Note on the Integration of the Wave Equation

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Introduction

Many authors have investigated the mixed problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2}+c(x)\frac{\partial u}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial u}{\partial x_j}+\sum_{i=1}^{n}a_i(x)\frac{\partial u}{\partial x_i}+a(x)u=0 .$$

(for the reference, see [5] p. 424). Yosida (in [4]) has attempted to integrate the equation, using Milgram-Lax theorem and the semi-group theory (cf. [2]). In this note we are going to demonstrate that the perturbation theory for semi-groups is applicable to prove the existence of the strong solution. All discussions will be given within the framework of Hilbert space theory.

1. Existence of the strong solution

Let $\mathfrak{H}$ be a Hilbert space, its scalar product and norm are denoted by $(\ ,\ )$ and $|\ |$, respectively. For a linear operator T, we denote its domain (or range) by $\mathfrak{D}(T)$ (or $\mathfrak{R}(T)$). Consider a positive definite self adjoint operator A in $\mathfrak{H}$, and its positive root $A^{1/2}$. Let $\mathfrak{D}\equiv\mathfrak{D}(A)$ and $\mathfrak{D}_{1/2}\equiv\mathfrak{D}(A^{1/2})$. We can regard $\mathfrak{D}_{1/2}$ as a Hilbert space with the scalar product $(u, v)+(A^{1/2}u, A^{1/2}v)$ for $u, v\in\mathfrak{D}_{1/2}$.

Now we consider a following $\mathfrak{H}$-valued homogeneous 2nd order ordinary differential equation:

$$\frac{d^2u}{dt^2}+C\frac{du}{dt}+Au+Bu=0 , \tag{1}$$

with the initial condition:

$$u(0)=u_1\in\mathfrak{D},\ u_t(0)=u_2\in\mathfrak{D}_{1/2} . \tag{2}$$

In the equation (1), C is a bounded operator on $\mathfrak{H}$ with its norm denoted by $|C|$, and B is a linear operator such that the domain $\mathfrak{D}(B)\supset\mathfrak{D}_{1/2}$ and that $|B|_{1/2}\equiv\sup\limits_{|A^{1/2}u|=1}|Bu|$ is finite.

We say that a $\mathfrak{H}$-valued function $u(t)$ is a solution of (1) and (2) if it is twice strongly differentiable in $\mathfrak{H}$, satisfying (1) and (2) (this implies especially that $u(t)\in\mathfrak{D}$ for any t). First the solution will be found in the slightly different sense. We note that (1) and (2) can be written in the matrix form by putting $u_1(t) \equiv u(t)$ and $u_2(t) \equiv u_t(t)$,

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i\begin{pmatrix} 0 & -iI \\ iA & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -B & -C \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3}$$

$$(u_1(0), u_2(0)) \in \mathfrak{D} \times \mathfrak{D}_{1/2}, \tag{4}$$

where I is the identity operator from $\mathfrak{D}_{1/2}$ to $\mathfrak{D}_{1/2}$. For $u, v \in \mathfrak{D}_{1/2}$, we define $(u, v)_{\mathfrak{E}} \equiv (A^{1/2}u, A^{1/2}v)$. The quantity $|u|_{\mathfrak{E}} \equiv (u, u)_{\mathfrak{E}}^{1/2}$ becomes a norm on $\mathfrak{D}_{1/2}$ since the positivity of A implies that 0 is not an eigenvalue of A. Let $\mathfrak{E}$ be the completion of $\mathfrak{D}_{1/2}$ by the norm $|\ |_{\mathfrak{E}}$. Naturally $\mathfrak{E}$ is a Hilbert space with the scalar product $(\ ,\ )_{\mathfrak{E}}$. We remark that if A is strictly positive i.e. there exists a positive constant α such that $(Au, u) \geqq \alpha(u, u)$ for all $u \in \mathfrak{D}$, then $\mathfrak{E}$ is homeomorphic to $\mathfrak{D}_{1/2}$.

Since $A^{1/2}$ is closed in $\mathfrak{H}$, A is closable as an operator from $\mathfrak{E}$ to $\mathfrak{H}$. We denote by $\tilde{A}$ the smallest closed extension of A in this sense. The set $\mathfrak{D}(\tilde{A})$ coincides with the completion of $\mathfrak{D}$ by the norm $|A^{1/2}u| + |Au|$. Obviously $\mathfrak{D} \subset \mathfrak{D}(\tilde{A}) \subset \mathfrak{E}$ and $\tilde{A}u = Au$ for $u \in \mathfrak{D}$. If A is strictly positive then $\mathfrak{D}(\tilde{A}) = \mathfrak{D}$ and $\tilde{A} = A$. Similarly $\tilde{I}$ is the smallest closed extension of I from $\mathfrak{H}$ into $\mathfrak{E}$. But in this case $\mathfrak{D}(\tilde{I}) = \mathfrak{D}(I) = \mathfrak{D}_{1/2}$ and $\tilde{I} = I$ by the closedness of $A^{1/2}$.

Let $\mathfrak{X} \equiv \mathfrak{E} \times \mathfrak{H}$. Its scalar product is naturally given by

$$((u, v)) \equiv (u_1, v_1)_{\mathfrak{E}} + (u_2, v_2),$$

for $u = \langle u_1, u_2 \rangle$, $v = \langle v_1, v_2 \rangle \in \mathfrak{X}$. We denote the norm of $\mathfrak{X}$ by $\|\ \|$.

Consider linear operators in $\mathfrak{X}$:

$$\mathscr{H}_0 \equiv \begin{pmatrix} 0 & -iI \\ iA & 0 \end{pmatrix} \text{ with } \mathfrak{D}(\mathscr{H}_0) = \mathfrak{D} \times \mathfrak{D}_{1/2},$$

and

$$\mathscr{H} \equiv \begin{pmatrix} 0 & -i\tilde{I} \\ i\tilde{A} & 0 \end{pmatrix} \text{ with } \mathfrak{D}(\mathscr{H}) = \mathfrak{D}(\tilde{A}) \times \mathfrak{D}_{1/2}.$$

LEMMA. *$\mathscr{H}$ is self adjoint in $\mathfrak{X}$.*

Proof. Since $\mathfrak{E}$ coincides with the completion of $\mathfrak{D}$ by $|\ |_{\mathfrak{E}}$, $\mathfrak{D}(\mathscr{H}_0)$ is dense in $\mathfrak{X}$. Definitions of scalar products $(\ ,\ )_{\mathfrak{E}}$ and $((\ ))$ assure the symmetricity of $\mathscr{H}_0$. Definitions of $\tilde{A}$ and $\tilde{I}$ imply that $\mathscr{H}$ is the smallest closed extension of $\mathscr{H}_0$. This means that $\mathscr{H}$ is symmetric. Therefore $\mathscr{H}$ is self adjoint if $\mathfrak{R}(\mathscr{H} \pm i)$ is dense in $\mathfrak{X}$. Notice $\mathfrak{R}(\mathscr{H} \pm i) \supset \mathfrak{R}(\mathscr{H}_0 \pm i)$. In order to get this lemma, we have only to show that $\mathfrak{R}(\mathscr{H}_0 \pm i)$ is dense in $\mathfrak{X}$. Now we put $v_1^{\pm} \equiv \mp i(A+1)^{-1}(u_1 \pm u_2)$ and $v_2^{\pm} \equiv i(A+1)^{-1}(Au_1 \mp u_2)$ for $u_1 \in \mathfrak{D}$, $u_2 \in \mathfrak{H}$. Since $\mathfrak{R}((A+1)^{-1}) = \mathfrak{D}(A) = \mathfrak{D}$, we have $v_1^{\pm}$, $v_2^{\pm} \in \mathfrak{D}$. By a direct calculation, we see that $(\mathscr{H}_0 \pm i)\begin{pmatrix} v_1^{\pm} \\ v_2^{\pm} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ holds. This shows that $\mathfrak{R}(\mathscr{H} \pm i) \supset \mathfrak{D} \times \mathfrak{H}$. As $\mathfrak{D} \times \mathfrak{H}$

dense in $\mathfrak{X}$, the assertion is proved.

By the assumption on B, B is a densely defined bounded operator from $\mathfrak{E}$ to $\mathfrak{H}$. Therefore we can regard B as a bounded operator from $\mathfrak{E}$ to $\mathfrak{H}$. The operator $\mathscr{B}\equiv\begin{pmatrix}0 & 0\\ -B & -C\end{pmatrix}$ is bounded in $\mathfrak{X}$, and its norm is estimated by

$$\|\mathscr{B}\|\leqslant\sqrt{2}\max(|B|_{1/2}, |C|)\ .$$

Taking account of the above argument we can rewrite (3), (4) as the following:

$$\frac{d}{dt}u=i\mathscr{H}u+\mathscr{B}u\ , \tag{5}$$

$$u(0)=u\in\mathfrak{D}(\mathscr{H})\ . \tag{6}$$

THEOREM 1.

For any $u\in\mathfrak{D}(\mathscr{H})$, *there exists a unique solution* $u(t)$ $(-\infty<t<\infty)$ of (5), (6) such that

1° $u(t)$ *is once continuously differentiable in* $\mathfrak{X}$;

2° $u(0)=u$;

3° $u(t)\in\mathfrak{D}(\mathscr{H})$ *satisfies* (5);

4° $\|u(t)\|\leqslant e^{\|\mathfrak{B}\|\,|t|}\,\|u(0)\|$ *for any* t.

Proof. Since $\mathscr{H}$ is self adjoint, for non zero real λ it holds $\|(\lambda-i\mathscr{H})^{-1}\|\leqslant|\lambda|^{-1}$. Formally we have

$$(\lambda-i\mathscr{H}-\mathscr{B})^{-1}=\sum_{n=0}^{\infty}(\lambda-i\mathscr{H})^{-1}\{\mathscr{B}(\lambda-i\mathscr{H})^{-1}\}^n\ .$$

This identity shows that $(\lambda-i\mathscr{H}-\mathscr{B})^{-1}$ exists for λ with $|\lambda|\leqslant\|\mathscr{B}\|$, and that it satisfies $\|(\lambda-i\mathscr{H}-\mathscr{B})^{-1}\|\leqslant(|\lambda|-\|\mathscr{B}\|)^{-1}$. Therefore we conclude that $i\mathscr{H}+\mathscr{B}$ with the domain $\mathfrak{D}(\mathscr{H})$ generates a unique strongly continuous group $\mathscr{T}_t$ satisfying $\|\mathscr{T}_t\|\leqslant e^{\|\mathfrak{B}\|\,|t|}$ by the well-known Hille-Yosida theorem ([1], [8]). Set $u(t)\equiv\mathscr{T}_t u_0$, then the assertion of the theorem follows.
(The above argument is a special case of the result concerning the bounded perturbation of the infinitesimal generator of a strongly continuous semi-group ([1] Chapt. XIII).)

Remembering that $\mathfrak{E}$ coincides with $\mathfrak{D}_{1/2}$ topologically when A is strictly positive, we have:

COROLLARY.

If A is strictly positive, then for any $\langle u_1, u_2\rangle\in\mathfrak{D}\times\mathfrak{D}_{1/2}$, *there exists a unique solution* $u(t)$ $(-\infty<t<\infty)$ *of (1), (2) such that*

1° $u(t)$ *is once continuously differentiable in* $\mathfrak{D}_{1/2}$ *and twice continuously differentiable in* $\mathfrak{H}$;

2° $u(t)=u_1$ *and* $u_t(0)=u_2$;

3° $\langle u(t), u_t(t)\rangle \in \mathfrak{D}\times\mathfrak{D}_{1/2}$ $(-\infty<t<\infty)$, *and* $u(t)$ *satisfies* (1);
4° *an energy inequality holds for any* t, *that is,*

$$\{|A^{1/2}u(t)|^2+|u_t(t)|^2\}^{1/2}\leqslant e^{\omega|t|}\{|A^{1/2}u(0)|^2+|u_t(0)|^2\}^{1/2},$$

where $\omega=\sqrt{2}\max(|B|_{1/2}, |C|)$.

2. Example

Let Ω be an open domain in R^n. Unless $\Omega=R^n$, we assume the boundary S of Ω consists of a C^2-closed surface. Let $\bar{\Omega}$ be the closure of Ω. Consider the following equation with the Dirichlet condition:

$$\frac{\partial^2 u}{\partial t^2}+c(x)\frac{\partial u}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial u}{\partial x_j}+\sum_{j=1}^{n}a_j(x)\frac{\partial u}{\partial x_j}+a(x)u=0, \tag{7}$$

$$u(0,x)=u_1(x),\ u_t(0,x)=u_2(x), \tag{8}$$

$$u(t,x)=0 \text{ for } x\in S. \tag{9}$$

Assume that real valued functions a_{ij} are continuously differentiable on $\bar{\Omega}$ and symmetric with respect to i and j, and that they satisfy the uniform ellipticity, namely there exists a constant $\alpha>0$ such that $\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\geqslant\alpha^2\sum_{i=1}^{n}\xi_i^2$ for any $x\in\Omega$ and $\xi\in R^n$. Functions a_j, a and c are bounded measurable functions on $\bar{\Omega}$, where $\beta^2=\sup_{1\le j\le n, x\in\bar{\Omega}}|a_j(x)|$, $\gamma=\sup_{x\in\bar{\Omega}}|a(x)-1|$ and $\delta=\sup_{x\in\bar{\Omega}}|c(x)|$ are all finite. We will take x-derivatives in (7) in the sense of distribution.

Let $A\equiv-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}\frac{\partial}{\partial x_j}+1$ with the Dirichlet dondition. It is well known that A is a self adjoint operator in $\mathfrak{H}=L^2(\Omega)$ satisfying $A\geqslant 1$. Let $\mathscr{E}(\Omega)$ (or $\mathscr{D}(\Omega)$) be the totality of C^∞-function on Ω (or C^∞-function which has compact support). We denote by $\mathscr{E}^m_{L^2}(\Omega)$ (or $\mathscr{D}^m_{L^2}(\Omega)$) a completion of $\mathscr{E}(\Omega)$ (or $\mathscr{D}(\Omega)$) by the norm of $\|\varphi\|_m=\{\sum_{|\alpha|\le m}\|D^\alpha\varphi\|^2_{L^2(\Omega)}\}^{1/2}$. Since $(u,v)_{\mathfrak{E}}=\sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial u}{\partial x_j},\frac{\partial u}{\partial x_i}\right)_{L^2(\Omega)}+(u,v)_{L^2(\Omega)}$ in our case, we know that $\mathfrak{E}=\mathfrak{D}_{1/2}=\mathscr{D}^1_{L^2}(\Omega)$ (equalities hold topologically), and that $\mathfrak{D}=\mathscr{E}^2_{L^2}(\Omega)\cap\mathscr{D}^1_{L^2}(\Omega)$ (see [3] Chapt III).

Let $(Cu)(x)\equiv c(x)u(x)$ and $(Bu)(x)\equiv\sum_{j=1}^{n}a_j(x)\frac{\partial u}{\partial x_j}(x)+(a(x)-1)u(x)$. We have $|C|\leqslant\delta$ and $|B|_{1/2}\leqslant\frac{\beta}{\alpha}+\gamma$. Applying Corollary of Theorem 1, we readily obtain the result.

THEOREM 2.
For any $u_1(x)\in\mathscr{E}^2_{L^2}(\Omega)\cap\mathscr{D}^1_{L^2}(\Omega)$ *and* $u_2(x)\in\mathscr{D}^1_{L^2}(\Omega)$, *there exists a unique strong solution* $u(t,x)$ $(-\infty<t<\infty)$ *of* (7), (8) *and* (9) *such that*
1° $u(t,x)$ *is twice strongly differentiable in* $L^2(\Omega)$ *with respect to* t;
2° $u(0,x)=u_1(x)$, $u_t(0,x)=u_2(x)$;

3° *for any* t, $u(t,x) \in \mathscr{E}^2_{L^2}(\Omega) \cap \mathscr{D}^1_{L^2}(\Omega)$ *and* $u_t(t,x) \in \mathscr{D}^1_{L^2}(\Omega)$;
4° *an energy inequality holds for any* t,

$$\left\{\sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial}{\partial x_j}u(t,\ ),\ \frac{\partial}{\partial x_i}u(t,\ )\right)_{L^2(\Omega)}+\|u(t,\ )\|^2_{L^2(\Omega)}+\|u_t(t,\ )\|^2_{L^2(\Omega)}\right\}^{1/2}$$
$$\leqq e^{\omega|t|}\left\{\sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial u_1}{\partial x_j},\ \frac{\partial u_1}{\partial x_i}\right)_{L^2(\Omega)}+\|u_1\|^2_{L^2(\Omega)}+\|u_2\|^2_{L^2(\Omega)}\right\}^{1/2},$$

where $\omega=\sqrt{2}\max\left(\frac{\beta}{\alpha}+\gamma,\ \delta\right)$.

If we assume sufficient smoothness on the boundary and coefficients, we can show that $u(t,x)$ is a genuine solution (see e.g., [3], [4]). The Neumann or Robin boundary condition can be treated analogously, for A is a semi-bounded self adjoint operator in this case.

References

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