Spectral Theory of the Perturbed Homogeneous Elliptic Operator with Real Constant Coefficients

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Introduction

The present paper is concerned with the unitary equivalence of two partial differential operators $P(D)$ and $P(D)+q(x)\cdot$ regarded as operators in the Hilbert space $\mathfrak{H}=L_2(R^n)$. The operator $P(D)$ is a homogeneous elliptic differential operator with real constant coefficients and $q(x)\cdot$ is a multiplicative operator by a real valued function $q(x)$. The unitary equivalence of continuous parts of the spectrum together with some additional information concerning the point spectrum of $P(D)+q(x)\cdot$ will be derived.

Instead of dealing with these operators directly, we consider their Fourier transforms. This method was developed by Faddeev in the case when $P(D)$ is the Laplacian in $L^2(R^3)$ ([1]). We will discuss the general case following after his treatment. In our case some refinement as well as suitable modifications is required. We remark that historically this method is closely connected with the Friedrichs model of the perturbation of continuous spectra ([2], [3]).

The results will be formulated in § 1. In § 2 we will discuss integrals of Cauchy type on R^n. The estimation of these integrals will play a fundamental role in our investigation. In § 3 proofs of the theorems will be given in the case of $n>m$ (m is the order of $P(D)$). If $n\leqq m$, we need some modification and it will be pointed out in the appendix.

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§ 1. Assumptions and results

We consider the operator acting in the space of functions f defined on $R^n (n\geqq 2)$ and defined formally by

$$L[f](k)\equiv p(k)f(k)+\int v(k-k')f(k')dk' ,$$

where $p(k)$ is a polynomial and v a function. In this paper the letter k is a vector in R^n and $|k|$ denotes the length of a vector k. Furthermore, we agree that the integral sign without any indication of the domain of integration indicates the integration over the whole R^n.

We suppose that the polynomial $p=p(k)$ is homogeneous and elliptic, whose coefficients are real. Let m be the degree of p. Using the usual multiple index α, the polynomial p is represented in the form

$$p=p(k)=\sum_{|\alpha|=m} c_\alpha k^\alpha, \text{ where } c_\alpha \text{ is real.}$$

The ellipticity of p means that $p(k)=0$ if and only if $k=0$. Therefore, it is sufficient to consider the problem in the case of $p(k)\geqq 0$ for each $k\in R^n$.

Incidentally, we remark that by means of Fourier transform L is transformed to

$$\tilde{L}=P(D)+q(x)\cdot,$$

where $P(D)$ is the differential operator determined by the formal insertion of $k=\left(\frac{\partial}{i\partial x_1}, \cdots, \frac{\partial}{i\partial x_n}\right)$ into $p(k)$ and $q(x)=(2\pi)^{-n}\int v(k)e^{-ikx}dk$.

We impose on v the following assumption.

ASSUMPTION 1. (1) *For any* k, $v(-k)=\overline{v(k)}$.
(2) *There exist positive constants* θ_0, μ_0, *and* C *such that*

$$\theta_0>\max\left(\frac{n}{2},\ n-m\right),\ \mu_0>\frac{1}{2},$$

and such that v *satisfies for any* k

$$|v(k)|\leqq C(1+|k|)^{-\theta_0},$$

and for $|k_1-k_2|\leqq 1$

$$|v(k_1)-v(k_2)|\leqq C(1+|k_1|)^{-\theta_0}|k_1-k_2|^{\mu_0}.$$

Let L_0 be the operator of multiplication by $p(k)$:

$$(L_0 f)(k)\equiv p(k)f(k)\ .$$

We consider L_0 as an operator in $\mathfrak{H}\equiv L^2(R^n)$. The domain $\mathfrak{D}$ of L_0 is given by

$$\mathfrak{D}\equiv\{f\,|(1+p)f\in\mathfrak{H}\}\ ,$$

and L_0 is a selfadjoint operator in $\mathfrak{H}$. We note that the spectrum of L_0 is absolutely continuous.

With the above assumption on v the operator $L[f]$ restricted to $\mathfrak{S}$, the set of all rapidly decreasing functions on R^n, gives a symmetric operator in $\mathfrak{H}$. We denote this operator by $L|_{\mathfrak{S}}$ and suppose the following assumption.

ASSUMPTION 2. *There exists a unique selfadjoint extension of* $L|_{\mathfrak{S}}$ *having* $\mathfrak{D}$ *as its domain.*

Hereafter L indicates this extension. Under these assumptions our results are formulated in the following theorems.

THEOREM 1. *The point spectrum of* L *is a bounded countable set on the real line having no limiting points except for zero. The multiplicity of any non-zero eigenvalue is finite.*

THEOREM 2. *The continuous spectrum of* L *is absolutely continuous and consists of the whole positive real line except for the point spectrum.*

THEOREM 3. *The absolutely continuous part of* L *is unitarily equivalent to* L_0.

We will close this section after giving some remarks concerning Assumption 2. The norm of $\mathfrak{H}$ is denoted by $\|\ \|$.

Since $L|_{\mathfrak{S}}$ commutes with the conjugation: $f(k) \longrightarrow \overline{f(-k)}$, at least one selfadjoint extension exists (Theorem 4.18 of [7]). Using the method developed in Kato's paper ([4]), we see that (2) of Assumption 1 assures Assumption 2 if $m > \frac{n}{2}$. Remembering that the validity of the inequality: $\|Vf\| \leqq a\|L_0 f\| + b\|f\|$, $f \in \mathfrak{D}$, with $0 \leqq a < 1$, $b \geqq 0$, yields the selfadjointness of the perturbed operator and the preservation of the domain (see (20) of [4]), we know that one of the sufficient, but trivial, conditions for Assumption 2 is $v \in L^1(R^n)$.

For $f \in \mathfrak{S}$ we write

$$(V_0 f)(k) \equiv \int v(k-k') f(k') dk' .$$

Since it follows from (2) of Assumption 1 that $V_0 f \in \mathfrak{H}$, Assumption 2 implies that

$$L = (L_0 + V_0)^{\sim}, \quad L^* = L \text{ and } \mathfrak{D}(L) = \mathfrak{D}(L_0) , \tag{1.1}$$

($\sim$ means the smallest closed extension, $*$ the adjoint and $\mathfrak{D}(L)$ the domain of L). We put $V = L - L_0$, whose domain is $\mathfrak{D}$.

LEMMA 1.1. *For any* $f \in \mathfrak{D}$, *we have*

$$(Vf)(k) = \int v(k-k') f(k') dk' \tag{1.2}$$

for almost every $k \in R^n$.

Proof. For the first step we notice that there exist positive constants a and b such that

$$\|Vf\| \leqq a\|Lf\| + b\|f\| \tag{1.3}$$

holds for $f \in \mathfrak{D}$ (see [6]).

By the relation of (1.1), for any $f \in \mathfrak{D}$ we can choose a sequence $\{f_j\}$ in $\mathfrak{S}$ such that $\lim_{j\to\infty} f_j = f$ and $\lim_{j\to\infty} (L_0 + V_0) f_j = Lf$ in $\mathfrak{H}$. The inequality (1.3) yields that $\lim_{j\to\infty} V_0 f_j = Vf$ in $\mathfrak{H}$. Choosing a suitable subsequence, if necessary, we may assume that $V_0 f_j(k) \to Vf(k)$ for almost every $k \in R^n$. Now we consider the equality

$$V_0 f_j(k) = \int v(k-k') f_j(k') dk' .$$

Since $v \in L^2(R^n)$ by Assumption 1, the right-hand side can be regarded as an inner product of $v(k-k')$ and $\overline{f_j}(k')$ for fixed k. Since $f_j \to f$ in $\mathfrak{H}$, the right-hand side converges to $\int v(k-k')f(k')dk'$ for fixed k. Therefore we conclude that (1.2) holds for almost every k.

By (2) of Assumption 1, the right-hand side of (1.2) is everywhere finite and Hölder continuous for $f\in\mathfrak{D}$. Actually it coincides with Vf as an L^2-element. Therefore we can regard that (1.2) holds for all k. Thus, by the provious lemma we see that, under the present assumptions, the domain of integral operator determined by v is bigger than $\mathfrak{D}$ and $Vf=Lf-L_0f$ is given by (1.2). This fact will be used afterwards.

§ 2. Estimates of integrals of Cauchy type on R^n.

In this section we use some conventional notations. The scalar product and the norm of n-dimensional vectors k and k' are denoted by (k, k') and $|k|$, respectively. For $k\neq 0$, let $\omega_k=|k|^{-1}\cdot k$. The unit sphere of R^n is denoted by Ω and its surface element $d\omega$.

Any k satisfying $p(k)=1$ can be represented as $k=\pi(\omega_k)\cdot\omega_k$. Since p is homogeneous, $\pi(\omega)$ is uniquely determined for each direction of R^n. The function $\pi(\omega)$ is a real valued measurable function on Ω. Since p is elliptic and homogeneous, it follows that there exist positive constants c_1 and c_2 such that

$$0<c_1\leqq\pi(\omega)\leqq c_2\ . \tag{2.1}$$

We note first that, if $\theta<n-1$, we have

$$\int_\Omega |\omega_0-(\omega_0,\ \omega)\omega|^{-\theta}d\omega=c_0\ , \tag{2.2}$$

for any fixed ω_0, where c_0 is dependent on θ but finite and common to all $\omega_0\in\Omega$. This is easily seen by rewriting the left-hand side of (2.2) in the polar coordinate system.

LEMMA 2.1. *For θ, $0<\theta<n-1$, there exists a positive constant C such that*

$$I(k,\ t)\equiv\int_\Omega (1+|k-t\pi(\omega)\omega|)^{-\theta}d\omega$$
$$\leqq C(1+|k|)^{-\theta_1}(1+|t|)^{-\theta_2}$$

for any $k\in R^n$, $t\in R^1$ and θ_1, θ_2 satisfying $0\leqq\theta_1, \theta_2\leqq\theta$, $\theta_1+\theta_2=\theta$.

Proof. $I(k,\ t)\leqq\int_\Omega |k-t\pi(\omega)\omega|^{-\theta}d\omega$

$$=|k|^{-\theta}\int_\Omega \left|\omega_k-|k|^{-1}\cdot t\pi(\omega)\omega\right|^{-\theta}d\omega\ .$$

Since the minimum of the distance between a point and a line is the length of the perpendicular from the point to the line,

$$\left|\omega_k-|k|^{-1}\ t\pi(\omega)\omega\right|\geqq|\omega_k-(\omega_k,\ \omega)\omega|\ .$$

Using (2.2), we obtain, for $0<\theta<n-1$,

$$I(k,\ t)\leqq|k|^{-\theta}\int_{\Omega}|\omega_k-(\omega_k,\ \omega)\omega|^{-\theta}d\omega=c_0|k|^{-\theta}.$$

Since the uniform boundedness of $I(k,\ t)$ is trivial, we have

$$I(k,\ t)\leqq C(1+|k|)^{-\theta}. \tag{2.3}$$

On the other hand

$$I(k,\ t)\leqq\int_{\Omega}\pi(\omega)^{-\theta}|\pi(\omega)^{-1}k-t\omega|^{-\theta}d\omega\ .$$

By (2.1), we obtain

$$I(k,\ t)\leqq c_1{}^{-\theta}\int_{\Omega}\left|\pi(\omega)^{-1}|k|\omega_k-t\omega\right|^{-\theta}d\omega\ .$$

Since the triangle with the vertices 0, $\pi(\omega)^{-1}\cdot|k|\omega_k$, and $t\cdot\omega$ is congruent to the triangle with the vertices 0, $t\omega_k$, and $\pi(\omega)^{-1}|k|\omega$, we get

$$\begin{aligned}\left|\pi(\omega)^{-1}|k|\omega_k-t\omega\right|&=|t|\cdot\left|\omega_k-(t\pi(\omega))^{-1}\cdot|k|\omega\right|\\&\geqq|t|\ |\omega_k-(\omega_k,\ \omega)\omega|\ .\end{aligned}$$

With the aid of (2.2), we have

$$I(k,\ t)\leqq c_0c_1{}^{-\theta}|t|^{-\theta}.$$

Therefore, for $0<\theta<n-1$,

$$I(k,\ t)\leqq C(1+|t|)^{-\theta}. \tag{2.4}$$

By (2.3) and (2.4), $I\equiv I(k,\ t)$ is estimated by

$$I(k,\ t)=I^{\frac{\theta_1}{\theta}}I^{\frac{\theta_2}{\theta}}\leqq C(1+|k|)^{-\theta_1}(1+|t|)^{-\theta_2}.$$

For any $\theta\geqq 0$, and μ, $0<\mu\leqq 1$, the norm $||f||_{\theta,\mu}$ of a function $f(k)$ on R^n is defined by

$$||f||_{\theta,\mu}\equiv\sup_{\substack{k_1,k_2\in R^n\\|k_1-k_2|\leqq 1}}(1+|k_1|)^{\theta}\left\{|f(k_1)|+\frac{|f(k_1)-f(k_2)|}{|k_1-k_2|^{\mu}}\right\}\ .$$

In 1-dimensional case we write $|\ |_{\theta,\mu}$ instead of $||\ ||_{\theta,\mu}$. Let $f(t)$ be a complex-valued function with finite $|f|_{\theta,\mu}$ for θ and μ, $0<\theta,\mu<1$. For such f, we write the Cauchy transform of f by

$$F(z)\equiv\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt$$

for non real complex number z. The following result is known ([1], [5]).

LEMMA 2.2. (1) *The function* $F(z)$, $Im\,z\gtrless 0$, *is continuously extendable up to the boundary of the upper and the lower half plane, respectively.*
(2) *For* θ', $0<\theta'<\theta$, *there exists a positive constant* C *independent of* f *such that*

$$|F(z)|\leqq C|f|_{\theta,\mu}(1+|z|)^{-\theta'}$$

for all z and such that, if z_1, z_2 belong to the same half plane and satisfy $|z_1-z_2|\leqq 1$, we have

$$|F(z_1)-F(z_2)|\leqq C|f|_{\theta,\mu}(1+|z_1|)^{-\theta'}|z_1-z_2|^{\mu}.$$

These two estimates also hold for the boundary value of F.

Let $w_j(k, k')$ $(j=0, 1)$ be a complex-valued function on $R^n\times R^n$ and suppose that there exist positive constants W_j, θ_j and μ_j, $0<\mu_j<1$ such that

$$W_j\equiv \sup_{|k_1-k_2|\leqq 1,\ |k_1'-k_2'|\leqq 1}(1+|k_1-k_1'|)^{-\theta_j}\left\{|w_j(k_1, k_1')|+\frac{|w_j(k_1, k_1')-w_j(k_2, k_2')|}{|k_1-k_2|^{\mu_j}+|k_1'-k_2'|^{\mu_j}}\right\}$$

is finite. (In this section θ_0, μ_0 are not necessarily the same as those in Assumption 1). For such w_j we consider the integral of Cauchy type for complex number z,

$$w(k, k', z)\equiv\int\frac{w_0(k, k'')w_1(k'', k')}{p(k'')-z}dk''.$$

For $Im\, z\neq 0$, this integral is well-defined if $\theta_0+\theta_1+m>n$.

In what follows except in the appendix we treat only the case $m<n$. Similar results for the case $m\geq n$ are easily obtained (See Appendix).

In the following two lemmas we give some estimates for w. We first note that under the assumptions of these lemmas, $w(k, k', z)$ can continuously be extended up to the boundary of the lower or upper half of the z-plane. The estimates given in the lemmas will agree to the boundary values of w. Furthermore, C denotes some constant independent of w_0 and w_1. Let $\theta=\min(\theta_0, \theta_1)$ and $\mu=\min(\mu_0, \mu_1)$.

LEMMA 2.3. (1) *If $\theta>n-m$, then*

$$|w(k, k', z)|\leqq CW_0W_1(1+|k-k'|)^{-\theta}. \tag{2.5}$$

(2) *If $n-m<\theta<n-1$, $\theta'<\theta$ and $0<\mu'<\mu$, then*

$$|w(k, k', z)|\leqq CW_0W_1(1+|k|)^{-\theta'}, \tag{2.6}$$

$$|w(k_1, k', z_1)-w(k_2, k', z_2)|\leqq CW_0W_1(1+|k_1|)^{-\theta'}\{|k_1-k_2|^{\mu'}+|z_1-z_2|^{\frac{\mu'}{m}}\} \tag{2.7}$$

for any k, k_1, $k_2\in R^n$, $|k_1-k_2|\leqq 1$ and z_1, z_2 belong to the same half plane.

Let us denote by $||w_1(\ , k')||_{\theta,\mu}$ the $||\ ||_{\theta,\mu}$ norm of $w_1(k, k')$ as a function of k for fixed k'. Analogously we use $||w(\ , k, z)||_{\theta,\mu}$ or $||w(\ , k, z_1)-w(\ , k, z_2)||_{\theta,\mu}$. It is obvious that $||w_1(\ , 0)||_{\theta_1,\mu_1}\leqq W_1$.

LEMMA 2.4 *For $n-m<\theta_0<n-1$ and $\theta_1\leqq\theta_0$, we choose θ_2 such that*

$$0<\theta_2<\theta_1+\theta_0-(n-m),\ \theta_2\leqq\theta_0.$$

Then for any ν satisfying $0<\nu<m^{-1}(\theta_0+\theta_1-(n-m)-\theta_2)$, and for any μ_2 satisfying $0<\mu_2<\mu_0$, we have the inequalities

$$||w(\ , 0, z)||_{\theta_2,\mu_2}\leqq C(1+|z|)^{-\nu}||w_1(\ , 0)||_{\theta_1,\mu_1}, \tag{2.8}$$

$$||w(\ ,0,z_1)-w(\ ,0,z_2)||_{\theta_2,\mu_2}\leqq C(1+|z_1|)^{-\nu}||w_1(\ ,0)||_{\theta_1,\mu_1}|z_1-z_2|^{\frac{\mu}{m}(1-\frac{\mu_2}{\mu_0})}, \quad (2.9)$$

where z_1 *and* z_2 *are as in* (2) *of Lemma* 2.3.

Proof of Lemmas 2.3. and 2.4. These lemmas can be proved along the same line. Introducing the new set of variables (p, ω) determined by $p=p(k'')$ and $\omega=\omega_{k''}$, we may rewrite dk as

$$dk=dp(m^{-1}\cdot p^{\frac{n-m}{m}}|\pi(\omega)|^n d\omega) . \quad (2.10)$$

We define a complex-valued function $\varphi=\varphi(p; k, k')$ of $p\in R^1$ by

$$\varphi(p; k, k')$$
$$\equiv\begin{cases} m^{-1}p^{\frac{n-m}{m}}\int_\Omega w_0(k, p^{\frac{1}{m}}\pi(\omega)\omega)w_1(p^{\frac{1}{m}}\pi(\omega)\omega, k')|\pi(\omega)|^n d\omega : & \text{for } p>0 \\ 0 & : \text{for } p\leqq 0. \end{cases}$$

Then by (2.10) and Fubini's theorem, we can write

$$w(k, k', z)=\int_{-\infty}^{\infty}\frac{\varphi(p; k, k')}{p-z}dp .$$

Similarly

$$w(k_1, k', z)-w(k_2\ k', z)=\int_{-\infty}^{\infty}\frac{\psi(p; k_1, k_2, k')}{p-z}dp ,$$

where $\psi(p; k_1, k_2, k')\equiv\varphi(p; k_1, k')-\varphi(p; k_2, k')$.

Taking into account of Lemma 2.2., we can reduce our problems to the estimation of $|\varphi|_{\alpha,\beta}$ or $|\psi|_{\alpha,\beta}$ for suitable α and β, $0<\alpha, \beta<1$. Lemma 2.1. is used in order to estimate integrals on Ω. We use sometimes k'' instead of $p^{\frac{1}{m}}\pi(\omega)\omega$ in this proof.

As $\pi(\omega)$ is bounded, we get, by Lemma 2.1,

$$\int_\Omega(1+|k-p^{\frac{1}{m}}\pi(\omega)\omega|)^{-\theta}|\pi(\omega)|^n d\omega\leqq C(1+|p^{\frac{1}{m}}|)^{-\theta}\leqq C(1+|p|)^{-\frac{\theta}{m}}.$$

Noticing that

$$(1+|k-k''|)^{-\theta_0}(1+|k''-k'|)^{-\theta_1}\leqq(1+|k-k''|)^{-\theta}(1+|k''-k'|)^{-\theta}$$
$$\leqq C(1+|k-k'|)^{-\theta}\{(1+|k-k''|)^{-\theta}+(1+|k''-k'|)^{-\theta}\} ,$$

we obtain, for $\theta>n-m$,

$$|\varphi|_{\frac{\theta-n+m}{m},\frac{\mu}{m}}\leqq CW_0W_1(1+|k-k'|)^{-\theta}.$$

This estimates assures (2,5) by Lemma 2.2. Inequalities (2,6) and (2,7) are valid since we have

$$|\varphi|_{\frac{\theta-n+m-\theta'}{m},\frac{\mu}{m}}\leqq CW_0W_1(1+|k|)^{-\theta}$$

and

$$|\psi|_{\frac{\theta-n+m-\theta'}{m},\ \frac{\mu(1-\gamma)}{m}} \leqq CW_0W_1(1+|k_1|)^{-\theta}|k_1-k_2|^{\mu_0\gamma}, \text{ for } 0<\gamma<1.$$

(Note that $(1+|k-k''|)^{-\theta_0}(1+|k''-k'|)^{-\theta_1} \leqq (1+|k-k''|)^{-\theta_0}$ and that $\int_\Omega (1+|k-p^{\frac{1}{m}}\pi(\omega)\omega|)^{-\theta}d\omega \leqq C(1+|k|)^{-\theta'}(1+p)^{-\frac{\theta-\theta'}{m}})$. In order to estimate the latter, we use the fact that, if

$$|f(x_1, y_1)-f(x_2, y_2)| \leqq C(|x_1-x_2|^{\alpha}+|y_1-y_2|^{\beta})$$

for $0<\alpha$, $\beta<1$, then for any γ, $0<\gamma<1$,

$$|f(x_1, y_1)-f(x_2, y_1)-f(x_1, y_2)+f(x_2, y_2)| \leqq C|x_1-x_2|^{\alpha\gamma}|y_1-y_2|^{\beta(1-\gamma)}$$

(C is independent of γ).

Let $\varphi_0=\varphi(p;\ k,\ 0)$ and analogously for ψ_0. Using the above fact, we obtain the estimates

$$|\varphi_0|_{\nu,\ \frac{\mu}{m}} \leqq C(1+|k|)^{-\theta_2}||w_1(\quad,\ 0)||_{\theta_1,\mu_1}$$

and

$$|\psi_0|_{\nu,\frac{\mu(1-\gamma)}{m}} \leqq C(1+|k_1|)^{-\theta_2}||w_1(\quad,\ 0)||_{\theta_1,\mu_1}|k_1-k_2|^{\mu_0\gamma}.$$

Therefore (2.8) is easily obtained from Lemma 2.2. To prove (2.9), we use the above fact after applying Lemma 2.2. to φ_0 and ψ_0.

Finally we remark that the continuiability of $w(k, k', z)$ mentioned before Lemma 2.3 are also assured by Lemma 2.2.

§ 3. Proofs of Theorems in the case of $n>m$.

1. We consider the set $\mathfrak{B}_{\theta,\mu}$ of all $f(k)$ with finite $||f||_{\theta,\mu}$, $\theta \geqq 0$, $1 \geqq \mu > 0$. This set becomes a Banach space with the norm $||\ \ ||_{\theta,\mu}$. The following lemma can be easily shown by repeated use of the Ascoli-Arzela lemma.

LEMMA 3.1. *If* $\theta>\theta' \geqq 0$ *and* $1 \geqq \mu > \mu' > 0$, $\mathfrak{B}_{\theta,\mu}$ *is naturally imbeded in* $\mathfrak{B}_{\theta',\mu'}$ *by the completely continuous injection.*

For θ, $\theta' \geqq 0$ and $1 \geqq \mu, \mu' > 0$, we denote by $L_{(\theta,\mu;\ \theta',\mu')}$ the space of all continuous linear operators from $\mathfrak{B}_{\theta,\mu}$ to $\mathfrak{B}_{\theta',\mu'}$. Its norm is denoted by $||\ \ ||_{(\theta,\mu;\ \theta',\mu')}$.

For non real complex number z, we write

$$A(z)f(k) \equiv \int \frac{v(k-k')}{p(k')-z} f(k')dk',$$

whenever the right-hand side has a meaning. For simplicity we assume, hereafter, that θ_0 of Assumption 1 is smaller than $n-1$. Applying Lemma 2.4 to $A(z)f$, we have the following lemma.

LEMMA 3.2. *If, for given* θ_1, $\mu_1 (\theta_0 \geqq \theta_1 > 0,\ 1 \geqq \mu_1 > 0)$, *we choose* θ_2 *as*

$$0<\theta_2<\theta_1+\theta_0-(n-m), \quad \theta_2 \leqq \theta_0 \tag{3.1}$$

and μ_2 *as* $0<\mu_2<\mu_0$, *then the following assertions hold.*
(1) $A(z)$ *can be regarded as an element of* $L_{(\theta_1,\mu_1;\ \theta_2,\mu_2)}$ *for each* z *and is a continuous mapping from the upper (or lower) half plane to* $L_{(\theta_1,\mu_1;\ \theta_2,\mu_2)}$.

$$A(x+iy)\quad (x,\,y \text{ real}) \text{ has a limit } A(x\pm i0)=\lim_{y\downarrow 0} A(x\pm iy) \quad \text{in } L_{(\theta_1,\mu_1;\ \theta_2,\mu_2)}\,.$$

(2) *For any* ν *with* $0<\nu<m^{-1}(\theta_0+\theta_1-(n-m)-\theta_2)$, *there exists a positive constant* C *such that*: i) *for any* z (*including the boundary point* $z=x\pm i0$), *we have*

$$||A(z)||_{(\theta_1,\mu_1;\ \theta_2,\mu_2)}\leqq C(1+|z|)^{-\nu}\,; \tag{3.2}$$

and ii) *for* z_1 *and* z_2 (*belonging to the same half plane including the real line*) *satisfying* $|z_1-z_2|\leqq 1$, *and* $\mu=\min(\mu_0,\ \mu_1)$, *we have*

$$||A(z_1)-A(z_2)||_{(\theta_1,\mu_1;\ \theta_2,\mu_2)}\leqq C(1+|z_1|)^{-\nu}|z_1-z_2|^{\frac{\mu}{m}\left(1-\frac{\mu_2}{\mu_0}\right)}. \tag{3.3}$$

Since $\theta_0>n-m$ by Assumption 1, we can choose θ_2 in (3.1) as $\theta_0\geqq\theta_2>\theta_1$. Combining above two lemmas, we see that for all z including $z=x\pm i\,0$, $A(z)$ is completely continuous from $\mathfrak{B}_{\theta,\mu}$ to $\mathfrak{B}_{\theta,\mu}$ if

$$0<\theta<\theta_0 \quad\text{and}\quad 0<\mu<\mu_0\,. \tag{3.4}$$

Applying Lemma 3.2 to $\lambda\varphi=A(z)\varphi$ $(\lambda\neq 0)$, we know that non-zero eigenvalues and associated eigenfunctions of $A(z)$ regarded as an operator in $\mathfrak{B}_{\theta,\mu}$ are common to all pairs of θ and μ satisfying (3.4). We use the term "*eighenfunction*, or *eigenvalue*, of $A(z)$" in this sense.

For fixed z the set of eigenvalues of $A(z)$ is denoted by s_z. By Lemma 3.2. s_z is defined also for $z=\lambda\pm i\,0$, where λ is real. We define a set of singular points of L as $S\equiv\{z|-1\in s_z\}$. The set S may include also ideal points as $\lambda+i\,0$ or $\lambda-i\,0$.

Lemma 1.1 implies that $L=L_0+V$ on $\mathfrak{D}$ and that for $f\in\mathfrak{D}$, V is an integral operator with the kernel $v(k-k')$. Therefore if $Im\,z\neq 0$, $A(z)f=V(L_0-z)f$ for $f\in\mathfrak{H}\cap\mathfrak{B}_{\theta,\mu}$. Noticing this fact, we can show the correspondence between S and $\sigma_p(L)$, the point spectrum of L, by the same method of Faddeev (See § 4 of [1], § 3 of [2]). Namely we have the following lemma.

LEMMA 3.3. (1) *Non-real* z *does not belong to* S.
(2) *For non-zero real* λ, $\lambda+i\,0\in S$ *if and only if* $\lambda-i\,0\in S$. *The function* $\varphi\in\mathfrak{B}_{\theta,\mu}$ (θ, μ *as in* (3.4)) *satisfies* $\varphi+A(\lambda+i\,0)\varphi=0$ *if and only if* $\varphi+A(\lambda-i\,0)\varphi=0$.
(3) *For non-zero real* λ, *the function* $\psi\in\mathfrak{D}$ *satisfies* $L\psi=\lambda\psi$ *if and only if* $\varphi=(p-\lambda)\psi$ *satisfies* $\varphi\in\mathfrak{B}_{\theta,\mu}$ (θ, μ as in (3.4)) *and* $\varphi+A(\lambda\pm i\,0)\varphi=0$.

We can prove this lemma using mainly Lemma 3.2. The details of the proof are omitted. We only remark that the conditions $\theta_0>\frac{n}{2}$ and $\mu_0>\frac{1}{2}$ are used there. Infact, $\theta_0>\frac{n}{2}$ is used to show that the relation: $\varphi+A(z)\varphi=0$, $\varphi\in\mathfrak{B}_{\theta,\mu}$, for $Im\,z\neq 0$, implies $\psi\equiv(L_0-z)^{-1}\varphi\in\mathfrak{D}$. The first assertion of this lemma follows from this fact. The condition $\mu_0>\frac{1}{2}$ is needed to prove that $\psi\in\mathfrak{D}$ for $z=\lambda\pm i\,0$

(λ real), which assures the final statement of the lemma.

The eigenspace corresponding to the eigenvalue -1 of $A(\lambda \pm i0)$ is finite dimensional since $A(\lambda \pm i0)$ is completely continuous. Therefore the multiplicity of non-zero eigenvalue of L is finite by (3) of Lemma 3.3. If we check the proof, it is seen that this fact also true for $\lambda=0$ in our case $n>m$.

Since $\lambda+i0 \in S$ if and only if $\lambda-i0 \in S$ by (2) of Lemma 3.3, we can and shall regard S as a subset of the complex plane.

LEMMA 3.4. *The set S is bounded, having zero as the only possible limiting point.*

Proof. The boundedness of S is an immediate consequence of the estimate (3.2). Other properties can be proved analogously as Lemma 3.9 and 3.10 of [2].

Since $S-\{0\}=\sigma_p(L)-\{0\}$, the proof of Theorem 1 is completed.

2. For non-real complex number z, we consider the operator

$$T(z) \equiv V - VR(z)V ,$$

where $V \equiv L-L_0$ and $R(z) \equiv (L-z)^{-1}$. Using $R_0(z) \equiv (L_0-z)^{-1}$, Faddeev characterized $T(z)$ as follows (See lemma 2.1 and 2.2 of [2]).

LEMMA 3.5. *For $Im\, z \neq 0$, $T(z)$ is uniquely determined on $\mathfrak{D}$ by the relation*

$$T(z) = V - VR_0(z)T(z) . \tag{3.5}$$

Using $T(z)$, the resolvent of L is represented by

$$R(z) = R_0(z) - R_0(z)T(z)R_0(z) . \tag{3.6}$$

In our case V is an integral operator with the kernel $v(k-k')$ by Lemma 1.1. If we assume that $T(z)$ is also an integral operator with the kernel $t(k, k' z)$, relation (3.5) suggests that $t(k, k', z)$ is a solution of the integral equation

$$t(k, k', z) = v(k-k') - \int \frac{v(k-k'')}{p(k'')-z} t(k'', k', z) dk . \tag{3.7}$$

This equation has a solution if $z \notin S$. In order to state this result clearly, let us put

$$\Pi_\varepsilon^+ \equiv \{z | Im\, z \geqq 0,\ d(z, S) \geqq \varepsilon\} ,$$

$d(z, S)$ being the distance between z and S. We agree that Π_ε^+ includes $\lambda+i0$ for real λ, $d(\lambda, S) \geqq \varepsilon$. We define Π_ε^- analogously. Put $\Pi^\pm \equiv \bigcup_{\varepsilon>0} \Pi_\varepsilon^\pm$. The following lemma holds.

LEMMA 3.6. *Let Π_ε and Π be either Π_ε^+ and Π^+ or Π_ε^- and Π^-. There exists a solution $t(k, k', z)$* of (3.7) *for $z \in \Pi$, satisfying the following estimates.*

For $\varepsilon>0$, $0<\theta<\theta_0$ and $0<\mu<\mu_0$, there exists a positive constant C depending on ε, θ and μ such that

$$|t(k, k', z)| \leqq C(1+|k-k'|)^{-\theta} \tag{3.8}$$

for $z \in \Pi_\varepsilon$ *and*

$$\begin{aligned}&|t(k_1, k_1', z_1)-t(k_2, k_2', z_2)| \\ &\leqq C(1+|k_1-k_1'|)^{-\theta}\{|k_1-k_2|^\mu+|k_1'-k_2'|^\mu+|z_1-z_2|^{\frac{\mu'}{m}}\}\end{aligned} \tag{3.9}$$

for $z_1, z_2 \in \Pi_\varepsilon$, $|z_1-z_2| \leqq 1$, $|k_1-k_2| \leqq 1$, $|k_1'-k_2'| \leqq 1$ *and* $\mu'=\mu\left(1-\frac{\mu}{\mu_0}\right)$.

Proof. Let $t_0(k, k', z) \equiv v(k-k')$, and, for $j \geqq 1$, let

$$t_j(k, k', z) \equiv -\int \frac{v(k-k'')}{p(k'')-z} t_{j-1}(k'', k', z)dk'' .$$

By Lemma 2.3.(1),

$$|t_j(k, k', z)| \leqq C(1+|k-k'|)^{-\theta_0} .$$

This estimate yields inductively that, for $T_j(z) \equiv (-1)^j(VR_0(z))^j V$ and $f \in \mathfrak{S}$,

$$T_j(z)f(k) = \int t_j(k, k', z)f(k')dk' .$$

Since $T_j(z)^* = T_j(\bar{z})$ on $\mathfrak{D}$, this representation implies

$$t_j(k, k', z) = \overline{t_j(k', k, \bar{z})} . \tag{3.10}$$

Lemma 2.3.(2) shows that $t_1(\ , k' z) \in \mathfrak{B}_{\theta,\mu}$ (k' and z are fixed) for $0<\theta<\theta_0-n+m$, $0<\mu<\mu_0$. Since $t_j(\ , k', z) = -A(z)t_{j-1}(\ , k', z)$, by the repeated use of Lemma 3.2, there exists a positive integer j_0 such that $t_j(\ , k', z) \in \mathfrak{B}_{\theta_0,\mu}(\mu<\mu_0)$ for $j \geqq j_0$. Moreover it is easily seen that, for $j \geqq j_0$, $||t_j(\ , k', z)||_{\theta_0,\mu} \leqq C$, C depending on j but not on k' and z. Similarly if $|k_1'-k_2'| \leqq 1$, then

$$||t_j(\ , k_1', z)-t_j(\ , k_2', z)||_{\theta_0,\mu} \leqq C|k_1'-k_2'|^\mu \quad \text{for} \quad j \geqq j_0 .$$

Therefore we have, for $j \geqq j_0$,

$$\begin{aligned}&|t_j(k, k', z)| \leqq C(1+|k|)^{-\theta_0} , \\ &|t_j(k_1, k_1', z)-t_j(k_2, k_2', z)| \leqq C(1+|k_1|)^{-\theta_0}\{|k_1-k_2|^\mu+|k_1'-k_2'|^\mu\} .\end{aligned}$$

But relation (3.10) permits us to replace $(1+|k|)^{-\theta_0}$ by $(1+|k'|)^{-\theta_0}$ in the above estimates. Therefore we may take $(1+|k|)^{-\theta_1}(1+|k'|)^{-\theta_2}(0 \leqq \theta_1, \theta_2 \leqq \theta_0, \theta_1+\theta_2=\theta_0)$ as an estimate function. This implies that $||t_j(\ , k', z)||_{\theta,\mu} \leqq C(1+|k'|)^{-(\theta_0-\theta)}$ for $0 \leqq \theta \leqq \theta_0$.

Now we fix θ and μ, $0<\theta<\theta_0$, $0<\mu<\mu_0$, and regard the equation

$$\hat{t}(k, k', z) = t_{j_0}(k, k', z) - \int \frac{v(k-k'')}{p(k'')-z} \hat{t}(k'', k', z)dk$$

as the following inhomogeneous equation in $\mathfrak{B}_{\theta,\mu}$

$$(1+A(z))\hat{t}(\ , k', z) = t_{j_0}(\ , k', z)$$

with the parameters k' and z. Since $A(z)$ is completely continuous, there exists $(1+A(z))^{-1} \in L_{(\theta,\mu;\, \theta,\mu)}$ if and only if $-1 \notin s_z$, e.g. $z \in \Pi$. It follows easily from

(3.2) that for $z \in \Pi_\varepsilon$ $\|(1+A(z))^{-1}\|_{(\theta,\mu;\,\theta,\mu)} \leqq C(\varepsilon, \theta, \mu)$. Therefore for $z \in \Pi_\varepsilon$ a solution $\hat{t}(\ \ , k', z)$ exists and satisfies

$$\|\hat{t}(\ \ , k', z)\|_{\theta,\mu} \leqq C(\varepsilon, \theta, \mu)(1+|k'|)^{-(\theta_0-\theta)} .$$

Similarly we have, for $z \in \Pi_\varepsilon$ and $|k_1'-k_2'| \leqq 1$,

$$\|\hat{t}(\ \ , k_1', z)-\hat{t}(\ \ , k_2', z)\|_{\theta,\mu} \leqq C(\varepsilon, \theta, \mu)(1+|k_1'|)^{-(\theta_0-\theta)}|k_1'-k_2'|^{\mu} .$$

Somewhat more careful treatment is required to estimate $t(\ \ , k', z_1)-t(\ \ , k' z_2)$. On fixing z_1 and z_2 we put for brevity that $\hat{u} \equiv \hat{t}(\ \ , k', z_1)-\hat{t}(\ \ , k', z_2)$ and $u_j \equiv t_j(\ \ , k', z_1)-t_j(\ \ , k', z_2)$. Then

$$\hat{u}=(1+A(z_1))^{-1}u_{j_0}+\{(1+A(z_1))^{-1}(A(z_2)-A(z_1))(1+A(z_2))^{-1}\}t_{j_0} .$$

In order to estimate u_j, it is sufficient to consider

$$\tau_j(k, k', z, \zeta) \equiv -\int \frac{v(k-k'')}{p(k'')-z} t_{j-1}(k'', k', \zeta)dk'' ,$$

since $t_j(k, k', z)=\tau_j(k, k', z, z)$. Applying Lemma 2.3.(2) to τ_j step by step, we conclude that

$$\|u_j\|_{\theta,\mu} \leqq C(1+|k|)^{-(\theta_0-\theta)}|z_1-z_2|^{-\frac{\mu_0'}{m}}$$

for $j \geqq j_0$, $|z_1-z_2| \leqq 1$, and $\mu_0' < \mu_0$. Applying this estimate and (3.3) to $\hat{u}$ we have

$$\|\hat{t}(\ \ , k', z_1)-\hat{t}(\ \ , k' z_2)\|_{\theta,\mu} \leqq C(\varepsilon, \theta, \mu)(1+|k'|)^{-(\theta_0-\theta)}|z_1-z_2|^{\frac{\mu'}{m}}$$

for $z_1, z_2 \in \Pi_\varepsilon$, $|z_1-z_2| \leqq 1$ and $\mu'=\mu\left(1-\dfrac{\mu}{\mu_0}\right)$.

We now put

$$t(k, k', z) \equiv \hat{t}(k, k', z)+\sum_{j=0}^{j_0-1} t_j(k, k', z) .$$

Then it is obvious that this satisfies (3.7). Estimating τ_j, we easily see that the Hölder coefficient of $t_j(k, k', z)$ with respect to k, k' and z may be bounded by $C(1+|k-k'|)^{-\theta_0}$. This fact and above three estimates of $\hat{t}$ assure (3.8) and (3.9).

LEMMA 3.7. *For* $f \in \mathfrak{S}$, *we have*

$$T(z)f(k)=\int t(k, k', z)f(k')dk' . \tag{3.11}$$

Proof. Let the right-hand side of (3.11) be denoted as Tf. Choose θ as $\theta > \dfrac{n}{2}$. Using estimate (3.8) we conclude $Tf \in \mathfrak{H}$ for $f \in \mathfrak{S}$. By Fubini's theorem, (3.7) assures that $Tf=Vf-VR_0(z)Tf$ as an element of $\mathfrak{H}$. Considering Lemma 3.5, we can write $(T(z)f-Tf)=-VR_0(z)(T(z)f-Tf)$. This is rewritten as $(L_0-z)\psi+V\psi=0$, where $\psi=R_0(z)(T(z)f-Tf) \in \mathfrak{D}$. Therefore $L\psi=z\psi$. Since L is selfadjoint, $\psi=0$. This implies $T(z)f=Tf$.

Now we proceed to the proof of Theorem 2. In what follows, (f, g) means the scalar product of $f, g \in \mathfrak{H}$.

Let a closed interval $[\alpha, \beta]$ be disjoint with $\sigma_p(L) \cup \{0\}$. Applying Lemma 2.4 to

$$T(z)R_0(z)f(k) = \int \frac{t(k, k', z)}{p(k')-z} f(k')dk', \tag{3.12}$$

we conclude that for $f \in \mathfrak{S}$ and k satisfying $\alpha \leqq p(k) \leqq \beta$ there exists a limiting value of $T(z)R_0(z)f(k)|_{z=p(k)\pm i\varepsilon}$ as $\varepsilon \downarrow 0$. We now define, for $f \in \mathfrak{S}$,

$$W_{*[\alpha,\beta]} f(k) \equiv \begin{cases} f(k) - T(z)R_0(z)f(k)|_{z=p(k)+i0} : \ if \alpha > 0 \text{ and } \alpha \leqq p(k) \leqq \beta, \\ 0 \qquad : \text{otherwise}. \end{cases}$$

The estimates for (3.12) referred to above also assure that $W_{*[\alpha,\beta]}f \in \mathfrak{H}$ for $f \in \mathfrak{S}$. Let $L = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ and $L_0 = \int_{-\infty}^{\infty} \lambda dE_0(\lambda)$. Clearly $E_0(\lambda)f(k)$ is equal to $f(k)$ if $p(k) \leqq \lambda$, and zero if $p(k) > \lambda$. For convenience we denote $(E(\lambda+0)-E(\mu-0))\mathfrak{H}$ by $\mathfrak{H}_{[\mu,\lambda]}$ and $(E(\lambda-0)-E(\mu+0))\mathfrak{H}$ by $\mathfrak{H}_{(\mu,\lambda)}$. $\mathfrak{H}^0_{[\mu,\lambda]}$ and $\mathfrak{H}^0_{(\mu,\lambda)}$ are defined analogously for $E_0(\lambda)$, although $\mathfrak{H}^0_{[\mu,\lambda]} = \mathfrak{H}^0_{(\mu,\lambda)}$ in this case.

We can prove the following lemma concerning $W_* = W_{*[\alpha,\beta]}$, using the same method of Lemma 8.7 of [1]. Relation (3.6) is fundamental in this proof.

LEMMA 3.8. *The operator W_* can uniquely be extended to a bounded operator from $\mathfrak{H}$ into $\mathfrak{H}^0_{[\alpha,\beta]}$. If the extension is also denoted by W_*, then for $f, g \in \mathfrak{H}$ and $\alpha \leqq \mu \leqq \beta$*

$$\int_\alpha^\mu d(E(\lambda)f, g) = \int_\alpha^\mu d(E_0(\lambda)W_*f, W_*g). \tag{3.13}$$

Relation (3.13) shows that $[\alpha, \beta]$ is contained in the absolutely continuous spectrum of L. Theorem 2 is proved by considering that $E_0(\lambda)=0$ for $\lambda<0$, and that $\sigma_p(L)$ has no limiting point except for zero.

3. To prove Theorem. 3, we consider

$$K_{\varepsilon_1,\varepsilon_2} f(k) \equiv \int \frac{t(k, k', p(k')+i\varepsilon_1)}{p(k)-p(k')-i\varepsilon_2} f(k')dk'$$

for $f \in \mathfrak{S} \cap \mathfrak{H}^0_{[\alpha,\beta]}$. Then Lemma 2.4 and Lemma 3.6 imply that, when $\varepsilon_1 \to +0$ or -0 and $\varepsilon_2 \to +0$ or -0, the function $K_{\varepsilon_1\varepsilon_2}f$ converges (as a function of k) uniformly to the square integrable limit function, say $K_{-0,-0}f(k)$. Therefore we can define a linear operator $W_{[\alpha,\beta]}$ from $\mathfrak{H}^0_{[\alpha,\beta]} \cap \mathfrak{S}$ into $\mathfrak{H}$ by

$$W_{[\alpha,\beta]} \equiv 1 - K_{-0,-0}$$

(1 is an identity operator). For $W = W_{[\alpha,\beta]}$, we have the following results, whose proof is parallel to that of lemma 8.2 of [1].

LEMMA 3.9. *The operator W can be uniquely extended to a partially isometric operator if we put $W=0$ on $\mathfrak{H}^{0\perp}_{[\alpha,\beta]}$. The intial and the final sets of W are $\mathfrak{H}^0_{[\alpha,\beta]}$ and $\mathfrak{H}_{[\alpha,\beta]}$, respectively. If we denote this operator also by W, then we have*

$$W^* = W_*,$$
$$W^* W = \int_\alpha^\beta dE_0(\lambda) \ . \tag{3.14}$$

Above two lemmas are extended to the case of an open interval (α, β) disjoint with $\sigma_p(L) \cup \{0\}$ as follows. For an arbitrary $f \in \mathfrak{H}^0_{[\alpha+\varepsilon, \beta-\varepsilon]}$, we put

$$W_{(\alpha,\beta)} f \equiv W_{[\alpha+\varepsilon, \beta-\varepsilon]} f .$$

It is easy to see that $W_{(\alpha,\beta)} f$ does not depend on the choice of ε. Furthermore we put $W_{(\alpha,\beta)} f = 0$ if $f \in \mathfrak{H}^0_{(\alpha,\beta)}{}^\perp$. Then $W_{(\alpha,\beta)}$ is uniquely extended to a partially isometric operator with the initial and final sets $\mathfrak{H}^0_{(\alpha,\beta)}$ and $\mathfrak{H}_{(\alpha,\beta)}$, respectively. This operator clearly satisfies (3.14) and instead of (3.13) the following relation holds:

$$((E(\mu-0) - E(\alpha+0))f, g) = \int_\alpha^\mu d(E_0(\lambda) W^* f, \ W^* g) \ . \tag{3.13'}$$

Let $I \equiv (0, \ \infty) - \sigma_p(L)$. The operator $\int_I dE(\lambda)$ is the projection of the absolutely continuous subspace $\mathfrak{H}_{ac}$ of $\mathfrak{H}$ with respect to L. Let $\lambda_j (j=1, 2, \cdots)$ be positive eigenvalues of L. By Theorem 1, we can assume $\lambda_1 > \lambda_2 > \cdots$. If the number of positive eigenvalues is $l < \infty$, we define

$$U \equiv \sum_{j=0}^{l} W_{(\lambda_{j+1} \lambda_j)} , \tag{3.15}$$

where $\lambda_0 = \infty$, $\lambda_{l+1} = 0$. If $\{\lambda_j\}$ is an infinite sequence, then $\lim_{j\to\infty} \lambda_j = 0$ by Theorem 1. Thus, for $f \in \mathfrak{H}$, $|| \sum_{j=l}^{l'} W_{(\lambda_{j+1}, \lambda_j)} f ||^2 = \sum_{j=l}^{l'} || \int_{\lambda_{j+1}}^{\lambda_j} dE(\lambda) f ||^2$ tends to 0 as $l, l' \to \infty$. Letting $l = \infty$ in (3.15), we can define U as a strong limit.

LEMMA 3.10. *U is an isometric operator from $\mathfrak{H}$ to $\mathfrak{H}_{ac}$, satisfying $LU = UL_0$.*

Proof. By (3.13′) and (3.14), the following relations are valid for $W_j = W_{(\lambda_{j+1}, \lambda_j)}$:

$$W_j^* W_j = \int_{\lambda_{j+1}}^{\lambda_j} dE_0(\lambda), \ W_j W_j^* = E(\lambda_j - 0) - E(\lambda_{j+1} + 0) \ .$$

From these, we see that U satisfies

$$U^* U = \int_{\lambda \in I} dE_0(\lambda) = \int_0^\infty dE_0(\lambda) = 1, \ UU^* = \int_{\lambda \in I} dE(\lambda) \ .$$

This implies that U is an isometric operator, whose initial and final set is $\mathfrak{H}$ and $\mathfrak{H}_{ac}$.

Next we show

$$\int_{-\infty}^\mu d(E(\lambda) f, \ g) = \int_{-\infty}^\mu d(E_0(\lambda) U^* f, \ U^* g) \tag{3.16}$$

for $f \in \mathfrak{H}_{ac}$, $g \in \mathfrak{H}$ and $\mu \in R^1$. Since the left-hand side of (3.16) is equal to

$$\sum_{\lambda_j<\mu}((E(\lambda_j-0)-E(\lambda_{j+1}+0))f,\ g)+((E(\mu)-E(\lambda_{j_0}+0)f,\ g)\,,$$

where λ_{j_0} is the largest of those λ_j which are smaller than μ, by (3.13′) we have

$$\int_{-\infty}^{\mu} d(E(\lambda)f,\ g)=\sum_{\lambda_j<\mu}\int_{\lambda_{j+1}}^{\lambda_j} d(E_0(\lambda)W_j{}^*f,\ W_j{}^*g)+\int_{\lambda_{j_0}}^{\mu} d(E_0(\lambda)W_{j_0}{}^*f,\ W_{j_0}{}^*g)\ .$$

Since the equality $W_j\int_{\lambda_{j+1}}^{\nu} dE_0(\lambda)W_j{}^*=U\int_{\lambda_{j+1}}^{\nu} dE_0(\lambda)U^*\ (\lambda_{j+1}\leqq\nu\leqq\lambda_j)$ is true by the definition of U, (3.16) is obtained.

Let f, g be arbitrary elements of $\mathfrak{H}$. Since $Uf\in\mathfrak{H}_{ac}$, we have by (3.16)

$$(E(\lambda)Uf,\ g)=\int_{-\infty}^{\lambda} d(E(\mu)Uf,\ g)=\int_{-\infty}^{\lambda} d(E_0(\mu)U^*Uf,\ U^*g)$$
$$=\int_{-\infty}^{\lambda} d(E_0(\mu)f,\ U^*g)=(UE_0(\lambda)f,\ g)\ .$$

Therefore we conclude that $E(\lambda)U=UE_0(\lambda)$. $LU=UL_0$ is an immediate consequence.

Lemma 3.10 implies Theorem 3.

If we define $W'_{[\alpha,\beta]}\equiv 1-K_{+0,+0}$, we can construct U' using $W'_{(\lambda_{j+1},\lambda_j)}$ in (3.15). Lemma 3.9 holds also for U'. We remark that $S=U^*U'$ becomes a scattering matrix (see § 10 of [1]).

Appendix: Remarks for the case of $n\leqq m$

In order to treat this case similarly to that of $n>m$, it is sufficient to have the results analogous to Lemma 2.3 and 2.4. Since $-1<\dfrac{n-m}{m}\leqq 0$, $\varphi(p\,;\ k,\ k')$ of (3.2) is discontinuous at $p=0$, so that estimates of $w(k,\ k',\ z)$ cannot be reduced to Lemma 2.2. Nevertheless we can perform the estimation of $w(k,k',z)$ by the following consideration.

Let $f(t)$ be a complex-valued function defined on the positive real line. For such f it is convenient to define

$$|f|^{+}_{\theta,\mu}\equiv\sup_{\substack{t_1,t_2>0\\|t_1-t_2|\leqq 1}}(1+t_1)^{\theta}\left\{|f(t_1)|+\frac{|f(t_1)-f(t_2)|}{|t_1-t_2|^{\mu}}\right\}$$

for $0\leqq\theta<1$, $\mu>0$. Choose f with finite $|f|^{+}_{0,\mu}$. If we assume $f(t)=0$ for $t<0$, then $F(z)=\int_{-\infty}^{\infty}\dfrac{t^{-\theta}f(t)}{t-z}dt$ is convergent for $0<\theta<1$ and $Im\,z\neq 0$. For $\varepsilon>0$ there exists a continuously differentiable function $\alpha_\varepsilon(t)$ such that $\alpha_\varepsilon(t)=t^{-\theta}$ if $t\geqq\varepsilon$, $\alpha_\varepsilon(t)=0$ if $t\leqq 0$ and such that $\sup_{t\in R_1}(|\alpha_\varepsilon(t)|+|\alpha_\varepsilon'(t)|)$ is finite. The estimate $|\alpha_\varepsilon f|_{0,\mu}\leqq C(\varepsilon)|f|^{+}_{0,\mu}$ is easily obtained.

If we decompose $F(z)$ as

$$F(z)=\int_{-\infty}^{\infty}\frac{\alpha_\varepsilon(t)f(t)}{t-z}dt-\int_{0}^{\varepsilon}\frac{\alpha_\varepsilon(t)f(t)}{t-z}dt+\int_{0}^{\varepsilon}\frac{t^{-\theta}f(t)}{t-z}dt\ ,$$

then the second and the third terms on the right are analytic in $|z|>\varepsilon$, whose maximum and Hölder norms are bounded by $|\alpha_\varepsilon f|_{\theta,\mu}$ for $|z|\geqq 2\varepsilon$. The first term on the right can be handled by Lemma 2.2. Using this, we see that (1) of Lemma 2.2 is valid for the present $F(z)$ except for $z=0\pm i0$. As for (2), it is valid, under the restriction $|z|$, $|z_1|$, $|z_2|\geqq\varepsilon$, if we replace C and $|f|_{\theta,\mu}$ by a constant $C(\varepsilon)$ depending on ε and $|f|^{+}_{0,\mu}$, respectively.

Similar result can be obtained for the Cauchy integral of $f(t)$ satisfying $|f|^{+}_{\theta,\mu}<0$ for $\theta>0$ and $f(t)=0$ for $t<0$.

The estimation of $w(k, k', z)$ can be easily reduced to these results. We can treat the case $m\geqq n$ analogously to $m<n$. Now whether $0\in S$ or $0\notin S$ has no meaning because z cannot be zero. It is, however, true that the only possible limiting point of S is zero.

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