Eisenstein Series and Representations of Some Finite Group

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Let k be the finite field with q elements ($q=p^n$. p is a prime number). We consider the representation of $G=SL(2,k)$, induced by the identity representation of $N=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\ x\in k\right\}$. This representation is described as follows:

$$H=\{f(a,b),\ \ a,b\in k;\ f(0,0)=0\}$$

$$T_g f(a,b)=f(\alpha a+\gamma b, \beta a+\delta b) \quad \text{for} \quad g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G\,.$$

We define the norm of $f\in H$ by $\|f\|^2=\sum_{a,b\in k}|f(a,b)|^2$. Then T_g is unitary.

For a multiplicative character χ of k, we put

$$f_\chi(a,b)=\sum_{t\in k^*}\overline{\chi(t)}f(ta,tb)\,.$$

Then, we have $f_\chi(ta,tb)=\chi(t)f_\chi(a,b)$, that is f_χ is a χ-homogeneous function.

We donote by H_χ the sub space of H consisting of χ-homogeneous functions. Then, as is easily seen, H_χ is a invariant subspace and

$$H=\sum_\chi \oplus H_\chi\,.$$

(orthogonal projection P_χ onto H_χ is given by $P_\chi f=f_\chi$)
Denoting by $U_g{}^\chi$ the restriction of T_g to H_χ, we have the following direct sum decomposition;

$$T_g=\sum_\chi \oplus U_g{}^\chi\,.$$

It is known [1] that if $\chi\neq\bar\chi$ (i.e. χ is not real), $(U_g{}^\chi,\ H_\chi)$ is irreducible and $U_g{}^\chi$ and $U_g{}^{\bar\chi}$ are equivalent to each other.

Therefore, there exists an isometry V of H_χ onto $H_{\bar\chi}$ such that

$$VU_g{}^\chi=U_g{}^{\bar\chi}V\,.$$

Moreover, V is unique up to a constant factor.

In this paper we shall give the explicit form of V by means of the functional equation of the Eisen stein series for some discontinuous group of Hilbert modular type.

1. Here we are concerned with some type of discontinuous group introduced by T. Kubota in his study about reciprocity law.

Let k be algebraic number field of degree $n=r_1+2r_2$ with r_1 real conjugates $k^{(l)}$ $(1\leqq l\leqq r_1)$ and r_2 pairs of complex conjugates $k^{(m)}$, $k^{(m+r_1)}$ $(r_1+1\leqq m\leqq r_1+r_2)$.

Let $\mathfrak{o}$ be the ring of integers in k. For an integral ideal $\mathfrak{f}$ in k, we put

$$\Gamma=\left\{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2, \mathfrak{o})\ \gamma\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\bmod \mathfrak{f}\right\}.$$

Γ can be regarded as a discrete subgroup of

$$G=SL(2, R)\times\cdots\times SL(2, R)\times SL(2, C)\times\cdots\times SL(2, C)$$

by the injection map: $\Gamma\ni\gamma\to(\gamma^{(1)}, \cdots)$, where we denote by $\gamma^{(i)}$ the i-th conjugate of γ.

Therefore Γ operates discontinuously on $X=K/G$

where $$K=SO(2)\times\cdots\times SO(2)\times SU(2)\times\cdots\times SU(2)$$

(a maximal compact subgroup of G). Moreover, the fundamental domain of Γ is of finite volume. X can be realized as follows:

$$X=H_r\times\cdots\times H_r\times H_c\times\cdots\times H_c\ .$$

where $$H_r=\{\tau=x+iy,\ y>0\}\quad\text{(upper half plane)}$$

$$H_c=\{\zeta=(z, v);\ z\in C,\ v>0\}\quad\text{(upper half space)}\ .$$

$SL(2, \boldsymbol{R})\ni g$ operates on H_r as a fractional linear transformation. If $g\in SL(2, C)$, g operates on H_c in the following way;

$$g\zeta=\left(\frac{v}{|cz+d|^2+|cv|^2}\ ,\ \frac{(az+b)(\overline{cz+d})+a\bar{c}v^2}{|cz+d|^2+|cv|^2}\right).$$

By combining these, we can see how G operates on X. (for details see Kubota [4]).

2. For a given prime number p and a positive integer n, there exists an algebraic number field of degree n such that (p) is prime in k. (For example, see Hasse [3]).

For this k and p, we define the group Γ as in 1.

Let $\mathfrak{d}$ be the different of k. Then, there exists an integral ideal $\mathfrak{a}'$ and $c\in k$, such that $\mathfrak{d}=c\mathfrak{a}'^2$. We put $\mathfrak{a}=\mathfrak{a}'^{-1}$. As $(p, \mathfrak{d})=\mathfrak{o}$, we have

$$\mathfrak{a}/p\mathfrak{a}\cong\mathfrak{o}/(p)\cong F_{p^n}\ .$$

Now, we define the theta series as follows.

For $t=(t_1, \cdots t_{r+r})$ $(t_j>0)$, $\hat{a}\in\mathfrak{a}\times\mathfrak{a}$ and $(\tau_1\cdots\tau_{r_1}\zeta_{r_1+1}, \cdots\zeta_{r_1+r_2})\in X$, we put

(1) $$\Theta(t, \hat{a})=\sum_{\substack{\alpha\in\mathfrak{a}\times\mathfrak{a}\\ \alpha\equiv a \bmod p}} \exp\left[-\pi\left\{\sum_{k=1}^{r_1}\frac{t_k}{|c_k|}\,\frac{|\alpha^{(k)}\tau_k+\beta^{(k)}|^2}{y_k}\right.\right.$$
$$\left.\left.+2\sum_{l=r_1+1}^{r_1+r_2}\frac{t_l}{|c_l|}\,\frac{|\alpha^{(l)}z_l+\beta^{(l)}|^2+|\alpha^{(l)}v_l|^2}{v_l}\right\}\right].$$

($\Theta(t, \hat{a})$ depends only on the class of $\hat{a}$ mod p, $\hat{a}=(a, b)$)

Then we have the following theta formula:

(2) $$\Theta(t, \hat{a})=\frac{1}{p^{2n}N(t)}\sum_{\Lambda \bmod p} e^{\frac{2\pi i}{p}S_p\frac{1}{c}\begin{vmatrix} a & b\\ \lambda & \mu\end{vmatrix}}\,\Theta\left(\frac{1}{p^2t}, \Lambda\right)$$

where we put $$N(t)=\prod_{k=1}^{r_1} t_k \prod_{l=r_1+1}^{r_1+r_2} t_l^2 .$$

Proof. We put

$$F(Z)=F(\zeta_1, \cdots, \zeta_{r_1+r_2})=\sum_{\substack{\alpha\in\mathfrak{a}\times\mathfrak{a}\\ \alpha\equiv 0 \pmod p}} \exp\left[-\pi\left\{\sum_{k=1}^{r_1}\frac{t_k}{|c_k|}\,{}^t(\zeta_k+\alpha^{(k)})C^{(k)}(\zeta_k+\alpha^{(k)})\right.\right.$$
$$\left.\left.+2\sum_{l=r_1+1}^{r_1+r_2}\frac{t_l}{|c_l|}\,{}^t(\zeta_l+\alpha^{(l)})C^{(l)}(\zeta_l+\alpha^{(l)})\right\}\right].$$

$$C^{(i)}=\begin{cases}\begin{pmatrix}\dfrac{|\tau_i|^2}{y_i}, & \dfrac{x_i}{y_i}\\[2mm] \dfrac{x_i}{y_i}, & \dfrac{1}{y_i}\end{pmatrix} & 1\leqq i\leqq r_1\\[6mm] \begin{pmatrix}\dfrac{|z_i|^2}{v_i}+v_i, & \dfrac{z_i}{v_i}\\[2mm] \dfrac{\bar{z}_i}{v_i}, & \dfrac{1}{v_i}\end{pmatrix} & r_1+1\leqq i\leqq r_1+r_2\end{cases}$$

$$\zeta_i\in \boldsymbol{R}^2\ (1\leqq i\leqq r_1), \quad \in \boldsymbol{C}^2\,(r_1+1\leqq i\leqq r_1+r_2) .$$

As we have

$$F(Z+\beta)=F(Z), \quad \text{for} \quad \beta\in\mathfrak{a}\times\mathfrak{a}, \quad \beta\equiv 0 \bmod p$$

we have

$$F(Z)=\sum_{M\in(\mathfrak{a}\delta)^{-1}\times(\mathfrak{a}\delta)^{-1}} C_M e^{\frac{2\pi i}{p}S_p{}^tMZ} \quad (\text{Fourier expansion of } F(Z))$$

where

$$S_p{}^tMZ=\sum_{k=1}^{r_1}{}^tM_k\zeta_k+\sum_{l=r_1+1}^{r_1+r_2}({}^tM_l\zeta_l+\overline{{}^tM_l\zeta_l}) .$$

C_M is calculated as follows:

$$C_M=\frac{2^{2r_2}}{|N(c)|p^{2n}}\int_{R^{2r_1}\times C^{2r_2}}\exp\left[-\pi\left\{\sum_k\frac{t_k}{|c_k|}{}^t\zeta_k C^{(k)}\zeta_k+2\sum_e\frac{t_e}{|c_e|}{}^t\bar{\zeta}_e C^{(e)}\zeta_e\right\}\right]$$
$$\times e^{-\frac{2\pi i}{p}S_p{}^tMZ}\,dZ=\frac{2^{2r_2}}{|N(c)|p^{2n}}\prod_{i=1}^{r_1+r_2}C_M{}^{(i)}\,.$$

If $\quad 1\leqq i\leqq r_1\,,\quad C_M^{(i)}=\int_{R^2}e^{-t_i\frac{\pi}{|c_i|}{}^txC^{(i)}x}\,e^{-\frac{2\pi i}{p}{}^tM_ix}\,dx$

$$=\frac{|c_i|}{t_i}\,e^{-\pi\frac{|c_i|}{p^2t_i}{}^tM_iC^{(i)}{}^{-1}_{Mi}}\,.$$

If $\quad r_1+1\leqq i\leqq r_1+r_2\,,\quad C_M^{(i)}=\int_{C^2}e^{-2\pi\frac{t_i}{|c_i|}{}^t\bar{Z}C^{(i)}Z}\,e^{-\frac{2\pi i}{p}({}^tM_iZ+\overline{{}^tM_iZ})}\,dZ$

$$=\left(\frac{|c_i|}{2t_i}\right)^2e^{-\frac{|c_i|}{p^2t_i}2\pi\,{}^t\bar{M}_i\overline{C^{(i)}}{}^{-1}_{M_i}}\,.$$

Hence we have

$$\Theta(t,\hat{a})=F(\hat{a})=\frac{1}{p^{2n}N(t)}\sum_M e^{-\frac{\pi}{p^2}\left\{\sum_{k=1}^{r_1}\frac{|c_k|}{t_k}{}^tM_kC^{(k)-1}M_k+2\sum_{l=r_1+1}^{r_1+r_2}\frac{|c_l|}{t_l}{}^t\bar{M}_l\bar{C}^{(l)-1}M_l\right\}}$$
$$\times e^{\frac{2\pi i}{p}S_p{}^tM}=\frac{1}{p^{2n}N(t)}\sum_{\Lambda\bmod p}e^{\frac{2\pi i}{p}S_p\frac{1}{c}\hat{a}\Lambda}\sum_{\substack{M\equiv\Lambda\\ M\in\mathfrak{a}\times\mathfrak{a}}}\exp\left[-\frac{\pi}{p^2}\left\{\sum_k\frac{1}{|c_k|t_k}\right.\right.$$
$$\left.\left.\times{}^tM_kC^{(k)-1}M_k+2\sum_l\frac{1}{|c_l|t_l}{}^t\bar{M}_l\overline{C^{(l)}}{}^{-1}M_l\right\}\right].$$

As we have

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}A\begin{pmatrix}0&-1\\1&0\end{pmatrix}={}^t\!A^{-1}\,.\quad\text{for}\quad A\in SL(2,C)\,,$$

we finally obtain the following:

$$\Theta(t,\hat{a})=\frac{1}{p^{2n}N(t)}\sum_{\Lambda\bmod p}e^{\frac{2\pi i}{p}S_p\frac{1}{c}\check{\Lambda}\hat{a}}\,\Theta\left(\frac{1}{p^2t},A\right),\qquad \check{A}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}A$$
$$=\frac{1}{p^{2n}N(t)}\sum_{\Lambda\bmod p}e^{\frac{2\pi i}{p}S_p\frac{1}{c}\begin{vmatrix}a&b\\ \lambda&\mu\end{vmatrix}}\,\Theta\left(\frac{1}{p^2t},A\right).$$

3. By definition, the Eisenstein series for Γ is

$$E((\tau,z),s;(a,b))=\sum_{\substack{\alpha,\beta\in\mathfrak{a}\\ \{\alpha,\beta\}\equiv\{a,b\}\bmod p\mathfrak{a}\\ \{\alpha,\beta\}\sim\{\varepsilon\alpha,\varepsilon\beta\}\\ \varepsilon=\text{unit},\ \varepsilon\equiv1\bmod(p)}}\prod_{k=1}^{r_1}\frac{y_k{}^s}{|\alpha^{(k)}\tau_k+\beta^{(k)}|^{2s}}\prod_{l=r_1+1}^{r_1+r_2}\frac{1}{(|\alpha^{(l)}z_l+\beta^{(l)}|^2+|\alpha^{(l)}v_l|^2)^s}\,.$$

where $(\tau,z)\in X$, $a,b\in\mathfrak{a}$, and s is a complex number whose real part is sufficiently large. As is known, using the theta series (1), $E((\tau, z), s; (a, b))$ can be extended to the whole complex plane as a meromorphic function.

For $\gamma \in SL(2, \mathfrak{o})$, we have

$$E(\gamma(\tau, z), s; (a, b)) = E((\tau, z), s; (a, b)\gamma) .$$

In particular, $E((\tau, z), s; (a, b))$ is Γ-invariant.

For a character $\chi \bmod (p)$ such that $\chi(-1)=1$ and $\chi \neq \bar{\chi}$, we put

$$E((\tau, z), s; (a, b), \chi) = \sum_{t \bmod (p)} \bar{\chi}(t)\, E((\tau, z), s; (ta, tb)) .$$

Let $\mathcal{E}_\chi$ be the space spaned by $E(\cdots, s; (a, b), \chi.)$

Then, the representation of $SL(2, \mathfrak{o})$ in $\mathcal{E}_\chi$ defined by

$$f((\tau, z)) \to f(\gamma(\tau, z))$$

induces a representation of $SL(2, F_{p^n}) = SL(2, \mathfrak{o})/\Gamma$, which is nothing but the representation $(U_g{}^\chi, H_\chi)$ defined in introduction.

By virtue of the theta formula (2), $E((\tau, z), s; \cdots, \chi)$ has a functional equation of the following form;

$$E((\tau, z), s; \cdots, \chi) = c(s)\, V_\chi E((\tau, z), 1-s; \cdots, \bar{\chi}) . \tag{3}$$

where V_χ is an isometry of $H_{\bar{\chi}}$ onto H_χ defined in the following way. ($c(s)$ is a scalar. Here we need not determine the explicit form of $c(s)$.)

$$\text{For} \quad f \in H , \quad V_\chi f(a) = \frac{1}{p^n} S_{\bar{\chi}} \{ \sum_{\lambda \in K} \chi(a-\lambda) f(\lambda) + f(\infty) \} \quad \text{if} \quad a \in K = F_{p^n}$$
$$= \frac{1}{p^n} S_{\bar{\chi}} \sum_{\lambda \in K} f(\lambda) . \qquad \text{if} \quad a = \infty .$$

(Here, for simplicity, we put $(a, 1) = a$, $(1, 0) = \infty$. f is determined by the values at these points.)

$$\text{where} \qquad S_\chi = \sum_{a \in K} e^{\frac{2\pi i}{p} S_p \xi a} \chi(a) . \qquad \text{(Gauss sum)}$$

If we extend χ to $\bar{K} = K \cup \{\infty\}$, by setting $\chi(\infty)=1$, the above formula can the above formula be written in a simpler form:

$$V_\chi f(a) = \frac{1}{p^n} S_{\bar{\chi}} \sum_{\lambda \in \bar{K}} \chi(a-\lambda) f(\lambda) , \quad a \varepsilon \bar{K} . \tag{4}$$

From (3), we obtain

$$U_g{}^\chi V_\chi = V_\chi U_g{}^{\bar{\chi}}$$

that is, V_χ is an isometry which gives the equivalence of $(U_g{}^{\bar{\chi}}, H_{\bar{\chi}})$ to $(U_g{}^\chi, H_\chi)$ (the intertwining operator).

Remark 1. In case K is the real number field, the representation corresponding to (U_g^χ, H) is given in the following form:

$$H=L^2(-\infty, \infty), \quad U_g^\rho f(x)=|cx+d|^{-(1+2i\rho)} f\left(\frac{ax+b}{cx+d}\right), \quad g \in SL(2, R).$$

(ρ is a real number). It is known (for example, see [2]) that (U_g^ρ, H) and $(U_g^{-\rho}, H)$ are equivalent to each other and the intertwining operator V is given as follows:

$$Vf(x)=\frac{1}{\Gamma(i\rho)}\int_{-\infty}^{\infty} |x-y|^{i\rho-1} f(y)dy .$$

This is quite similar to (4). (It is to be noticed that the Gauss sum is the "gamma function" for finite fields and $|\chi|^s$ is a multiplicative character of R (4) may be obtained in the same way as in [2]. But the situation is more complicated, because we can not ignore "the point at infinity".

For example, the "spectrum" of $\left\{U_\xi^\chi : \xi=\begin{pmatrix}1 & \xi \\ 0 & 1\end{pmatrix}\right\}$ is not simple. Consequently, the operator which commutes with U_ξ^χ is not "diagonal".

Remark 2. In case $\chi(-1)=-1$, instead of $\Theta(t; \hat{a})$ we consider the following:

$$\tilde{\Theta}(t; \hat{a})=AF(\hat{a})$$

where $$A=\frac{\partial}{\partial\xi}-\tau_1\frac{\partial}{\partial\eta}, \quad F(Z) \text{ is as in 2. } ((\xi, \eta)=\zeta_1) .$$

Then, we have the formula analogous to (2), by means of which the intertwining operator can be determined also in this case.

4. Eisenstein series plays an important role in the theory of (infinite dimensional) unitary representations.

For example, the structure of the continuous spectrum of a discrete subgroup of $SL(2, R)$ is closely connected with the functional equation of the Eisenstein series for this group.

Moreover, we need the Eisenstein series to construct the trace formula, which gives the asymptotic distribution of the point spectrum (see Tanaka [5] Here we give the explicit form of the functional equation of the (restricted) Eisenstein series for Γ_p (principal congruence subgroup of Stufe p).

1) For a character mod $p\chi(\chi(-1)=-1)$, we put

$$E(\tau, s; a_1, a_2)=\sum_{\substack{(m,n)=1 \\ m\equiv a_1, n\equiv a_2 (p)}} \frac{y^s}{|m\tau+n|^{2s}}$$

$$E_1(\tau, s; \chi)=\sum_{t=1}^{p-1}\bar{\chi}(t)E(\tau, s; 0, t)$$

$$E_{q+2}(\tau, s; \chi)=\sum_{t=1}^{p-1}\bar{\chi}(t)E(\tau, s; t, qt) \qquad 0\leqq q\leqq p-1$$

$$\boldsymbol{E}(\tau, s; \chi)=\begin{pmatrix} E_1(\tau, s; \chi) \\ \vdots \\ E_{p+1}(\tau, s; \chi) \end{pmatrix}.$$

Then, we have

$$\boldsymbol{E}(\tau, s; \chi)=\varphi(s, \chi)T_\chi \boldsymbol{E}(\tau, 1-s; \bar{\chi}).$$

where $$\varphi(s, \chi)=\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\sqrt{p}\,\Gamma(s)}\frac{L(2s-1, \bar{\chi})}{L(2s, \bar{\chi})} \qquad \chi \not\equiv 1$$

($L(s, \chi)$=Dirichlet L-function with the character χ)

$$=\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} \qquad \chi \equiv 1$$

($\zeta(s)$=Riemann zeta function)

$$T_\chi=\frac{1}{\sqrt{p}}\begin{bmatrix} 0, & 1, & \cdots, & & 1 \\ 1, & 0, & \chi(p-1), & \cdots, & \chi(1) \\ \vdots & \chi(1) & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \chi(p-1) \\ 1 & \chi(p-1), & \cdots & \chi(1), & 0 \end{bmatrix}. \qquad \chi \not\equiv 1$$

$$=\frac{1}{p^{2s}-1}\begin{bmatrix} p-1, & p^{2s-1}-1, & \cdots, & p^{2s}-1 \\ \vdots & \ddots & & \vdots \\ p^{2s-1}-1, & \cdots\cdots, & & p-1 \end{bmatrix}. \qquad \chi \equiv 1$$

2) If $\chi(-1)=-1$, we put

$$E(\tau, s; a_1, a_2)=\sum_{\substack{(m,n)=1 \\ m\equiv a_1, n\equiv a_2 \ (p)}} \frac{y^s}{|m\tau+n|^{2s}}\frac{m\tau+n}{|m\tau+n|}.$$

($E_k(\tau, s; \chi)$ and $\boldsymbol{E}(\tau, s; \chi)$ are defined as in 1))

Then, we have

$$\boldsymbol{E}(\tau, s; \chi)=\varphi(s, \chi)T_\chi \boldsymbol{E}(\tau, 1-s; \bar{\chi}).$$

where $$\varphi(s, \chi)=\frac{i\sqrt{\pi}\,\Gamma(s)}{\sqrt{p}\,\Gamma\left(s+\frac{1}{2}\right)}\frac{L(2s-1, \bar{\chi})}{L(2s, \bar{\chi})}.$$

T_χ is the same as in 1).

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