Functional Studies of Automata (I)

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There are a number of well-known theories which are concerned with the construction of complicated " machinery " from a small number of basic elements—for instance, the combinatorial construction of (k-valued) logical networks. Minsky and many others considered the construction of binary automata and have obtained important results.

In this paper we propose a general framework which integrates these theories, especially those of Kudrjavtiev, Loomis and Ibuki. Several variants of the notions of 'universality' and of 'functional completeness' are formulated within the framework.

Several theorems related to these notions shall be given in the subsequent paper: functional studies of automata (II).

1. Physical Background

We shall start with an informal exposition of the " machinery " to be considered.

1.1. *I–O* devices

An *I–O device* is a machinery having a number of " input terminals " and an " output terminal ". It receives " input signals " from the input terminals and emits an " output signal " through the output terminal.

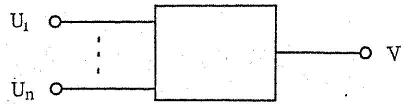

Figure 1.
$U_1, \cdots, U_n$: input terminals, V: output terminal

We assume that:

1) there are k distinct " signals " denoted by

$$0, 1, 2, \cdots, k-1.$$

2) input and output signals are functions of a variable t which moves in a discrete set of "moments" denoted by

$$0, 1, 2, 3, \cdots .$$

Each input (output) signal is one of the above-mentioned signals $0, 1, \cdots, k-1$.

Let $U_1, \cdots, U_n$ be input terminals and V the output terminal of an I-O device A.

We represent by $u_i(t)$ (by $v(t)$) the signal passing through a terminal U_i (V, respectively) at a moment t.

u_i and v represent sequences of signals, that is, functions from

$$N = \{0, 1, 2, \cdots\}$$

to

$$(k) = \{0, 1, \cdots, k-1\} .$$

We also assume the following conditions:

3) The sequence v of output signals is uniquely determined by the sequences $u_1, \cdots, u_n$ of input signals.

In other words, the input-output behavior of a device A is deterministic and can be represented by a mapping from

$$(k)^* \times \cdots \times (k)^* \qquad (n \text{ factors})$$

to

$$(k)^*$$

where $(k)^*$ is the whole set of functions from $\boldsymbol{N}$ to (k).

4) The value $v(t_0)$ of v at a momens t_0 depends only on the values

$$u_1(0), \cdots, u_1(t_0)$$

$$u_2(0), \cdots, u_2(t_0)$$

and

$$u_n(0), \cdots, u_n(t_0) .$$

In short, an output signal is independent of the future input.

1.2. Connection of *I–O* devices

Let A_0, A_1 be I-O devices having two input terminals. We consider a new device A obtained from these devices.

The construction of a new device can be specified by showing how to connect the terminals of A, A_0 and A_1. For instance:

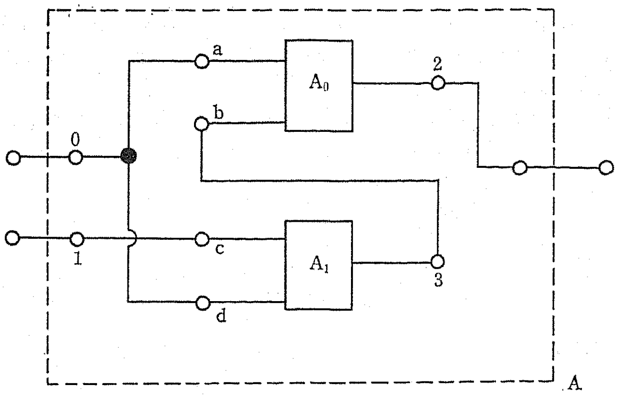

Figure 2.

The input terminals of A and the output terminals of A_0, A_1 are called *senders* since they supply to other terminals external input signals and output signals. The output terminal of A and input terminals of A_i's are called *receivers* since each of them must be connected to one of the senders.

We can therefore represent a connection among these terminals by a mapping h from the set of receivers to the set of senders.

Let us denote the input terminals of A by 0 and 1 and the output terminals of A_0 and of A_1 by 2 and 3. The connection of terminals shown in Figure 2 is then represented by the following table:

Table 1.

Output of A	A_0		A_1	
	a	b	c	d
2	0	3	1	0

1.3. **Automaton**

An automaton

$$A=(S, U, V, s_0, f, g)$$

is a six-tuple of finite sets S, U, V, an element s_0 of S and functions f, g defined as following:

$$f: S\times U\to S$$

$$g: S\times U\to V.$$

Under some modest conditions, the behavior of an electronic circuit is represented by an automaton. Each element of S represents an internal state of this circuit, g represents the input-output behavior and f represents the transition of the internal state of this circuit.

f is called *state transition function* and g is called *output function*.

We can assume, without loss of generality, that

$$S=(k)^m, \qquad U=(k)^n, \qquad V=(k)^r \qquad (1)$$

and that

$$s_0=(0, 0, \cdots, 0) \qquad (2)$$

We shall denote

$$f(x_1, \cdots, x_m, y_1, \cdots, y_n)=(x_1', \cdots, x_m')$$

$$f_i(x_1, \cdots, x_m, y_1, \cdots, y_n)=x_i'$$

$$g(x_1, \cdots, x_m, y_1, \cdots, y_n)=(z_1, \cdots, z_r)$$

$$g_j(x_1, \cdots, x_m, y_1, \cdots, y_n)=z_j .$$

We shall call (k)-*automaton* an automaton satisfying the conditions (1) and (2).

2. Mathematical Framework

2.1. Admissible Functions

We shall denote by $M(X, Y)$ the whole set of functions from X to Y.

$$(k)=\{0, 1, \cdots, k-1\} \qquad (k\geqq 2)$$

$$\boldsymbol{N}=\{0, 1, 2, \cdots\}$$

$$(k)^*=M(\boldsymbol{N}, (k)) .$$

Let X be a non null set.

$$\Omega_n(X)=M(X^n, X)$$

$$\Omega(X)=\bigcup_{n=1}^{\infty} \Omega_n(X) .$$

DEFINITION 1. C_s is an operator defined over $(k)^*$ as follows:

$$(C_s\cdot u)(t)=\begin{cases} u(t) & \text{for } t<s \\ 0 & \text{for } t\geqq s \end{cases}$$

C_s is called *cut-off operator*.

C_s is extended immediately to $[(k)^*]^n$:

$$C_s\cdot(u_1, \cdots, u_n)=(C_s\cdot u_1, \cdots, C_s\cdot u_n)\,.$$

DEFINITION 2. Let F be a function in $\Omega_n(k)^*$:

$$F\colon [(k)^*]^n \rightarrow (k)^*\,.$$

F is said to be *admissible* if it satisfies the following condition:

$$(\forall s>0)\ C_s\cdot F=C_s\cdot F\cdot C_{s-1}\,.$$

Remark.

$$(C_s\cdot F)\cdot C_r=C_s\cdot(F\cdot C_r)\,.$$

F is said to be *weakly admissible* if

$$C_sF=C_sFC_s$$

for any $s\geqq 0$.

Remark. F is admissible (or weakly admissible) if ths value $v(t_0)$ of $v=F(u_1, \cdots, u_n)$ at a moment t_0 depends only on the values $u_1(t), \cdots, u_n(t)$ for $t<t_0$ (or for $t\leqq t_0$, respectively).

The input-output behavior of an *I-O* device is represented by a weakly admissible function (Assumption 4), 1. 1.)

We shall denote the whole set of admissible functions (weakly admissible functions) by $\mathscr{A}$ (by $\mathscr{B}$, respectively).

2. 2. General Operation of Composition

In the following, we shall abbreviate $\Omega_n(k)$ by Ω_n and $\Omega_n(k)^*$ by Ω_n^*.

As it has been mentioned in 1. 2, construction of a new device A by a number of (elementary) devices $A_1, \cdots, A_s$ can be specified by a mapping h from the set of 'receivers' to the set of 'senders'. In the following, we shall represent senders and receivers by integers and pairs of integers in the following manner:

Representation of senders

1) the i-th input terminal of A......i
2) the output terminal of A_j.........$n+j$,

where n is the number of input terminals of A.

Representation of receivers

1) the output terminal of A...........$(0, 0)$
2) the j-th input terminal of A_i......(i, j).

DEFINITION 3. A *connector* h is a mapping from a subset of $\boldsymbol{N}\times\boldsymbol{N}$ to a subset of $\boldsymbol{N}$.

DEFINITION 4. Let

$$F_1, \cdots, F_s$$

be admissible functions having respectively $n(1), \cdots, n(s)$ variables.

Let h be a connector whose domain D and image I are the sets as following:

$$D=\{(0,0)\}\cup\{(i,j);\ 1\leqq i\leqq s,\ 1\leqq j\leqq n(i)\}$$

$$I=\{1,2,3,\cdots,n+s\}$$

where n is a certain positive integer.

As we shall see below, we can associate to any n-tuple of sequences

$$u_1,\cdots,u_n\in(k)^*$$

the sequences

$$w_1,\cdots,\cdots,w_{n+s}\in(k)^*$$

which satisfy the following conditions.

1) $$w_i=u_i \quad \text{for} \quad 1\leqq i\leqq n$$

2) $$w_{n+i}=F_i(w_{h(i,1)},\cdots,w_{h(i,n(i))}) \quad \text{for} \quad 1\leqq i\leqq s\,.$$

We denote by

$$[F_1,\cdots,F_s]_h$$

the function of Ω_n^* defined as follows:

$$[F_1,\cdots,F_s]_h(u_1,\cdots,u_n)=w_{h(0,0)}\,.$$

Proof of the existence and uniqueness of w_{n+i}

Since F_i's are admissible,

$$C_t w_{n+i}=C_t F_i C_{t-1}(w_{h(i,1)},\cdots\ \ )$$

for any $t>0$. We have therefore

a) $$w_{n+i}(0)=(C_1 w_{n+i})(0)$$
$$=[C_1F_iC_0(w_{h(i,1)},\cdots)](0)$$
$$=F_i(O,O,\cdots,O)(0)$$

where $O\in(k)^*$ denotes the constant sequence such that

$$O(t)=0$$

for all $t\geqq 0$.

In a similar way, we can obtain

b) $$w_{n+i}(t)=[F_i(C_{t-1}\cdot w_{h(i,1)},\cdots,C_{t-1}\cdot w_{h(i,n(i))})](t)\,.$$

Now the values

$$w_{n+1}(t), \cdots, w_{n+s}(t)$$

are recursively determined by b) starting from the values

$$w_{n+1}(0), \cdots, w_{n+s}(0)$$

which are determined by a). The uniqueness of w_{n+i}'s is now evident.

On the other hand, the sequences w_{n+i}'s defined by a) and b) satisfy obviously the condition 2). The existence of w_{n+i} is therefore obvious.

The function $[F_1, \cdots, F_s]_h$ is called *function composed from* $F_1, \cdots, F_s$ *according to a connector* h.

Remark. If some F_i's are weakly admissible, the composed function may *not* be defined.

DEFINITION 5. Let $\mathfrak{F}$ be a subset of $\mathcal{A}$.

We denote by $[\mathfrak{F}]$ the set of all functions which can be obtained from $\mathfrak{F}$ by the operation of composition.

LEMMA 1. *Any function composed from* $\mathfrak{F}$ *is admissible.*

COROLLARY. $\mathfrak{F} \subseteq [\mathfrak{F}] \subseteq \mathcal{A}$.

The proof is immediate from the definition 4.

2.3. Other Operation of Composition

2.3.1. *Loop-free composition*

DEFINITION 6. A connector h is said to be *loop-free* if it satisfies the following condition:

$$1 \leqq h(i, j) < n+i \qquad \text{for any} \quad i \neq 0 .$$

LEMMA 2. *Let* h *be the connector which we considered in the definition* 4. *If* h *is loop-free, then*

1) *the connector* h_r *defined as follows is also loop-free:*

a) $h_r: \{(i, j) \in D\,;\ i \leqq r\} \rightarrow \{1, 2, \cdots, n+r\}$

b) $h_r(0, 0) = n+r$

c) $h_r(i, j) = h(i, j) \quad$ *for* $\quad i \neq 0$.

2) $$w_r = [F_1, \cdots, F_r]_{h_r}$$

where w_r *is the sequence defined in the definition* 4.

The proof is immediate.

2.3.2. *Free Closure*

Let X be an arbitrary set.

DEFINITION 7.
1) Suppose that $f\in\Omega_p(X)$ and that

$$g_1, \cdots, g_p\in\Omega_q(X)$$

$f(g_1\times\cdots\times g_p)$ is the function of $\Omega_q(X)$ which is defined as follows:

$$f(g_1\times\cdots\times g_p)(x_1, \cdots, x_q)$$
$$=f[g_1(x_1, \cdots, x_q), \cdots, g_p(x_1, \cdots, x_q)]\,.$$

2) $$f\circ G=\{f(g_1\times\cdots\times g_q);\ g_i\in G\}$$

where $f\in\Omega_p(X)$ and $G\subseteq\Omega_q(X)$.

Remark. $$f\circ G\subseteq\Omega_q(X)$$

3) $$F\circ G=\bigcup_{f\in F}\bigcup_{q=1}^{\infty}f\circ(G\cap\Omega_q(X))\,.$$

LEMMA 3.

1) $$(F\circ G)\circ H=F\circ(G\circ H)$$

2) $$F\subseteq F',\ G\subseteq G'\Rightarrow F\circ G\subseteq F'\circ G'\,.$$

DEFINITION 8. A projection P_i^N is a function defined over X^N as follows:

$$P_i^N(x_1, \cdots, x_N)=x_i\,.$$

We denote:

$$\wp_n=\{P_i^n;\ 1\leqq i\leqq N\}$$
$$\wp=\bigcup_{n=1}^{\infty}\wp_n\,.$$

DEFINITION 9. Let F be a subset of $\Omega(X)$.
$\bar F$ is the smallest subset of $\Omega(X)$ which satisfies the following conditions.

1) $$\bar F\supseteq F$$

2) $$\bar F\supseteq\bar F\circ(F\cup\wp)\,.$$

Remark. 2) is equivalent to the following condition.

2′) $$f\in\bar F\cap\Omega_p,\ g_1, \cdots, g_p\in(F\cup\wp)\cap\Omega_q\Rightarrow g(g_1\times\cdots\times g_p)\in\bar F\,.$$

PROPOSITION 1. *Let* $\mathfrak{F}\subseteq\Omega^*$.
$\bar{\mathfrak{F}}$ *is identical to the whole set of functions which can be obtained by iterative application of loop-free composition from* $\mathfrak{F}$.

Proof. a) Suppose that $F\in\Omega_p(k)^*$, $G_1, \cdots, G_p\in\Omega_q(k)^*$. Then we have

$$F(G_1\times\cdots\times G_p)=[G_1, \cdots, G_p, F]_h$$

where h is the function defined as follows:

$$h(0, 0)=p+q+1$$

$$h(i, j)=j \quad \text{for} \quad 1\leqq i\leqq p, \quad 1\leq j\leq q$$

$$h(p+1, j)=q+j \quad \text{for} \quad 1\leq j\leq p$$

Every function in $\overline{\mathcal{F}}$ can be therefore obtained from $\mathcal{F}$ by loop-free composition.

b) Suppose that F is the function of n variables composed from $F_1, \cdots, F_s$ according to a loop-free connector h.

If $h(0, 0)\leqq n$, then

$$F=P^n_{h(0,0)}.$$

If $h(0, 0)=n+r$, then

$$F=G_r$$

where

$$G_r=[F_1, \cdots, F_r]_{h_r}$$

(see the lemma 2).

Now we shall show that $G_r\in\overline{\mathcal{F}}$

1) $r=1$:

$$G_r=F_1(P^n_{h(1,1)}\times\cdots\times P^n_{h(1,n(1))}).$$

2) Suppose that $G_i\in\overline{\mathcal{F}}$ for $i<r$.

$$G_r=F_r(H_1\times\cdots\times H_{n(r)})$$

where

$$H_j=\begin{cases}P^n_{h(r,j)} & \text{if} \quad h(r, j)\leqq n\\ G_{h(r,j)-n} & \text{if} \quad h(r, j)>n.\end{cases}$$

Since $h(r, j)-n<r$, this completes the proof.

For the other operations of composition, the reader is referred to Minnik and to Naemura.

3. Basic Problems

3.1. Representability

We shall start with introducing new operators over $(k)^*$ to define the notion of representability.

Definition 1. Let p be a positive integer and $u\in(k)^*$.

$$(R_p \cdot u)(t) = \begin{cases} u(t') & \text{for } t = pt' \\ 0 & \text{otherwise} \end{cases}$$

$$(R_p^{-1} \cdot u)(t) = u(pt) .$$

Let q be an arbitrary integer

$$(D^q \cdot u)(t) = \begin{cases} u(t-q) & \text{for } t-q \geqq 0 \\ 0 & \text{otherwise} . \end{cases}$$

These operators are immediately extended over $[(k)^*]^n$:

$$D^q \cdot (u_1, \cdots, u_n) = (D^q u_1, \cdots, D^q u_n) \text{ etc.}$$

Examples. Suppose that $u(0)=0$, $u(1)=1$, $u(2)=2$ and $u(3)=0$. Then

1) $D^1 \cdot u(0)=0 , \quad D^1 \cdot u(1)=0 , \quad D^1 \cdot u(2)=1, \cdots$

2) $D^{-1} \cdot u(0)=1 , \quad D^{-1} \cdot u(1)=2 , \quad D^{-1} \cdot u(3)=0, \cdots .$

D^q is therefore a "shift operator".
In the following, we shall abbreviate D^1 by D.

3) $R_2 \cdot u(0)=u(0)=0 , \quad R_2 \cdot u(1)=0 ,$

$R_2 \cdot u(2)=u(1)=1 , \quad R_2 \cdot u(3)=0 ,$

$R_2 \cdot u(4)=u(2)=2, \cdots .$

4) $R_2^{-1} \cdot u(0)=u(0)=0 , \quad R_2^{-1} \cdot u(1)=u(2)=2 ,$

$R_2^{-1} \cdot u(2)=u(4), \cdots .$

R_p represents the function of an "encoder" which emits signals bearing information at intervals of p unit time.

R_p^{-1} represents the function of a "decoder" which reconstructs original sequence of signals.

5) $R_1 = R_1^{-1} = D^0 = I$

where I is the identity operator: $I \cdot u = u$

Lemma 1. *Let p be a positive integer*

1) $D^{-p} \cdot D^p = I$

2) $R_p^{-1} \cdot R_p = I .$

The proof is immediate.

Remark. $D^p \cdot D^{-p} \neq I , \quad R_p \cdot R_p^{-1} \neq I .$

DEFINITION 2. Let F, G be functions in $\Omega_n(k)^*$.

G is said to be *weakly representable by* F, $G \dashv F$ in symbol, if the following condition is satisfied:

$$(\exists p \geqq 1)(\exists d \geqq 0)(\exists c \geqq 0)\ R_p^{-1} \cdot D^{-(d+c)} \cdot F = G \cdot R_p^{-1} \cdot D^{-c}.$$

If $p=1$, G is said to be *representable by* F.
If $p=1$ and $c=0$, G is said to be *strongly representable* by F.
The number p is called *information rate*. d is called *delay* and c is called *initialization time*. The triple

$$(p, d, c)$$

is called *index* of F with respect to G.

3. 2. Neuman type problem

Every function h in $\Omega(k)$ induces naturally a function in $\Omega(k)^*$ as follows:

$$v(t) = h(u_1(t), \cdots, u_n(t)).$$

We denote this function over $(k)^*$ by h^*:

$$v = h^*(u_1, \cdots, u_n).$$

DEFINITION 3. Let $\mathfrak{F}$ be a subset of $\Omega(k)^*$.
$\mathfrak{F}$ is said to be *(strongly) (k)-complete* if for every function h in $\Omega(k)$ h^* is (strongly) representable by a function H belonging to $\mathfrak{F}$.
$\mathfrak{F}$ is said to be *(k)-universal* if for every function h in $\Omega(k)$ h^* is weakly representable by a function belonging to $[\mathfrak{F}]$.

DEFINITION 4. Let $\mathfrak{F} \subseteq \Omega(X)$.

$$\mathfrak{F} \text{ is complete} \longleftrightarrow \overline{\mathfrak{F}} = \Omega(X).$$

Here arises an interesting problem: what are the necessary and sufficient conditions for $\mathfrak{F}$ to be (k)-complete, strongly (k)-complete, (k)-universal or complete?

We shall call such a problem *Neumann type problem*.

Von Neumann investigated in 1956 sets of binary devices and remarked that a device which reckons with certain time lag the Sheffer's function:

$$S(x, y) = 1 - x \cdot y$$

could *not* generate all logical functions. In our terminology, he has shown that the set $\{D \cdot S^*\}$ is *not* (2)-complete while the set $\{S\}$ is complete in $\Omega(2)$.

Remark. It is obvious that: strongly (k)-complete $\Rightarrow$ (k)-complete
$\Rightarrow$ (k)-universal.

3. 3. Kleene Type Problem

Let

$$A=(S, U, V, s_0, f, g)$$

be an automaton such that

1) $$S=(k)^m, \qquad U=(k)^n, \qquad V=(k)^r$$

2) $$s_0=(0, 0, \cdots, 0).$$

Hereafter we shall call such an automaton *(k)-automaton with n input and r output.*

Obviously, f, g are mappings from $(k)^{m+n}$ to $(k)^m$ and to $(k)^r$, respectively.

DEFINITION 5. The *output sequence function* G of an automaton A is the mapping from $M(N, U)$ to $M(N, V)$ which is defined as following.

Let u be an arbitrary element of $M(N, U)$.

1) $$s(0)=s_0$$

2) $$s(t+1)=f(s(t), u(t))$$

3) $$v(t)=g(s(t), u(t))$$

4) $$G(u)=v.$$

Remark. If A is a (k)-automaton with n input and r output, then G is a mapping from $[(k)^*]^n$ to $[(k)^*]^r$.

DEFINITION 6. Let $\mathfrak{F}$, $\mathfrak{G}$ be subsets of $\Omega(k)^*$. $\mathfrak{G}$ is said to be *uniformly representable by* $\mathfrak{F}$ if the following condition is satisfied:

$$(\exists (p, d, c))(\forall G\in \mathfrak{G})(\exists F\in\mathfrak{F})$$

G is weakly represented by F whose index is (p, d, c).

DEFINITION 7. Let A be a (k)-automaton with n input and r output and G its output sequence function. Let $\mathfrak{F}$ be a subset of $\Omega(k)^*$.

A is said to be *realizable* if

$$P_1^r \cdot G, \cdots, P_r^r \cdot G$$

are uniformely representable by $\mathfrak{F}$.

$\mathfrak{F}$ is said to be *universal* if any automaton is realizable by $[\mathfrak{F}]$.

Kleene Type Problem: What are the necessary and sufficient conditions for $\mathfrak{F}\subseteq\Omega(k)^*$ to be universal?

Remark. Kleene has shown that any binary automaton could be constructed by so-called 'majority organs' which might have inhibitory inputs.

Minsky has shown that the set of following devices could generate any binary automaton.

A: conjunction device

B: disjunction device

J: non-monotone I-O device.

A few years later, Arden considered a similar problem in a rather general framework and obtained under certain assumptions the following result.

For any $\mathcal{F} \subseteq \mathcal{E}(2)^*$, $\mathcal{F}$ (2)-universal $\Rightarrow \mathcal{F}$ universal. (see 4.1)

This result can be generalized to the case $\mathcal{F} \subseteq \mathcal{A}(k)^*$ as it shall be shown in the subsequent paper: functional studies of automata (II).

4. Neumann Type Problem on Elementary Functions

In the following, we shall introduce a variant of the notion of completeness, $\sim$-completeness, which is defined for a set of 'elementary functions'.

4.1. Elementary Functions

Definition 1. A function $G \in \Omega(k)^*$ is said to be *elementary* if there exists a function $g \in \Omega(k)$ which satisfies the following condition:

$$(\exists s \geqq 0): D^{-s} \cdot G = g^* .$$

The number s is called *delay* of G. s is unique unless g is constant.

We denote by $\mathcal{E}(k)^*$ the whole set of elementary functions in $\Omega(k)^*$.

Remark. The operation carried out by a logic element of an electronic circuit is usually represented by an elementary function.

An elementary function with delay 1 is called *unit delay function.*

An elementary function with delay 0 is said to be *combinatorial.* A function $G \in \Omega(k)^*$ is combinatorial if there exists a function $g \in \Omega(k)$ such that $G = g^*$. If we identify g and g^*, then the set of all combinatorial functions can be denoted by $\Omega(k)$.

4.2. Spectrum over (k)

A spectrum S is a series of sets

$$S_0, S_1, S_2, \cdots .$$

We denote:

$$S = (S_i) .$$

Definition 2.

$$S \subseteqq S' \Longleftrightarrow (\forall_i \geqq 0) \qquad S_i \subseteqq S'_i .$$

Definition 3. Let $\mathcal{F}$ be a set of elementary functions:

$$\mathcal{F} \subseteq \mathcal{E}(k) .$$

The *spectrum of* $\mathscr{F}$ is the series of subsets of $\Omega(k)$ defined as follows:

$$S_i=\{g\in\Omega(k)\,;\ \exists\, G\in\mathscr{F},\ g^*=D^{-i}\cdot G\}\,.$$

4. 3. **~-completeness**

We consider here a spectrum over $\Omega(k)$, that is, a series of subsets of $\Omega(k)$.

DEFINITION 4. Let $S=(S_i)$ be a spectrum over $\Omega(k)$.

We denote by $\tilde{S}$ the minimum spectrum which satisfies the following conditions.

1) $$\tilde{S}\supseteq S$$

2) $$(\tilde{S})_n\supseteq(\tilde{S})_n\circ\wp$$

3) $$(\tilde{S})_n\supseteq(\tilde{S})_{n-d}\circ(\tilde{S})_d \quad \text{for} \quad 0\leqq d\leqq n$$

where $\circ$ denotes the product defined in 2. 3. 2.

DEFINITION 5. (Kudrjavtiev)

Let $\mathscr{F}$ be a set of elementary functions and S the spectrum of $\mathscr{F}$.

$\mathscr{F}$ is said to be *~-complete* if

$$\bigcup_{i=0}^{\infty}(\tilde{S})_i=\Omega(k)\,.$$

Remark. 1) ~-complete $\Rightarrow$ (k)-complete.

The reciprocal is *not* true. For instance, the following set is (2)-complete (Ibuki) but not ~-complete (Kudrjavtiev).

$$K=\{f\in\Omega(2)\,;\ f(0,0,\cdots,0)=f(1,1,\cdots,1)\}$$

2) If $S_i=\phi$ for $i>0$, then $\mathscr{F}$ ~-complete $\Longleftrightarrow \bar{S}_0=\Omega(k)\Longleftrightarrow S_0$ complete.

3) In general,

$$\bigcup_{i=0}^{\infty}(\tilde{S})_i\subseteq\{g\,;\ g^*\in\bar{\mathscr{F}}\}\subseteq\{g\,;\ g^*\in[\mathscr{F}]\}\,.$$

5. Comments and Further Problems

Neumann type problems can be classified accoding to the choice of the following alternatives.

1) Closure operation: $[\mathscr{F}]$, $\bar{\mathscr{F}}$, $\tilde{S}$.

2) Representability: Strongly representable, representable, weakly representable.

3) Restriction on the set $\mathscr{F}$ to be considered: $\mathscr{F}\subseteq\Omega^*$, $\mathscr{F}\subseteq\mathscr{A}$, $\mathscr{F}\subseteq\mathscr{E}$, $\mathscr{F}\subseteq\Omega$ etc.

4) Value of k: $k=2,3,\cdots$.

Comparison of these alternatives is desirable as well as the following related problems:

1) characterization of minimal complete sets.

2) characterization of maximal incomplete sets.

In the case when $\mathcal{F}\subseteq\Omega$, the situation is satisfactory.

1) Post has given all maximal incomplete sets contained in $\Omega(2)$ and thus solved the Neumann type problem for the case when $\mathcal{F}\subseteq\Omega(2)$.

2) Ibuki has shown that there exist exactly 42 distinct types of minimal complete sets in $\Omega(2)$.

3) Słupecki has given a certain criterion for $\mathcal{F}\subseteq\Omega(k)$ to be complete which are refined by Salomaa and Bulter.

4) Jablonski has given all maximal incomplete sets contained in $\Omega(3)$.

5) Rosenberg has given the characterization of all maximal incomplete sets in $\Omega(k)$ for $k\geqq 3$.

For the case when $\mathcal{F}\subseteq\mathcal{E}$ and $k=2$, Kudrjavtiev has given the characterization of maximal $\sim$-incomplete sets. Ibuki considered the set of unit delay functions and characterized the maximal (2)-incomplete sets and minimal (2)-complete sets.

The author has given a certain criterion for $\mathcal{F}\subseteq\mathcal{E}$ to be $\sim$-complete and ennumerated all maximal $\sim$-incomplete sets of unit delay functions for the case $k=3$.

For the case when $\mathcal{F}\subseteq\mathcal{A}$, we know very little. However, Minsky and Loomis investigated the case $k=2$ and obtained certain conditions for $\mathcal{F}$ to be (2)-complete.

The following problems remain to be solved.

1) Comparison of (k)-universality, (k)-completeness and $\sim$-completeness.

2) Characterization of maximal $\sim$-incomplete sets of elementary functions.

3) Characterization of (k)-complete sets of elementary (or admissible) functions.

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