int_0^\infty \rho(t)dt=1$ and $\rho\geqslant 0$.

The equalities $(A.2)$ and $(A.3)$ imply that $D^2G(t)x=S_tx$ if $0<t<1$ and $x\in\mathcal{R}(\mathcal{T})$. Hence if we take a sequence $\{\rho_n>0\}\subset\mathcal{D}^+$ satisfying $\int_0^1 \rho_n(t)dt=1$, converging to δ_t, $0<t<1$, we have that $\{S_t:\ 0<t<1\}$ is equicontinuous on $\mathcal{R}(\mathcal{T})$. Hence, by continuity in t, $\{S_t: 0\leqslant t\leqslant 1\}$ is equicontinuous on $\mathcal{R}(\mathcal{T})$. Since $\mathcal{R}(\mathcal{T})$ is dense in E, $\{\bar{S}_t:\ 0\leqslant t\leqslant 1\}$ is an equicontinuous family in $L(E)$, where $\bar{S}_t\in L(E)$ is the closure of S_t.

Let us define

$$T_t=\bar{S}_{t-[t]}[\bar{S}_1]^{[t]} \qquad 0\leqslant t<\infty.$$

Then $\{T_t\colon t\geqslant 0\}$ forms an locally equicontinuous family in $L(E)$ satisfying $(T_0.2)$ and $(T_0.3)$. It holds that for any $\varphi\in\mathcal{D}$ and $x\in E$,

$$\mathcal{T}(\varphi)x=\int_0^\infty \varphi(t)T_t x dt \qquad (A.4)$$

In fact, $(A.4)$ holds for $x\in\mathcal{R}(\mathcal{T})$ by $(A.3)$ since T_t is the closure of S_t for any $t\geqslant 0$. Since $\mathcal{R}(\mathcal{T})$ is dense in E, $(A.3)$ holds for any $x\in E$.

The condition $(\mathcal{D}.2)$ for $\mathcal{T}$ implies the condition $(T_0.1)$ for $\{T_t\}$. Hence $\{T_t\colon t\geqslant 0\}$ is a locally equicontinuous semi-group. By Proposition 4.1, the generator of $\{T_t\colon t\geqslant 0\}$ coincides with the generator of $\mathcal{T}$, which is the operator A.

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