On the Spectrum of Some Hamiltonian Operator

By Teruo USHIJIMA

Institute of Mathematics and Department of Pure and Applied Sciences,
College of General Education, University of Tokyo

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1. In the previous paper [3], the author investigated the spectral properties of $L=L_0+V$ in $L^2(R^n)$, where L_0 is a multiplying operator by the real homogeneous elliptic polynomial $p(k)$ and V is a convolution operator with the kernel $v(k)$. The fundamental tool of this investigation was the T-matrix method of Faddeev [1]. In the present paper we note that the same method can be applied to a Hamiltonian operator: $\sum_{j=1}^{n}\left(\frac{\partial}{i\partial x_j}+b_j(x)\right)^2+q(x)$ in $L^2(R^n)$, where n[1] $\geqq 3$. We consider only its momentum representation:

$$(Lf)(k)\equiv k^2f(k)+\int v(k,k')f(k')dk'^{\,2)}.$$

When we need no modification or change, we use the terminology and notations in [3], without noticing explicitly.

We impose the following conditions (1) and (2) on v.

(1) $v(k,k')=\overline{v(k',k)}$.

(2) *Let* $m(k,k')=1+\min(|k|,|k'|)$. *There exist constants* C, θ_0 *and* $\mu_0\left(\theta_0>n, \mu_0>\frac{1}{2}\right)$ *such that the function* v *satisfies*

$$|v(k,k')|\leqq Cm(k,k')(1+|k-k'|)^{-\theta_0}$$

and

$$|v(k,k')-v(k_1,k_1')|$$
$$\leqq Cm(k,k')(1+|k-k'|)^{-\theta_0}\{|k-k_1|^{\mu_0}+|k'-k_1'|^{\mu_0}\}$$

for any k, k_1 $(|k-k_1|\leqq 1)$ *and* k', k_1' $(|k'-k_1'|\leqq 1)$.

If we put $\hat{b}_j(k)=(2\pi)^{-n}\int b_j(x)e^{-ikx}dx$ and $\hat{q}(k)=(2\pi)^{-n}\int q(x)e^{-ikx}dx$, we have

1) We denote the dimension of independent variables by n throughout this paper.
2) The symbol k is an independent variable in R^n and k^2 its scalar product. We write only the integral sign whenever the domain of integration is the whole R^n.

$$v(k, k')=\sum_{j=1}^{n}\Big[((k)_j+(k')_j)\hat{b}_j(k-k')$$

$$+\int\hat{b}_j(k-k'')\hat{b}_j(k''-k')dk''\Big]+\hat{q}(k-k')$$

($(k)_j$ is the j-th component of k). Therefore if b_j and q are real-valued, and if $\hat{b}\in\mathfrak{B}_{\theta_0+1,\mu_0}$ and $\hat{q}\in\mathfrak{B}_{\theta_0,\mu_0}$, assumptions (1) and (2) are fulfilled.[3]

Consider the free Hamiltonian in $\mathfrak{H}=L^2(R^n)$, namely $(L_0f)(k)\equiv k^2f(k)$, with the domain $\mathfrak{D}\equiv\{f\,|\,(1+k^2)f\in\mathfrak{H}\}$. We note that L_0 is self adjoint, and that its spectrum lies on the whole positive real line and is absolutely continuous.

LEMMA 1. *Put* $(Vf)(k)\equiv\int v(k,k')f(k')dk'$. *Then the operator* $L=L_0+V$ *with the domain* $\mathfrak{D}$ *is self adjoint.*

Proof. By (2), we have

$$\left|\int v(k,k')f(k')dk'\right|\leqq C\int(1+|k-k'|)^{-\theta_0}(1+|k'|)|f(k')|dk'.$$

The inequality, $1+|k'|\leqq C(\varepsilon)+\varepsilon|k'|^2$, holds for any $\varepsilon>0$. Since $\theta_0>n$, the Fourier image of $(1+|k|)^{-\theta_0}$ is a bounded function. Therefore we conclude that there exist a, $0<a<1$ and $b>0$ such that $\|Vf\|\leqq a\|L_0f\|+b\|f\|$ for $f\in\mathfrak{D}$. We have the assertion by the argument of [2].

Our results are the following theorems.

THEOREM 1. *The point spectrum of* L *is an at most countable set on the real line, which is bounded below. Zero is the only possible limiting point of negative eigenvalues. Positive eigenvalues, if they exist, have not a finite limiting point. The multiplicity of any eigenvalue is finite.*

THEOREM 2. *The continuous spectrum of* L *is absolutely continuous and consists of the whole positive real line except for the point spectrum.*

THEOREM 3. *The absolutely continuous part of* L *is unitarily equivalent to* L_0.

2. For a complex number κ, $Re\kappa\geqq0$ and $Im\kappa\neq0$, we consider $A(\kappa)\equiv VR_0(\kappa^2)$ $(R_0(\kappa^2)\equiv(L_0-\kappa^2)^{-1})$ as an operator from $\mathfrak{B}_{\theta,\mu}$ to $\mathfrak{B}_{\theta',\mu'}$. Once the estimate of $\|A(\kappa)\|_{(\theta,\mu;\,\theta',\mu')}$ is obtained, we can treat the problem along the same methods as in [1] and [3]. Therefore, we need only to prove the next lemma.

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3) In [3] we assumed that the function, which was used to estimate v, took the form $(1+|k-k'|)^{-\theta_0}\left((\theta_0>\frac{n}{2}\right)$, since $v(k,k')=\hat{q}(k-k')$. We note that the function $m(k,k')$ corresponds to first derivatives in L.

LEMMA 2. *For the given θ_1, μ_1 $(\theta_1 \leqq n-1, 1 \geqq \mu_1 > 0)$, the following assertions hold if we choose θ_2 as $0<\theta_2-\theta_1<1$, μ_2 as $0<\mu_2<\mu_0$ and $\mu<\min(\mu_0, \mu_1)$.*

1° The operator $A(\kappa)$ belongs to $L(\theta_1, \mu_1; \theta_2, \mu_2)$, and $A(\sigma+i\tau)(\sigma\geqq 0)$ has a limit $A(\sigma\pm i0)=\lim_{\tau\downarrow 0} A(\sigma\pm i\tau)$ in $L(\theta_1, \mu_1; \theta_2, \mu_2)$.

2° There exists a positive constant C such that for any κ (including the boundary point $\kappa=\sigma\pm i0$, $\sigma\geqq 0$) we have

(3) $$\|A(\kappa)\|_{(\theta_1,\mu_1;\,\theta_2,\mu_2)} \leqq C(1+Re\kappa)^{\theta_2-\theta_1},$$

and such that for κ_1 and κ_2 (belonging to the same half plane including the half real axis) satisfying $|k_1-k_2|\leqq 1$, we have

(4) $$\|A(\kappa_1)-A(\kappa_2)\|_{(\theta_1,\mu_1;\,\theta_2,\mu_2)} \leqq C(1+Re\kappa)^{\theta_2-\theta_1}|\kappa_1-\kappa_2|^{\mu\left(1-\frac{\mu_2}{\mu_0}\right)}.$$

3° For any $\theta\leqq n-1$, $\mu<\mu_0$ and $0<\varepsilon<1$, there exists $C>0$ such that

(5) $$\|A(\kappa)\|_{(\theta,\mu;\,\theta,\mu)} \leqq C(1+|Im\kappa|)^{-\varepsilon}.$$

Proof. For $f\in\mathfrak{B}_{\theta,\mu}(\theta\leqq n-1)$, we have

$$A(\kappa)f(k)=\int\frac{v(k,k')}{k'^2-\kappa^2}f(k')dk'.$$

Let Ω be a unit surface of R^n, ω a unit vector, $d\omega$ surface element and t real. We define

$$\varphi(t;\kappa,k)\equiv\begin{cases}\dfrac{t^{n-1}}{t+\kappa}\displaystyle\int_\Omega v(k,t\omega)f(t\omega)d\omega & \text{for } t\geqq 0\\ 0 & \text{for } t<0.\end{cases}$$

Then we can rewrite $A(\kappa)f(k)$ as

$$A(\kappa)f(k)=\int_{-\infty}^{\infty}\frac{\varphi(t;\kappa,k)}{t-\kappa}dt.$$

Since we assume $Re\kappa\geqq 0$, we have $|t+\kappa|\geqq|t|$ for $t\geqq 0$.

Therefore, for $t\geqq 0$ we have

$$|\varphi(t;\kappa,k)| \leqq C\|f\|_{\theta,\mu}|t|^{n-2}(1+|t|)^{-\theta+1}\int(1+|k-t\omega|)^{-\theta_0}d\omega.$$

The integral $I\equiv\int_\Omega(1+|k-t\omega|)^{-(n-1)-\varepsilon}d\omega(\varepsilon>0)$ is estimated by

$$I\leqq C(1+|k|)^{-\alpha}(1+|t|)^{-\beta}(1+||k|-|t||)^{-\varepsilon}$$

for any α and β, $\alpha, \beta \geqq 0$, $\alpha+\beta=n-1$, where C depends on ε but not on α, β (this fact will be proved in 3). Then we have

$$\begin{aligned}&|\varphi(t\,;\kappa, k)|\\ &\quad\leqq C\|f\|_{\theta,\mu}(1+|k|)^{-\theta}(1+||k|-t|)^{-(\theta_0-n+1)}|t|^{n-2}(1+|t|)^{-n+2}\\ &\quad\leqq C\|f\|_{\theta,\mu}(1+|k|)^{-\theta}(1+||k|-t|)^{-1}\,.\end{aligned}$$

Noticing that $\dfrac{d}{dt}\left(\dfrac{t^{n-1}}{t+\kappa}\right)$ is uniformly bounded in the neighbourhood of zero for $Re\kappa\geqq 0$ if $n\geqq 3$, we can calculate the Hölder continuity of φ as

$$\begin{aligned}&|\varphi(t+l\,;\kappa, k)-\varphi(t\,;\kappa, k)|\\ &\quad\leqq C\|f\|_{\theta,\mu}(1+|k|)^{-\theta}(1+||k|-t|)^{-1}(1+|t|)^{-1}|l|^{\hat{\mu}}\end{aligned}$$

where $|l|\leqq 1$ and $\hat{\mu}=\min(\mu_0, \mu)$.

Put $\phi(t)=\phi(t\,;\kappa, k)=\varphi(t+|k|\,;\kappa, k)$. Then we have

$$|\phi|_{1,\hat{\mu}}\leqq C(1+|k|)^{-\theta}\|f\|_{\theta,\mu}\,.$$

Consider the function

$$F(k\,;\kappa, \kappa')\equiv\int_{-\infty}^{\infty}\frac{\varphi(t\,;\kappa', k)}{t-\kappa}dt=\int_{-\infty}^{\infty}\frac{\phi(t\,;\kappa', k)}{t-(\kappa-|k|)}dk\,.$$

Applying Lemma 2.2 of [3], we have

$$|F(k\,;\kappa, \kappa')|\leqq C\|f\|_{\theta,\mu}(1+|k|)^{-\theta}(1+|\kappa-|k||)^{-\varepsilon}$$

and

$$\begin{aligned}&|F(k\,;\kappa, \kappa')-F(k\,;\kappa_1, \kappa')|\\ &\quad\leqq C\|f\|_{\theta,\mu}(1+|k|)^{-\theta}(1+|\kappa-|k||)^{-\varepsilon}|\kappa-\kappa_1|^{\hat{\mu}}\end{aligned}$$

for $0<\varepsilon<1$ and $|\kappa-\kappa_1|\leqq 1$. Since we have $(1+|\kappa-|k||)^{-1}\leqq 2(1+Re\kappa)(1+|k|)^{-1}$ for $Re\kappa\geqq 0$, we can estimate $A(\kappa)f(k)=F(k\,;\kappa, \kappa)$ as

$$|A(\kappa)f(k)|\leqq C\|f\|_{\theta,\mu}(1+Re\kappa)^{\varepsilon}(1+|k|)^{-\theta-\varepsilon}\,.$$

We need to estimate the Hölder coefficient of $A(\kappa)f(k)$ as a function of k in order to have (3). But this estimation can be obtained similarly. We have (4) by the analogous method. Finally we note that the inequality, $|\kappa-|k||\geqq|Im\kappa|$, is used to derive (5).

By this lemma we see that $A(\kappa)$ is completely continuous from $\mathfrak{B}_{\theta,\mu}$ to $\mathfrak{B}_{\theta,\mu}$ if

$$0<\theta\leqq n-1 \text{ and } 0<\mu<\mu_0\,. \tag{6}$$

The estimate (3) shows that non-zero eigenvalues and associated eigenfunctions of $A(\kappa)$ as an operator in $\mathfrak{B}_{\theta,\mu}$ are common to all pairs of (θ, μ) satisfying (6).

Therefore, we can define a set of singular points of L as $S\equiv\{\kappa\mid -1\in s_\kappa\}$ where s_κ is the spectrum of $A(\kappa)$ in $\mathfrak{B}_{\theta,\mu}$. We have the following lemma corresponding to Lemma 3.3 of [3].

LEMMA 3. *(1) The function $\psi\in\mathfrak{D}$ satisfies $L\psi=-\omega^2\psi$ $(\omega>0)$ if and only if $\varphi=(k^2+\omega^2)\psi$ satisfies $\varphi\in\mathfrak{B}_{\theta,\mu}$ and $\varphi+A(\pm i\omega)\varphi=0$.*

(2) For positive σ, $\sigma+i0\in S$ if and only if $\sigma-i0\in S$. The eigen space of $A(\sigma+i0)$ corresponding to eigenvalue -1 is equal to that of $A(\sigma-i0)$.

(3) The function $\psi\in\mathfrak{D}$ satisfies $L\psi=\omega^2\psi$ $(\omega>0)$ if and only if $\varphi=(k^2-\omega^2)\psi$ satisfies $\varphi\in\mathfrak{B}_{\theta,\mu}$ and $\varphi+A(\omega\pm i0)\varphi=0$.

(4) The set S does not contain a point $\kappa=\sigma+i\tau$ $(\sigma>0, \tau\neq 0)$.

(5) If 0 is an eigenvalue of L, then $0\pm i0\in S$.

We can prove Theorem 1 with the aid of Lemmas 2 and 3. Namely, the lower boundedness of $\sigma_p(L)$, the point spectrum of L, is an immediate consequence of (5). By one-to-one correspondence of $\omega\in S$ to $\omega^2\in\sigma_p(L)$ in Lemma 3, we conclude any eigenvalue of L has a finite multiplicity since $A(\omega)$ is completely continuous. The limiting point argument is the same as [1], but we cannot exclude the possibility of the sequence $\{\lambda_j\mid\lambda_j<0, \lambda_j\in\sigma_p(L)\}$ tending to 0 in this case.

Next, consider the integral equation

$$t(k,k';z)=v(k,k')-\int\frac{v(k,k'')}{k''^2-z}t(k'',k';z)dk''. \tag{7}$$

For ε, $0<\varepsilon<1$, we write

$$\Pi_\varepsilon^+\equiv\{z\mid Im z\geqq 0,\ |z|\leqq\varepsilon^{-1},\ d(z,\sigma_p(L)\cup\{0\})\geqq\varepsilon\},$$

and Π_ε^- analogously. Put $\Pi^\pm\equiv\bigcup_{0<\varepsilon<1}\Pi_\varepsilon^\pm$. Let Π_ε and Π be either Π_ε^+ and Π^+ or Π_ε^- and Π^-. We can establish the following lemma along the similar treatment of Lemma 3.6 of [3].

LEMMA 4. *There exists a solution $t(k,k';z)$ of (7) for $z\in\Pi$, satisfying the following estimates. For $\varepsilon>0$, $0<\mu<\mu_0$ and $\nu<\mu\left(1-\frac{\mu}{\mu_0}\right)$, there exists a constant C depending on ε,μ and ν such that*

$$|t(k,k',z)|\leqq Cm(k,k')(1+|k-k'|)^{-n+1}$$

and

$$|t(k,k',z)-t(k_1,k_1',z_1)|$$
$$\leqq Cm(k,k')(1+|k-k'|)^{-n+1}\{|k-k_1|^\mu+|k'-k_1'|^\mu+|z-z_1|^\nu\}$$

for $z,z_1\in\Pi_\varepsilon$, $|z-z_1|\leqq 1$, $|k-k_1|\leqq 1$, and $|k'-k_1'|\leqq 1$.

Put $K_M \equiv \{k \mid |k| \leqq M\}$, the subset of R^n. We may assume the estimate of $t(k, k', z)$ takes the form

$$C(k, k', z)(1+|k-k'|)^{-n+1}$$

where $C(k, k', z)$ is uniformly bounded in $(k, k', z) \in R^n \times K_M \times \Pi_\varepsilon$ or $K_M \times R^n \times \Pi_\varepsilon$. Noticing this fact, we can conclude that Lemmas 3.7 to 3.10 of [3] are valid for this case if we replace $p(k)$ by k^2. Therefore, we have Theorem 2 and 3.

3. First we note that for the fixed ε the integral I depends only on $|k|$ and $|t|$, and that $I=I(|k|, |t|)=I(|t|, |k|)$. Put $\lambda \equiv |t| \cdot |k|^{-1}$.

In order to have the inequality, it is sufficient to prove

$$I \leqq C(\varepsilon)(1+|k|)^{-n+1}(1+||k|-|t||)^{-\varepsilon} \tag{8}$$

in the case of $0<\lambda \leqq 1$.

If $0<\lambda \leqq \frac{1}{2}$, it is easy to show $I \leqq C(\varepsilon)(1+|k|)^{-(n-1)-\varepsilon}$, calculating the integral by the polar coordinate. Since $|k| \geqq |k|-|t| \geqq |t|$ in this case, we have (8).

Assume $\frac{1}{2} \leqq \lambda \leqq 1$. Put $l=n+\varepsilon$, then

$$I=\text{Const}\int_0^\pi \frac{\sin^{n-2}\theta d\theta}{1+|k|^l|1+\lambda^2-2\lambda\cos\theta|^{\frac{l}{2}}}$$

$$=\text{Const}\int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{1+|k|^l|1+\lambda^2-2\lambda t|^{\frac{l}{2}}}dt.$$

Let $s=1-t$, then

$$I=\text{Const}\int_0^2 \frac{s^{\frac{n-3}{2}}(2-s)^{\frac{n-3}{2}}}{1+|k|^l|(1-\lambda)^2+2\lambda s|^{\frac{l}{2}}}dt.$$

Noticing that

$$1+|k|^l \cdot |(1-\lambda)^2+2\lambda s|^{\frac{l}{2}} \geqq C(1+|k|^2 \cdot |(1-\lambda)^2+2\lambda s|)^{\frac{l}{2}}$$

$$\geqq C(|k|^2(2\lambda s))^{\frac{n-3}{2}}\{1+|k|^2|(1-\lambda)^2+2\lambda s|\}^{\frac{2+\varepsilon}{2}},$$

we have

$$I \leqq \text{Const}|k|^{-n+3}\int_0^2 \frac{ds}{\{1+|k|^2 \cdot |(1-\lambda)^2+2\lambda s|\}^{\frac{2+\varepsilon}{2}}}.$$

Put $r=1+|k|^2 \cdot |(1-\lambda)^2+2\lambda s|$, then

$$\int_0^2 = (2\lambda|k|^2)^{-1}\int_{1+||k|-|t||^2}^{1+||k|+|t||^2} r^{-1-\frac{\varepsilon}{2}}dr$$

$$=(\varepsilon\lambda)^{-1}|k|^{-2}((1+||k|-|t||^2)^{-\frac{\varepsilon}{2}}-(1+||k|+|t||^2)^{-\frac{\varepsilon}{2}})$$

$$\leqq \text{Const}|k|^{-2}(1+||k|-|t||)^{-\varepsilon}.$$

Therefore, we have

$$I \leqq \text{Const}|k|^{-n+1}(1+||k|-|t||)^{-\varepsilon}.$$

From this inequality we have (8) for $\frac{1}{2} \leqq \lambda \leqq 1$ and $|k| \geqq 1$. It is trivial that (8) holds in the case $\frac{1}{2} \leqq \lambda \leqq 1$ and $|k| \leqq 1$.

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