There Exists a Hopf Algebra Whose Antipode is not Injective

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We showed in [2] the existence of a Hopf algebra over an arbitrary field whose antipode is not bijective. In this note we strengthen this result as in the title. We adopt the terminology and definitions in [1].

Let k be a field. The forgetful functor from the category of Hopf algebras over k to that of coalgebras over k has a left adjoint functor H. In other words for any coalgebra C there exist a Hopf algebra $H(C)$ and a coalgebra map $u: C \to H(C)$ such that $\mathrm{Hom}(u, H)$: $\mathrm{Hopf}(H(C), H) \to \mathrm{Coalg}(C, H)$ is a bijection for any Hopf algebra H. $H(C)$ is called the *free Hopf algebra* generated by C. We proved the following facts in [2].

1. Let A be an algebra over k. If we put

$$L(C, A) = \{(f_i)_{i\geqq 0} \in \mathrm{Hom}(C, A)^N \mid f_{i+1} = f_i^{-1} \text{ in } \mathrm{Hom}(C^{\mathrm{op}^i}, A)\}$$

then the map $f \mapsto (f \circ S^i \circ u)_{i\geqq 0}$ is a bijection from $\mathrm{Alg}(H(C), A)$ onto $L(C, A)$, where C^{op^i} denotes C (resp. the opposite of C) for i even (resp. odd) and S is the antipode of $H(C)$.

2. Let $M_n(k)$ be the $n \times n$ matrix algebra over k and $M_n(k)^*$ be its dual coalgebra. Then the antipode of $H(M_n(k)^*)$ is not bijective for $n > 1$.

Now we reduce the problem.

PROPOSITION 1. *If there exist a coalgebra C over k, a coideal I of C, an algebra A over k and an element $(f_i)_{i\geqq 0}$ of $L(C, A)$ such that*

$$f_0|I \neq 0 \qquad f_i|I = 0 \quad \text{for} \quad i > 0$$

then there exists a Hopf algebra whose antipode is not injective.

Proof. Let $u: C \to H(C)$ be a free Hopf algebra. $J = \sum_{i>0} S^i(u(I))$ is a coideal of $H(C)$ such that $S(J) \subset J$. So the ideal K of $H(C)$ generated by J is a Hopf ideal of $H(C)$. The algebra map from $H(C)$ to A determined by $(f_i)_{i\geqq 0}$ is zero on J, so is on K. Hence f_0 can be factorized as

$$f_0: C \longrightarrow H(C) \longrightarrow H(C)/K \longrightarrow A,$$

By hypothesis and definition we have

$$I \not\subset \mathrm{Ker}(C \longrightarrow H(C)/K)$$

$$I \subset \mathrm{Ker}(C \longrightarrow H(C)/K \xrightarrow{S} H(C)/K).$$

This means that the antipode S of $H(C)/K$ is not injective.

COROLLARY 2. *If there exist an algebra A, a finite dimensional algebra M, a subalgebra N of M and elements*

$$x_0 \in M \otimes A - N \otimes A,\ x_1, x_2, x_3, \cdots \in N \otimes A$$

such that $x_{i+1} = x_i^{-1}$ in $M \otimes A^{\mathrm{op}^i}$, then there exists a Hopf algebra whose antipode is not injective.

Proof. Take $C = M^*$ and $I = \mathrm{Ker}(M^* \to N^*)$. Then $(x_i)_{i \geq 0}$ belongs to $L(M^*, A)$ and $x_0 | I \neq 0$, $x_i | I = 0$ for $i > 0$.

COROLLARY 3. *If there exist an algebra A over k and matrices $x_i \in M_m(A)$, $Y_i \in M_n(A)$, $i = 0, 1, 2, \cdots$, and $Z_0 \in M_{n \times m}(A)$ such that we have*

$$\begin{pmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{pmatrix}^{-1} = \begin{pmatrix} * & W \\ * & * \end{pmatrix} \quad \textit{for some } W \neq 0$$

$$\textit{and } X_{i+1}^{-1} = {}^tX_i, \qquad Y_{i+1}^{-1} = {}^tY_i,$$

where ${}^tX$ is the transpose of X, then we have the same conclusion.

Proof. Put $M = M_{m+n}(k)$,

$$N = \left\{ \begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix};\ X \in M_m(k),\ Y \in M_n(k) \quad \text{and} \quad Z \in M_{n \times m}(k) \right\}$$

and

$$x_0 = \begin{pmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{pmatrix}^{-1}.$$

Then we have $x_0 \in M \otimes A - N \otimes A$ and there exists an element $(x_0, x_1, x_2, \cdots)$ of $L(M^*, A)$ such that $x_1, x_2, \cdots$ belong to $N \otimes A$.

In what follows we fix an integer $n > 1$ and put $A = H(M_n(k)^*)$. Let X be the element of $M_n(A)$ which corresponds to $u: M_n(k)^* \to A$ under the identification

$$\mathrm{Hom}(M_n(k)^*, A) = M_n(k) \otimes A = M_n(A).$$

Then $S(X)$ corresponds to $S \circ u$. So $S(X) = X^{-1}$ in $M_n(A)$. The fact S is not bijective is equivalent to that that the transpose ${}^tX$ is not invertible. Hence the map

$${}^tX:\ A^n \longrightarrow A^n,\ (x_1, \cdots, x_n) \longmapsto (x_1, \cdots, x_n) \cdot {}^tX$$

is not bijective, because $\mathrm{End}({}_A A^n)=M_n(A)$. We may and shall assume that the antipode S of A is injective, because if contrary the theorem would have been proved.

REMMA 4. *If S is injective then ${}^tX\colon A^n\to A^n$ is injective.*

Proof. S is an anti-algebra map. If $x\cdot{}^tX=0$, then

$$0={}^t(S(x\cdot{}^tX))=S(X)\cdot{}^tS(x)=X^{-1}\cdot{}^tS(x)\,.$$

So $x=0$.

COROLLARY 5. *If the antipode S of $H(M_n(k)^*)$ is injective, then there exist a vector space V over k and elements X_i of $M_n(\mathrm{End}_k(V))$, $i=0,1,2,\cdots$, such that $X_{i+1}^{-1}={}^tX_i$ and*

$$X_0\colon V^n\longrightarrow V^n, \qquad (v_1,\cdots,v_n)\longmapsto(v_1,\cdots,v_n)X_0$$

is injective but not surjective.

COROLLARY 6. *There exists an infinite cardinal α_0 such that for any $\alpha\geqslant\alpha_0$, there exist V and X_i as in* COROLLARY 5 *such that*

$$\alpha=\dim_k V=\dim_k \mathrm{Coker}\, X.$$

Proof. Let V and X_i be as in COROLLARY 5. It is sufficient to take $\alpha_0=\dim_k V$ and $\bigoplus_\alpha V$, $\bigoplus_\alpha X_i$.

LEMMA 7. *There exist a vector space V over k and matrices $X_i\in M_n(\mathrm{End}_k(V))$, $i=0,1,2,\cdots$, such that $X_{i+1}^{-1}={}^tX_i$ and $X_0\colon V^n\to V^n$ is surjective but not injective.*

Proof. Let V and X_i be as in COROLLARY 5. Put $V^*=\mathrm{Hom}_k(V,k)$ and $f^*=\mathrm{Hom}_k(f,k)$ for $f\in\mathrm{End}_k(V)$. If we identify $(V^n)^*=(V^*)^n$, then we have $X^*=(x_{ji}^*)_{ij}$ for $X\in M_n(\mathrm{End}_k(V))$. Hence we have

$$(X_{i+1}^*)^{-1}=(X_{i+1}^{-1})^*=({}^tX_i)^*={}^t(X_i^*).$$

Because $X_0^*\colon V^{*n}\to V^{*n}$ is surjective but not injective, the proof is complete.

REMARK 8. The analogue of COROLLARY 6 is also valid.

THEOREM 9. *Let k be a field. There exists a Hopf algebra over k whose antipode is not injective.*

Proof. Let n be an integer >1. If the antipode S of $A=H(M_n(k)^*)$ is not injective, then the proof is complete. So we assume that S is injective. Let α be a sufficiently large cardinal. There exist a vector space V over k and matrices X_i, $Y_i\in M_n(\mathrm{End}(V))$, $i=0,1,2,\cdots$, such that

$$X_{i+1}^{-1}={}^tX_i \qquad Y_{i+1}^{-1}={}^tY_i$$

$$X_0\colon V^n \longrightarrow V^n \text{ is injective}$$

$$Y_0\colon V^n \longrightarrow V^n \text{ is surjective}$$

and

$$\alpha=\dim_k Y=\dim_k \operatorname{Coker} X_0=\dim_k \operatorname{Ker} Y_0.$$

If we decompose $V^n=M\oplus N=P\oplus Q$ such that

$$X_0\colon V^n \xrightarrow[\cong]{} M \quad\text{and}\quad P=\operatorname{Ker} Y_0,$$

then there exists an isomorphism from P onto N. This means that there exists a matrix $Z_0\in M_n(\operatorname{End}_k(V))$ such that

$$\begin{pmatrix} X_0 & 0 \\ Z_0 & Y_0 \end{pmatrix}^{-1}=\begin{pmatrix} * & W \\ * & * \end{pmatrix} \qquad \text{for some } W\neq 0$$

By COROLLARY 3 the proof is complete.

References

1. Sweedler, M. E., Hopf algebras, Benjamin, New York, 1969.
2. Takeuchi, M., Free Hopf Algebras Generated by Coalgebras, J. Math. Soc. Japan, Vol. 23 (1971), pp. 561-582.