Murase's Quasi-Matrix Rings and Generalizations

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In a series of papers in this journal Ichiro Murase introduced the notion of a quasi-matrix ring over a division ring [7]. He showed that under rather general conditions an Artinian, generalized uniserial ring is a quasi-matrix ring, and conversely.

In an attempt to obtain a more natural representation of a quasi-matrix ring an interesting class of prime, Noetherian rings was encountered. We call these rings infinite quasi-matrix rings. Every quasi-matrix ring is a homomorphic image of an infinite quasi-matrix ring.

Infinite quasi-matrix rings are in a sense generalizations of polynomial rings. In fact, these two classes of rings share many properties: As already mentioned, they are prime and (left and right) Noetherian. Further, they are Jacobson semisimple; every proper homomorphic image is Artinian and generalized uniserial; and every two-sided ideal is principal.

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1. Preliminaries.

Let K be a ring with identity and let S be a semigroup denoted multiplicatively. The *semigroup ring* $K(S)$ *of* S *over* K is the ring whose underlying group is the free K-module with basis S; multiplication is defined by the rule

$$\left(\sum_s f(s)s\right)\left(\sum_s g(s)s\right)=\sum_t\left(\sum_{rs=t} f(r)g(s)\right)t$$

where $f, g\colon S\to K$ are finitely non-zero.

A larger class of rings may be obtained if we consider semigroups with zeros: Let z be a multiplicative zero for S. Clearly then Kz is an ideal of $K(S)$. The *contracted semigroup ring* $K[S]$ *of* S *over* K is the quotient ring of $K(S)$ modulo Kz. One easily sees that $K[S]$ may be constructed as the free K-module with basis $S-\{z\}$ in which multiplication is given by the rule above, subject to the identification of the zero, 0, of the module and the zero, z, of S.

Of course, every ring R may be thought of as either the semigroup ring of the one element semigroup $S=\{1\}$ over itself or as the contracted semigroup ring of $\{1,0\}$ over itself. And, in general, any semigroup ring over a ring K may be thought of as a contracted semigroup ring over the same ring K: one simply adjoins an external zero to the original semigroup. The converse, however, is false; e. g., the ring K_n of $n\times n$ matrices over a division ring K is the contracted semi-group ring of the semigroup $\{e_{ij}\}\cup\{0\}$ of matrix units over K. On the other hand, if a semigroup S contains more than one element, then the semigroup ring $K(S)$ contains the non-trivial ideal of elements $\sum f(s)s$ where $\sum f(s)=0$. Hence, if $n>1$ we cannot have $K(S)$ isomorphic to K_n, [5].

In the theory of semigroups, terms such as subsemigroup, ideal, direct product, etc., usually have the same meaning as the corresponding terms: subring, ideal, direct sum, etc., in ring theory, if one forgets about the additive structure of a ring. The quotient of a semigroup by an ideal, however, requires a separate treatment:

Let I be an ideal of a semigroup S. Let S/I be the set consisting of those elements s in S which are not in I together with an element 0 which is assumed to be not an element of S. S/I becomes a semigroup with zero, 0, if we define the product of two elements s,t to be their product st in S if $s,t\in S$ and $st\notin I$, and to be 0 in all other cases. S/I is called the *Rees factor semigroup of* S *modulo* I. Clearly S/I is a homomorphic image of S.

One easily verifies that the canonical homomorphism of S onto S/I may be extended to a ring homomorphism from the contracted semigroup ring $K[S]$ onto the contracted semigroup ring $K[S/I]$. The kernel of the latter homomorphism is easily seen to be $K[I]$, and so we have

$$K[S]/K[I]\cong K[S/I]$$

whenever I is an ideal of S.

Remark. Because of the above comment concerning semigroup rings, we cannot hope that all ideals of a semigroup ring $K(S)$ be of the form $K(I)$ for some ideal I of S. On the other hand, for certain semigroups with zeros it is the case that every ideal of $K[S]$ has the form $K[I]$ for some ideal I of the semigroup S. E. g., any subsemigroup S of the semigroup of ordinary $n\times n$ matrix units satisfies this property, if $e_{11},\cdots,e_{nn}\in S$ and if K is a division ring. Murase's quasi-matrix units semigroups also satisfy this property. It should, however, be noted that the infinite quasi-matrix semigroups introduced below do not have this property.

2. Quasi-Matrix Units Semigroup Rings.

Let $MU(n)$ denote the usual semigroup of all $n\times n$ matrix units

$$\begin{array}{cccc} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ & \cdots\cdots\cdots & & \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{array}$$

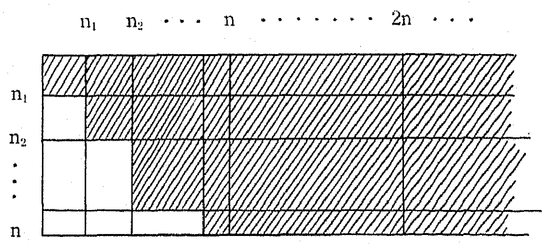
together with 0. This semigroup may be enlarged by extending each row infinitely to the right. More precisely, let $S(n)$ denote the semigroup with elements

(1)
$$\begin{array}{cccccc} e_{11} & e_{12} & \cdots & e_{1n} & e_{1\,n+1} & \cdots \\ e_{21} & e_{22} & \cdots & e_{2n} & e_{2\,n+1} & \cdots \\ & & \cdots\cdots\cdots\cdots & & & \\ e_{n1} & e_{n2} & \cdots & e_{nn} & e_{n\,n+1} & \cdots \end{array}$$

together with a symbol 0. Define products as follows:

$$e_{hi}e_{jk}=\begin{cases} e_{h,i-j+k} & \text{if } i\equiv j \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

By a *complete, block infinite quasi-matrix semigroup* (of degree n) we mean a subsemigroup of $S(n)$ which contains all the elements in an area such as that represented by the shaded portion of the following figure:

(2)


where the integers n_i are fixed but arbitrary subject to the condition $0<n_1<n_2<\cdots<n$. Note that in a sense this is analogous to a complete, block triangular matrix ring.

The simplest case is that in which $n_i=i$, i.e., each diagonal block is 1×1. In this case the semigroup consists of elements

(3)
$$\begin{array}{cccccc} e_{11} & e_{12} & \cdots & e_{1n} & e_{1\,n+1} & \cdots \\ & e_{22} & \cdots & e_{2n} & e_{2\,n+1} & \cdots \\ & & \cdots\cdots\cdots\cdots & & & \\ & & & e_{nn} & e_{n\,n+1} & \cdots \end{array}$$

together with 0. For simplicity we shall deal mostly with semigroups of the form in (3) which we shall denote by $QM(n)$ and refer to as an *infinite quasi-matrix semigroup.*

We shall be primarily interested in the contracted semigroup ring $K[QM(n)]$ over a division ring K. We call this an *infinite quasi-matrix ring* (*of degree n over K*). Using (2) instead of (3) one obtains a *complete, block infinite quasi-matrix ring.*

THEOREM 1. *Let I be any non-zero ideal of $QM(n)$. Let S denote the Rees factor semigroup of $QM(n)$ modulo I. Then, for a division ring K, $K[S]$ is a quasi-matrix ring in the sense of Murase. Conversely, every such ring can be obtained in this manner.*

Proof. Let p_i be n integers such that $1 \leq p_1 \leq p_2 \leq \cdots \leq p_n \leq p_1 + n$. Let I denote the subset of $QM(n)$ consisting of all e_{ij} with $j \geq p_i$ together with 0. This can be easily seen to be an ideal. The Rees factor semigroup T of $QM(n)$ modulo I consists of e_{ij} where $j < p_i$ together with an element 0. Multiplication in T is given by $e_{hi}e_{jk} = e_{h,i-j+k}$ if $i \equiv j \pmod{n}$ and $i-j+k < p_h$, and all other products yield 0.

Now $K[T]$ is precisely what Murase called a quasi-matrix ring.

We leave it to the reader to verify that every ideal of $QM(n)$ has this form.

Remark. Murase also defined quasi-matrix rings of general form. From our point of view these correspond to the contracted semigroup rings of Rees quotients of the semigroups indicated by (2).

To facilitate computations in these rings we now introduce a new way of representing them.

Let $K[x]$ denote the ring of polynomials in x over a division ring K. Let A denote the ring of all $n \times n$ matrices (a_{ij}) over $K[x]$ such that x divides a_{ij} whenever $i > j$. Symbolically we write

$$A = \begin{pmatrix} K[x] & K[x] & \cdots & K[x] \\ (x) & K[x] & \cdots & K[x] \\ \cdots & \cdots & \cdots & \cdots \\ (x) & (x) & \cdots & K[x] \end{pmatrix} \tag{4}$$

THEOREM 2. *The infinite quasi-matrix ring $K[QM(n)]$ is isomorphic to the ring A of matrices of the form* (4).

Proof. Note that we always consider $QM(n)$ as a subsemigroup of $S(n)$ given by (1). Let us consider the following element of $K[QM(n)]$,

$$y = \sum_{i=1}^{n} e_{i\,i+n}\,.$$

Then we have $y^k=\sum_{i=1}^{n} e_{i\,i+kn}$, and so

$$y^k e_{ij}=e_{ij}y^k=e_{i\,j+kn},$$

as can be readily seen by the multiplication rule of $S(n)$. Hence the elements of $QM(n)$ can be written as follows.

$$\begin{array}{llllllllll}
e_{11} & e_{12} & \cdots & e_{1n} & ye_{11} & ye_{12} & \cdots & ye_{1n} & y^2e_{11} & \cdots \\
 & e_{22} & \cdots & e_{2n} & ye_{21} & ye_{22} & \cdots & ye_{2n} & y^2e_{21} & \cdots \\
 & & & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\
 & & & e_{nn} & ye_{n1} & ye_{n2} & \cdots & ye_{nn} & y^2e_{n1} & \cdots
\end{array}$$

Since the elements e_{ij} $(1\leq i\leq n,\ 1\leq j\leq n)$ are essentially the usual matrix units, it is clear that $K[QM(n)]$ is isomorphic to A defined by (4). Let E_{ij} $(1\leq i\leq n,\ 1\leq j\leq n)$ be the matrix units for A. Then the above isomorphism is given by

$$(5)\qquad x^kE_{ij}\ \rightarrow\ y^ke_{ij}=e_{i\,j+kn}.$$

Theorem 2 enables us to handle an infinite quasi-matrix ring with almost the same facility as the full matrix ring. One would conjecture that many properties of $K[x]$ which are inherited by the full matrix ring $(K[x])_n$ would also be inherited by $K[QM(n)]$. This is indeed true. In fact, we have the following theorem.

THEOREM 3. *Every infinite quasi-matrix ring over a division ring satisfies the following properties:*

(i) *The Jacobson radical is zero.*

(ii) *Every proper homomorphic image is Artinian and generalized uniserial.*

(iii) *Every two-sided ideal is principal.*

(iv) *It is prime, and left and right Noetherian.*

This theorem will be established in the next two sections in some cases as corollaries to more general propositions.

3. Pattern Rings and Generalizations.

In this section R will always denote a ring with identity and R_n will be the ring of all $n\times n$ matrices over R. We shall denote the matrix units of R_n conventionally by e_{ij} $(1\leq i\leq n, 1\leq j\leq n)$. Although these notations originally meant elements of $S(n)$, there will be no fear of confusion.

Let $\leq$ be any quasi-ordering, i.e., transitive, reflexive relation on the set $\{1, 2, \cdots, n\}$. Let I be any ideal of R. Then $A=(I, R, \leq, n)$ will denote the subring of R_n consisting of all matrices $\sum r_{ij}e_{ij}$ where $r_{ij}\in I$ if $i\not\leq j$. If, for example, $\leq$ is the usual ordering of the integers, then A will be the ring of matrices indicated by:

$$(6)\qquad \begin{pmatrix} R & R & \cdots & R \\ I & R & \cdots & R \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ I & I & \cdots & R \end{pmatrix}$$

If $I=0$, this is just the ring of all upper triangular matrices over R.

Another example is the ring corresponding to the quasi-ordering $\leq$ on $1, 2, 3, 4$, defined by $4\leq 2\leq 1$, $4\leq 3\leq 1$ and $i\leq i$ for $i=1, 2, 3, 4$. This yields the ring of all 4×4 matrices of the form:

$$\begin{pmatrix} R & I & I & I \\ R & R & I & I \\ R & I & R & I \\ R & R & R & R \end{pmatrix}$$

In case $I=0$, the rings obtained by this process are the same as the pattern rings considered by Mitchell [6]. The case in which $I=0$ and R is a field or division ring was investigated by Clark [2], [3], and [4].

Our immediate motivation for considering these rings is that if we specialize (6) by taking $R=K[x]$ and $I=(x)$, then we obtain the infinite quasi-matrix ring (4).

One easily verifies that the ring $A=(I, R, \leq, n)$ is a subring of R_n which contains all of the matrix units $e_i=e_{ii}$. Furthermore, if $i\leq j$, then $e_{ij}\in A$. If $i\not\leq j$, then we can only say that $re_{ij}\in A$ for $r\in I$.

In the remainder of this section we adhere to the following notation: I is an ideal of R; $\leq$ is a quasi-ordering of $1, 2, \cdots, n$, and $r(I)$ (resp. $l(I)$) denotes the right (resp. left) annihilators of I in R.

THEOREM 4. *If $r(I)=0$ or $l(I)=0$, and if R is (Jacobson) semisimple, then $A=(I, R, \leq, n)$ is also.*

Proof. Let J denote the radical of A. Then, as is well known, since e_i is idempotent, the radical of e_iAe_i is e_iJe_i. Since $R\cong e_iAe_i$ and R is semisimple, we have $e_iJe_i=0$.

Suppose that re_{pq} is a non-zero element of e_pJe_q. Now, if $r(I)=0$, $Ir\neq 0$ and so there is an element $x\in I$ such that $xr\neq 0$. Since $x\in I$, $xe_{qp}\in A$, and so $xre_{qq}=(xe_{qp})(re_{pq})$ is a non-zero element of e_qJe_q. This contradiction shows that $e_pJe_q=0$ for all p, q. Hence $J=0$. A similar argument applies if $l(I)=0$.

COROLLARY. *Let R be a semisimple ring containing no zero divisors. Then $(I, R, \leq, n)$ is semisimple for all non-zero ideals I of R.*

Example. To see that the annihilator condition in the above theorem is necessary, consider the ring $R=K\oplus K$ where K is a field and let $I=K\oplus(0)$. Then A may be represented as the ring of all matrices

$$\begin{pmatrix} K & 0 & K & 0 \\ 0 & K & 0 & K \\ K & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix},$$

since R is isomorphic to the ring

$$\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$

One easily shows that A is not semisimple.

THEOREM 5. *If R is a principal ideal ring, then so is any subring A of R_n which contains Re_{ii} for all $i=1, 2, \cdots, n$.*

Proof. (We call R a principal ideal ring if every two-sided ideal is generated by a single element.)

Let B be an ideal of A. Then the sets

$$(7) \qquad B_{ij}=\{r\in R \mid re_{ij}\in B\}$$

are ideals of R. Let $B_{ij}=Rb_{ij}R$. Then B is generated by $\sum b_{ij}e_{ij}$.

THEOREM 6. *If R is Artinian (Noetherian), then so is any subring A of R_n which contains R.*

Proof. As usual, we identify R with the R-multiples of the identity in R_n. Consider R_n as a left R-module as usual. Then R_n is finitely generated as an R-module and A is a submodule. If R is Artinian (Noetherian), then so is any finitely generated R-module. A submodule of an Artinian (Noetherian) module is again Artinian (Noetherian). Thus, since a left ideal of A is a left R-module and also a submodule of R_n, the conclusion is clear.

THEOREM 7. *Let R be a ring such that every proper homomorphic image is Artinian, then $A=(I, R, \leq, n)$ has this property also if $r(I)=l(I)=0$.*

Proof. Let B be a non-zero ideal of A and let B_{ij} be as in (7). Suppose $B_{pq}\neq 0$ for some p, q. Let b be a non-zero element of B_{pq}. By the annihilator hypothesis we can find x, y in I such that xby is not zero. Since x and y are in I, xe_{ip} and ye_{qj} are in A for any i and j. Hence $(xby)e_{ij}=(xe_{ip})(be_{pq})(ye_{qj})$ is a non-zero element of B, and $xby \in B_{ij}$. Thus, if B is not zero, B_{ij} is not zero for all i, j.

Now as a left R-module $A = \oplus M_{ij}$ where $M_{ij} = Re_{ij}$ or $M_{ij} = Ie_{ij}$. Also, $B = \oplus C_{ij}$ where $C_{ij} = B_{ij}e_{ij}$. It follows that as a left R-module $A/B \cong \oplus (M_{ij}/C_{ij})$. Since M_{ij}/C_{ij} is isomorphic to R/B_{ij} or I/B_{ij}, it is clear that as a left R-module A/B is Artinian; hence as a ring A/B is Artinian.

THEOREM 8. *Let R be a ring containing no zero divisors, and let I be a non-zero ideal of R. Then $A = (I, R, \leq, n)$ is prime.*

Proof. Assume that $aAb = 0$ for some $a, b \in A$. Now if a and b are both non-zero, we have $e_i a e_j$ and $e_p b e_q$ non-zero for some i, j, p, q. Let x be a non-zero element of I, then xe_{jp} is in A. But then $(e_i a e_j)(xe_{jp})(e_p b e_q)$ is clearly a non-zero element of $e_i(aAb)e_q$. This contradiction shows that $a = 0$ or $b = 0$. Hence A is prime.

4. Quotients of Infinite Quasi-Matrix Rings.

In this section, K will denote a division ring and $K[x]$ the ring of polynomials in x over K. Obviously every (two-sided) ideal of $K[x]$ is of the form $K[x]f = fK[x]$ for some monic polynomial f whose coefficients are in the center C of K. We denote such an ideal by (f).

LEMMA 1. *Let $A = ((x), K[x], \leq, n)$ where $\leq$ is the usual order of the integers. Then every ideal of A is of the form*

$$(8) \qquad \begin{pmatrix} (f_{11}) & (f_{12}) & \cdots & (f_{1n}) \\ (f_{21}) & (f_{22}) & \cdots & (f_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ (f_{n1}) & (f_{n2}) & \cdots & (f_{nn}) \end{pmatrix},$$

where each f_{ij} is a monic polynomial in $C[x]$ such that

(a) if $j < h$, $f_{ij} = f_{ih}$ *or* xf_{ih},

(b) if $i < k$, $f_{kj} = f_{ij}$ *or* xf_{ij}.

Conversely, any additive subgroup of A of the form (8) *which satisfies* (a) *and* (b) *is an ideal of A.*

Proof. Let I be an ideal of A. Then as in Theorem 5, I is a group direct sum of $I_{ij}e_{ij}$ where I_{ij} is an ideal of $K[x]$. Thus $I_{ij} = (f_{ij})$ for a monic polynomial f_{ij} in $C[x]$. This is what we mean by having the form (8).

Now let us observe that if g and h are monic polynomials in $C[x]$, then the following statements are equivalent:

(i) $g = h$ or $g = xh$

(ii) $xh \in (g)$ and $g \in (h)$.

We now recall that $e_{ij} \in A$ if $i \leq j$ and that xe_{ij} is always in A.

To prove (a) note that

$$xf_{ih}e_{ij}=(f_{ih}e_{ih})(xe_{hj})\in I.$$

It follows that $xf_{ih}\in(f_{ij})$. If $j\leq h$ we have $e_{jh}\in A$ and hence

$$f_{ij}e_{ih}=(f_{ij}e_{ij})e_{jh}\in I.$$

This implies that $f_{ij}\in(f_{ih})$. Thus we have $xf_{ih}\in(f_{ij})$ and $f_{ij}\in(f_{ih})$, and so (a) holds by the above comment.

A similar argument establishes (b).

Now from (a) it is clear that

$$(f_{i1})\subseteq(f_{i2})\subseteq\cdots\subseteq(f_{in}) \tag{9}$$

for each i, and

$$(f_{1j})\supseteq(f_{2j})\supseteq\cdots\supseteq(f_{nj}) \tag{10}$$

for each j. It is also clear that each f_{ij} is either equal to f_{in} or xf_{in}. In particular, for all p, q, $x(f_{iq})\subseteq(f_{ip})$. Similarly, each f_{ij} is equal to f_{1j} or xf_{1j} and $x(f_{qj})\subseteq(f_{pj})$ for all p, q.

Let us prove that a subgroup of A^+ of the form (8), which satisfies (a) and (b), is an ideal. Let $\sum g_{ij}e_{ij}\in A$. To show that (8) is a left ideal it suffices to show that for each r, s, f_{rs} divides $\sum_k g_{rk}f_{ks}$. To see this, note that if $k\geq r$, then $(f_{ks})\subseteq(f_{rs})$; so we need only consider those $k<r$. But if $k<r$, then $g_{rk}\in(x)$ and therefore xf_{ks} divides $g_{rk}f_{ks}$. Thus, $g_{rk}f_{ks}\in(xf_{ks})$, which by the above comment is contained in (f_{rs}).

A similar argument shows that (8) is a right ideal.

Definition. We will say that an ideal H having the form (8) is *homogeneous* if each f_{ij} is a power of x or equal to 1.

LEMMA 2. *Let A be as in Lemma* 1 *and let I be an ideal of A. Then $I=Hf$ where H is homogeneous and f is a monic polynomial in* C$[x]$ *such that* $(x,f)=1$.

Proof. Write I in the form (8), and let $f_{1n}=x^sf$ where $(f, x)=1$. From Lemma 1 each f_{ij} is equal to x^sf, $x^{s+1}f$ or $x^{s+2}f$. For $j<i$, since $f_{ij}\in(x)$ it is clear that in case $s=0$, f_{ij} must be equal to one of xf or x^2f. The point is that after dividing out f what is left is a homogeneous ideal in A. That it is an ideal follows from the converse of Lemma 1.

LEMMA 3. *Let A and I be as in Lemma* 2. *Then,*

$$A/I\cong A/H\oplus A/Af_1\oplus A/Af_2\oplus\cdots\oplus A/Af_r,$$

where H is homogeneous and the f_i are pairwise relatively prime monic polynomials in $C[x]$.

Proof. As in Lemma 2, let $I=Hf$ where f is not divisible by x. Now, to show that $A/I \cong A/H \oplus A/Af$ it suffices to show that $Hf = H \cap Af$ and that $A=H+Af$. The first part is clear since a matrix is in $H \cap Af$ if and only if its (i,j)-th entry is in an intersection $(x^t) \cap (xf)=(x^tf)$ or $(x^t) \cap (f)=(x^tf)$, where (x^t) means the (i,j)-th entry in H. To show the second part it suffices to show that $e_i=e_{ii} \in H+Af$ for each i. Note that for some $j \geq 0$, $x^j e_i \in H$. In any case, $fe_i \in Af$. Since $(f, x)=1$, it is clear that $e_i \in H+Af$.

To complete the proof it suffices to show that if $f=f_1f_2$ where $(f_1, f_2)=1$ in $C[x]$, then $A/Af \cong A/Af_1 \oplus A/Af_2$. This follows by essentially the same argument used in the previous paragraph.

THEOREM 9. *Let $A=((x), K[x], \leq, n)$ where $\leq$ is the usual order of the integers. Then every proper homomorphic image of A is Artinian and generalized uniserial.*

Proof. It suffices by Lemma 3 to prove that A/H and A/Af are generalized uniserial, where H is a homogeneous ideal in A and f is a power of an irreducible monic polynomial in $C[x]$. That they are Artinian follows from Theorem 7.

We first consider the isomorphism (5) between $A=((x), K[x], \leq, n)$ and $K[QM(n)]$. Under this isomorphism H corresponds to an ideal of $K[QM(n)]$, which has a basis (over K) a semigroup ideal of $QM(n)$. Thus, A/H has the form $K(S)$ where S is as in Theorem 1, and hence it is a quasi-matrix ring over K. Therefore A/H is generalized uniserial by Murase [7].

Let $f=g^m$ where g is an irreducible monic polynomial in $C[x]$. Then the ideal (g) is maximal in $K[x]$, and so from Lemma 1 it is clear that Ag is a maximal two-sided ideal of A. Hence A/Ag is simple. Since A/Ag is the quotient of $A'=A/Af$ by the nilpotent ideal Ag/Af, it follows that A' is a primary ring. Since obviously every two-sided ideal of A' is a principal ideal, A' is uniserial by a theorem of Asano [1].

Finally, we conclude Theorem 3.

Proof of Theorem 3. It is well known that $K[x]$ has no zero divisors and satisfies the properties (i)—(iv). It follows from Theorem 2 that an infinite quasi-matrix ring A is isomorphic to a ring $((x), K[x], \leq, n)$ where $\leq$ is the usual order of $1, 2, \cdots, n$. Thus, from Theorems 4—9, we conclude immediately that the properties (i)—(iv) hold for A.

Remark. Every complete, block, infinite quasi-matrix ring (2) may be represented as the ring of all block matrices over $K[x]$ of the form

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} \end{pmatrix}$$

where B_{ij} is an $n_i \times n_j$ matrix over $K[x]$ such that every entry of B_{ij} is divisible by x if $i>j$. This is clearly of the form $((x), K[x], \leq, n)$ if one defines the quasi-order $\leq$ appropriately. One may establish fairly easily that Theorem 3 also holds for this class of rings. However, the details of the analogue of Theorem 9 are somewhat more cumbersome.

Question. To what extent do the four properties in Theorem 3 characterize such rings? It is clear that rings of the form $((p), R, \leq, n)$ where (p) is a prime ideal of a principal ideal domain R share all these properties with the possible exception of the latter part of (ii).

Rings satisfying the condition that all proper homomorphic images are Artinian are investigated by Ornstein [8]. However, Ornstein's work concerns mostly non-prime rings which, except in trivial cases, have non-zero Jacobson radicals.

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