On the Decomposition of Some Unitary Representation of the Lorentz Groupe

By Akio ORIHARA

Institute of Mathematics, College of General Education, University of Tokyo

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Introduction

Let $(\mathfrak{X}, U_g)$ be a irreducible unitary representation of class 1 of $G_2=SL(2, C)$, that is,

$$\mathfrak{X}=L^2(C)$$

$$U_g f(z)=|cz+d|^{-2-i\sigma} f\left(\frac{az+b}{cz+d}\right) \qquad \text{for} \quad g=\begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{and} \quad f\in\mathfrak{X}$$

where σ is a real number parametrising the representation. (Here we consider only the principal series).

By restricting this to $G_1=SL(2, R)$, we obtain a unitary representation of G_1, which is also denoted by $(\mathfrak{X}, U_g)$.

In order to decompose this representation into irreducible factors, we define an isometry of $\mathfrak{X}$ onto $H=\{F(z)\,;\, \iint |F(z)|^2 \frac{dxdy}{y^2}<\infty\}$ by

$$(If)(z)=y^{1+\frac{i}{2}\sigma} f(z) \qquad (z=x+iy)\,.$$

Then, if we put

$$T_g=IU_gI^{-1}, \qquad T_gF(z)=F\left(\frac{az+b}{cz+d}\right).$$

Obviously, (T_g, H) is the direct sum of two representations, each of which is equivalent to the quasi-regular representation of G_1.

On the other hand, the quasi-regular representation is decomposed into irreducible representation of class 1 (with multiplicity 1) by the well known method. Thus the problem is completely solved. (For more general case, see Romm [4].)

In this paper we shall show that for the Lorentz group of n-th order, the similar result can be obtained (see § 4).

We shall also discuss the application to special functions (see § 4 (5), (6) (7)).

§ 1. Irreducible representations of class 1

Let $G=G_n$ be the Lorentz group, that is, the connected component of the orthogonal group with respect to the quadratic form $-x_0^2+x_1^2+\cdots+x_{n+1}^2$.

As is known, irreducible unitary representations of class 1 are parametrised by positive real numbers.

These representations are constructed in the following way.

1) $\mathfrak{X}=L^2(S^n, d\omega)$ ($S^n=n$-dimensional sphere $d\omega=$uniform measure on S^n)

$$U_g f(\omega)=e^{\left(\frac{n}{2}+i\rho\right)(\omega,g)} f(\omega_g) \quad (\rho \text{ is a positive real number})$$

where ω_g and $t(\omega, g)$ are defined as follows.

A is known, any semi-simple Lie group (with finite center) is (topologically) the product of its three subgroups:

$$G=NAK \qquad \text{(Iwasawa decomposition)}$$

where K is a maximal compact subgroup, N (or A) is a simply connected nilpotent (or abelian) subgroup.

In our case K is conjugate to the subgroup $\left\{\begin{pmatrix}1 & \\ & k\end{pmatrix}, k\in S0(n+1)\right\}$ and A is one-dimensional.

Therefore, for any $g\in G$ and $k\in K$, kg can be expressed in the following form:

$$kg=na_t k \qquad (n\in N,\ a_t\in A,\ k\in K)\,.$$

We put $k=k_g$, $t=t(k, g)$.

Then, it is easy to see that $k\to k_g$ induces a transformation of S^n: $\omega\to\omega_g$ and $t(k, g)$ is a function on $S^n\times G$. (for details, see [2]).

2) Another realization (see [1])

$$\mathfrak{X}=L^2(R^n)$$

$$U_g f(x)=f(x_g)\Delta(x, g)$$

where

$$\Delta(x, g)=\tfrac{1}{2}(g_{0,0}-g_{0,n+1})(|x|^2+1)+\sum_{j=1}^{n} x_j(g_{j,0}-g_{0,n+1})$$

$$+\tfrac{1}{2}(g_{n+1,0}-g_{n+1,n+1})(|x|^2-1)$$

and

$$(x_g)_k=\tfrac{1}{2}g_{0,k}(|x|^2+1)+\sum_{j=1}^{n} x_j g_{jk}+\tfrac{1}{2}g_{n+1,k}(|x|^2-1)/\Delta(x, g) \qquad (1\leqq k\leqq n)$$

($\Delta(x, g)$ is positive for any $x\in R^n$ and $g\in G$).

§ 2. Quasi-regular representation

We define a unitary representation of G in $H=L^2(K/G)$ by

$$T_g f(x)=f(xg)\,.$$

This representation is called the quasi-regular representation of G. As $X=K/G$ is homomorphic to $A\times N$,

$$A=\left\{a_t=\begin{pmatrix} cht & & & sht \\ & 1 & & \\ & & \ddots & \\ & & 1 & \\ sht & & & cht \end{pmatrix}\right\},\quad N=\left\{n_x=\begin{pmatrix} 1+\varDelta, & x_1 & \cdots & x_n, & \varDelta \\ x_1 & & & & x_1 \\ \vdots & & & & \vdots \\ x_n & & & & x_n \\ -\varDelta, & -x_1 & \cdots & -x_n, & 1-\varDelta \end{pmatrix} \varDelta=\frac{1}{2}\sum_{i=1}^{n} x_i^2\right\}$$

we can express the operation of G on X in terms of the coordinates $(x_1\cdots x_n, t)$.

From the relation, $a_t n_x g=k a_{t'} n_{x'}$, $k=\begin{pmatrix}1 & \\ & k\end{pmatrix}$, $k\in SO(n+1)$ we have

$$(cht+e^{-t}\varDelta)g_{0,k}+e^{-t}\sum_{j=1}^{n} x_j g_{jk}+(sht+e^{-t}\varDelta)g_{n+1,k}$$

$$=\begin{cases} cht+e^{-t}\varDelta & (k=0) \\ e^{-t}x_k & (1\leqq k\leqq n) \\ sht+e^{-t}\varDelta & (k=n+1) \end{cases}$$

(Compare the first row of the matrices on both sides).

Therefore we obtain the following:

$$y'=\frac{y}{\frac{1}{2}(g_{0,0}-g_{0,n+1})(|x|^2+y^2+1)+\sum_{j=1}^{n} x_j(g_{j,0}-g_{j,n+1})+\frac{1}{2}(g_{n+1,0}-g_{n+1,n+1})(|x|^2+y^2-1)}$$

$$x'_k=\frac{\frac{1}{2}g_{0,k}(|x|^2+y^2+1)+\sum_{j=1}^{n} x_j g_{jk}+\frac{1}{2}g_{n+1,k}(|x|^2+y^0-1)}{\frac{1}{2}(g_{0,0}-g_{0,n+1})(|x|^2+y^2+1)+\sum_{j=1}^{n} x_j(g_{j,0}-g_{j,n+1})+\frac{1}{2}(g_{n+1,0}-g_{n+1,n+1})(|x|^2+y^2-1)}$$

where we put $y=e^t$ and $y'=e^{t'}$.

Next, we determine the invariant measure on X. We put

$$J(g)=\frac{\partial(x'_1\cdots x'_n y')}{\partial(x_1\cdots x_n y)}\bigg|_{x_1=\cdots=x_n=0,\, y=1}\,.$$

Then, it is easy to see that $J(g)=J(kg)$ (therefore J is a function on X. We put $J(x,y)=J)a_t n_x))$ and $\frac{1}{J(x,y)}dx_1\wedge\cdots\wedge dx_n\wedge dy_y$ is a G-invariant differential form on X.

As $J(x,y)=y^{n+1}$, we see that the measure μ defined by

(1) $$\int_{R^n}\int_0^\infty f(x,y)d\mu(x,y)=\int_{R^n}\int_0^\infty f(x,y)\frac{1}{y^{n+1}}dx_1\cdots dx_{n+1}dy$$

is a invariant measure on X. Consequently, the quasi-regular representation can be described as follows.

$$H=L^2(R^n\times(0,\infty),\mu)$$

where x' and y' are given by

$$T_g f(x,y)=f(x',y')\,.$$

Remark. By the transformation:

$$\xi_0=cht+e^{-t}\varDelta,\ \xi_k=e^{-t}x_k\ (1\leqq k\leqq n),\ \xi_{n+1}=sht+e^{-t}\varDelta$$

X is mapped homomorphically onto

$$\varXi=\{\xi=(\xi_0,\cdots,\xi_{n+1})\,;\ \xi_0^2-\xi_1^2-,\cdots-\xi_{n+1}^2=1,\ \xi_0>0\}\,.$$

As

$$\frac{\partial(\xi_1,\cdots,\xi_{n+1})}{\partial(x_1,\cdots,x_n,t)}=\begin{vmatrix} e^{-t}0\cdots 0, e^{-t}x_1 \\ 0\ e^{-t}\cdots 0, -e^{-t}x_2 \\ 0\cdots e^{-t}, -e^{-t}x_n \\ e^{-t}x_1\cdots e^{-t}x_n, cht-e^{-t}\varDelta \end{vmatrix}=\begin{vmatrix} e^{-t} & & 0 \\ & 0 & \\ & e^{-t} & \\ *\cdots\cdots* & & cht+e^{-t}\varDelta \end{vmatrix}$$

$$=e^{-nt}(cht+e^{-t}\varDelta)=\frac{1}{y^n}\xi_0$$

we see that the measure μ defined above corresponds to the invariant measure ν on $\varXi$ defined by

$$d\nu(\xi)=\frac{d\xi_1\cdots d\xi_{n+1}}{\xi_0}\,.$$

§3. Decomposition of the quasi-regular representation

For $f\in C_{\infty}^{0}(X)$, we put

$$\check{f}_\rho(k)=\int_N\int_{-\infty}^{\infty}f(na_tk)e^{-(\frac{n}{2}+i\rho)t}dtdn\,.$$

Then,

$$(T_g\check{f})_\rho(k)=\int_N\int_{-\infty}^{\infty}f(na_tkg)e^{-(\frac{n}{2}+i\rho)t}dtdn$$

$$=\int_N\int_{-\infty}^{\infty}f(na_tna'_{t'}k_g)e^{-(\frac{n}{2}+i\rho)t}dndt$$

$$=\int_N\int_{-\infty}^{\infty}f(na_{t+t'}k_g)e^{-(\frac{n}{2}+i\rho)t}dtdn$$

$$=e^{-(\frac{n}{2}+i\rho)t(k,g)}\check{f}_\rho(k_g)=U_g^\rho\check{f}_\rho(k)\,.$$

It is known ([2], [3]) that the map: $f\to\check{f}$ can be extended to an isometry of

$$H=L^2(X)$$

onto

$$H=\int_0^\infty \mathfrak{X}_\rho \omega(\rho)d\rho$$

where

$$\mathfrak{X}_\rho=\mathfrak{X}=L^2(S^n)$$

and

$$\omega(\rho)=\frac{\Gamma\left(\frac{n}{2}+i\rho\right)\Gamma\left(\frac{n}{2}-i\rho\right)}{2^n\pi^{1+\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)}\rho sh\rho\,.$$

Hence we have

$$T_g=\int_0^\infty U_g^\rho\omega(\rho)d\rho\,. \tag{2}$$

Remark. If f is K-invariant (i. e. $f(xk)=f(x)$), there exist a function F such that

$$f(x_1,\cdots,x_n,y)=F\left(\frac{|x|^2+y^2+1}{2y}\right).$$

In this case, $\check{f}_\rho(k)=\check{f}_\rho$ (independent of k) and

$$\check{f}_\rho=\int_G f(g)\varphi(g)dg=\frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\int_0^\infty F(cht)\varphi_\rho(a_t)sh^n t\,dt \tag{3}$$

where

$$\varphi_\rho(ka_tk)=\varphi_\rho(a_t)=\frac{2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{sh^{\frac{n-1}{2}}t}\mathfrak{P}^{1-\frac{n+1}{2}}_{-\frac{1}{2}+i\rho}(cht)$$

(zonal spherical function).

(Here we normalise the Haar measure of G by $\int_K dk=1$ and (1)).

§ 4. Decomposition of U_g

We put $G'=\{g\in G;\ g_{nk}=g_{kn}=\delta_{kn}\}$ $(\cong G_{n-1})$. For $f\in\mathfrak{X}$, we define $f^{(i)}\in H'$ $(i=1,2)$ by

$$f^{(1)}(x,y)=y^s f(x_1,\cdots,x_{n-1},y)$$
$$f^{(2)}(x,y)=y^s f(x_1,\cdots,x_{n-1},y)\qquad \left(s=\frac{1}{2}+i\sigma\right).$$

Then, $I:f\to(f^{(1)},f^{(2)})$ is an isometry of $\mathfrak{X}$ onto $H'\oplus H'$ and

$$IU_gI^{-1}=T'_g\oplus T'_g$$

where (T'_g, H') is the quasi-regular representation of G' (see § 2).

To see this we have only to note that if $g\in G$,

$(xg)_n=$

$$\frac{x_n}{\frac{1}{2}(g_{0,0}-g_{0,n+1})(|x|^2+x_n^2+1)+\sum_{j=1}^{n-1}x_j(g_{j,0}-g_{j,n+1})+\frac{1}{2}(g_{n+1,0}-g_{n+1,n+1})(|x|^2+x_n^2-1)}$$

$(xg)=$

$$\frac{\frac{1}{2}g_{0,k}(|x|^2+x_n^2+1)+\sum_{j=1}^{n-1}x_jg_{jk}+\frac{1}{2}g_{n+1,k}(|x|^2+x_n^2-1)}{\frac{1}{2}(g_{0,0}-g_{0,n+1})(|x|^2+x_n^2+1)+\sum_{j=1}^{n-1}x_j(g_{j,0}-g_{j,n+1})+\frac{1}{2}(g_{n+1,0}-g_{n+1,n+1})(|x|^2+x_n^2-1)}$$

$$(1\leqq k\leqq n-1.\quad x^2=\sum_{j=1}^{n-1}x_j^2).$$

Hence, combining with (2) we can complete the decomposition of U_g.

Thus, for $f\in\mathfrak{X}$, there correspond $\check{f}_\rho^{(i)}\in\mathfrak{X}'$ $(i=1,2)$ such that

$$(U_gf,f)=\sum_{i=1}^{2}\int_0^\infty(U'^{\rho}_g\check{f}_\rho^{(i)},\check{f}_\rho^{(i)})\omega(\rho)d\rho\,. \tag{4}$$

In particular, if f is M-invariant $(M=G\cap K)$

$$\check{f}_\rho^{(1)}=\int_0^\infty y^{-(\frac{n-1}{2}+i\rho)}\int_{R^{n-1}}y^sf(x_1,\cdots,x_{n-1},y)\frac{1}{y}dx_1,\cdots,dx_{n-1}dy\cdot\check{f}_0$$

$$\check{f}_\rho^{(2)}=\int_0^\infty y^{-(\frac{n-1}{2}+i\rho)}\int_{R^{n-1}}y^sf(x_1,\cdots,x_{n-1},y)\frac{1}{y}dx_1,\cdots,dx_{n-1}dy\cdot\check{f}_0$$

where f_0 is a M-invariant vector in $\mathfrak{X}'$ with norm 1. (We denote by $(U'^{\rho}_g,\mathfrak{X}')$ the representation of class 1 of G' parametrised by ρ).

We put

$$f(x)=\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}\right)^{\frac{1}{2}}\frac{1}{(1+|x|^2)^s}\qquad\left(s=\frac{1}{2}+i\sigma\right).$$

Then, f is a K-invariant vector in $(\|f\|=1)$ (see [1]).

In this case, $\check{f}_\rho^{(1)}=\check{f}_\rho^{(2)}=c(\rho)\check{f}_0$ where

$$c(\rho)=\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}\right)^{\frac{1}{2}}\int_0^\infty y^{-(\frac{n-1}{2}+i\rho)}\frac{dy}{y}\int_{R^{n-1}}\frac{y^s}{(1+|x|^2+y^2)^s}dx\,.$$

As we have

$$\int_{R^{n-1}}\frac{dx}{(a^2+|x|^2)^s}=a^{(n-1)-2s}\frac{\pi^{\frac{n-1}{2}}}{\Gamma(s)}\Gamma\left(s-\frac{n-1}{2}\right)$$

$$c(\rho)=\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}\right)^{\frac{1}{2}}\frac{\pi^{\frac{n-1}{2}}}{\Gamma(s)}\Gamma\left(s-\frac{n-1}{2}\right)\int_0^\infty y^{-(\frac{n-1}{2}+i\rho)+s-1}(1+y^2)^{\frac{n-1-2s}{2}}dy$$

$$=\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}\right)^{\frac{1}{2}}\frac{\pi^{\frac{n-1}{2}}}{2}\frac{\Gamma\left(\frac{s}{2}-\frac{n-1}{4}+\frac{i\rho}{2}\right)\Gamma\left(\frac{s}{2}-\frac{n-1}{4}-\frac{i\rho}{2}\right)}{\Gamma(s)}.$$

On the other hand, by (3)

$$c(\rho)=\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}\right)^{\frac{1}{2}}\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty(2cht)^{-s}\frac{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{sh^{\frac{n-2}{2}}t}\mathfrak{P}^{1-\frac{n}{2}}_{-\frac{1}{2}+i\rho}(cht)sh^{n-1}tdt.$$

Hence we have

$$(5)\qquad \int_1^\infty x^{-s}(x^2-1)^{\frac{n-2}{4}}\mathfrak{P}^{1-\frac{n}{2}}_{-\frac{1}{2}+i\rho}(x)dx$$

$$=\frac{2^{s-\frac{n}{2}-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{s}{2}-\frac{n-1}{4}+\frac{i\rho}{2}\right)\Gamma\left(\frac{s}{2}-\frac{n-1}{4}-\frac{i\rho}{2}\right)}{\Gamma(s)}.$$

From (4) we obtain

$$\varphi_\sigma(g)=2\int_0^\infty|c(\rho)|^2\varphi_\rho'(g)\omega(\rho)d\rho\qquad(g\in G')$$

where $\varphi'_\rho(g)$ is the zonal spherical function of G'.

Therefore we have

$$(6)\qquad \mathfrak{P}^{1-\frac{n+1}{2}}_{-\frac{1}{2}+i\rho}(cht)=\frac{1}{\pi(2\pi)^{\frac{3}{2}}}\int_0^\infty(sht)^{\frac{1}{2}}$$

$$\mathfrak{P}^{1-\frac{n}{2}}_{-\frac{1}{2}+i\rho}(cht)\left|\frac{\Gamma\left(\frac{s}{2}-\frac{n-1}{4}+\frac{i\rho}{2}\right)\Gamma\left(\frac{s}{2}-\frac{n-1}{4}-\frac{i\rho}{2}\right)}{\Gamma(s)}\right|^2$$

$$\left|\Gamma\left(\frac{n-1}{2}+i\rho\right)\right|^2\rho sh\rho d\rho.$$

Remark. It is known that a irreducible representation of class of $SO(n+1)$ is decomposed into representations of class 1, if it is restricted to $SO(n)$. In connection with this, we can obtain a formula concerning Gegenbauer polynomial, which is analogous to (6) (see [3]).

So far we excluded the case where $n=1$. Here we consider this case. Irreducible unitary representations of class 1 of $G'=SL(2,R)$ are given as follows.

$$U_g^sf(x)=|cx+d|^{-2s}f\left(\frac{ax+b}{cx+d}\right),\qquad g=\begin{pmatrix}a&c\\b&d\end{pmatrix}\in G,\qquad f\in L^2(-\infty,\infty)=\mathfrak{X},$$

$$s=\frac{1}{2}+i\sigma.$$

For $f \in \mathfrak{X}$, we put

$$\tilde{f}^{(+)}(\rho)=\int_0^\infty f(x)x^{s-i\rho-1}dx$$

$$\tilde{f}^{(-)}(\rho)=\int_0^\infty f(-x)x^{s-i\rho-1}dx .$$

Then, it is easy to see that

$$(\widetilde{U^s_{a_t}f})^{(\pm)}(\rho)=e^{i\rho t}\tilde{f}^{(\pm)}(\rho) \qquad a_t=\begin{pmatrix} e^{\frac{t}{2}} & \\ & e^{-\frac{t}{2}} \end{pmatrix}$$

and

$$\|f\|^2=\frac{1}{2\pi}\int_{-\infty}^\infty [|\tilde{f}^{(+)}(\rho)|^2+|\tilde{f}^{(-)}(\rho)|^2]\, d\rho .$$

If

$$f(x)=\frac{1}{\sqrt{\pi}}\frac{1}{(1+x^2)^s}, \qquad \tilde{f}^{(\pm)}(\rho)=\frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{s+i\rho}{2}\right)\Gamma\left(\frac{s-i\rho}{2}\right)}{\Gamma(s)} .$$

Therefore we have

$$P_{-\frac{1}{2}+i\rho}(cht)=(U^s_a f, f)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\rho t}[|\tilde{f}^{(+)}(\rho)|^2+|\tilde{f}^{(-)}(\rho)|^2]\, d\rho$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^\infty \left|\frac{\Gamma\left(\frac{s+i\rho}{2}\right)\Gamma\left(\frac{s-i\rho}{2}\right)}{\Gamma(s)}\right|^2 e^{i\rho t}d\rho .$$

Similarly, if

$$f(x)=\frac{1}{(1-x^2)^s_+},$$

$$\tilde{f}^{(\pm)}(\rho)=\int_0^1 \frac{x^{s-i\rho-1}}{(1-x^2)^s_+}dx=\frac{1}{2}\frac{\Gamma\left(\frac{s+i\rho}{2}\right)\Gamma\left(\frac{s-i\rho}{2}\right)}{\Gamma(s)}\frac{\sin \pi\frac{s+i\rho}{2}}{\sin \pi s} .$$

Therefore

$$\left.\begin{matrix} (t>0) & Q_{-\frac{1}{2}+i\rho}(cht) \\ (t<0) & Q_{-\frac{1}{2}-i\rho}(cht) \end{matrix}\right\} =(U^s_{a_t}f, f)$$

$$=\frac{1}{4\pi}\int_{-\infty}^\infty e^{i\rho t}\left|\frac{\Gamma\left(\frac{s+i\rho}{2}\right)\Gamma\left(\frac{s-i\rho}{2}\right)}{\Gamma(s)}\right|^2 \frac{ch\pi(\sigma+\rho)}{2ch^2\pi\sigma}d\rho .$$

Remark. In the case of G_n, we put

$$\tilde{f}(\rho, \omega)=\int_0^\infty r^{s-i\rho-1}f(r, \omega)dr(\omega\in S^{n-1}), \qquad \text{for } f\in\mathfrak{X} .$$

Then we have

$$(\widetilde{U^s_{a_t}f})(\rho, \omega)=e^{i\rho t}\tilde{f}(\rho, \omega)$$

and

$$\|f\|^2=\frac{1}{2\pi}\int_{S^{n-1}}\int|\tilde{f}(\rho,\omega)|^2 d\rho d\omega\,,$$

For

$$f(x)=\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}\right)^{\frac{1}{2}}\frac{1}{(1+|x|^2)^s}\,,$$

$$\tilde{f}(\rho,\omega)=\tilde{f}(\rho)=\frac{1}{2}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{s+i\rho}{2}\right)\Gamma\left(\frac{s-i\rho}{2}\right)}{\Gamma(s)}\,,$$

Hence we obtain the following:

(7)
$$\frac{2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{sh^{\frac{n-1}{2}}t}\mathfrak{P}^{-\frac{n-1}{2}}_{-\frac{1}{2}+i\sigma}(cht)$$

$$=(U^s_{a_t}f,f)=\frac{1}{2\pi}\cdot\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_{-\infty}^{\infty}e^{i\rho t}|\tilde{f}(\rho)|^2 d\rho$$

$$=\frac{1}{4\pi}\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)^2}\int_{-\infty}^{\infty}e^{i\rho t}\left|\frac{\Gamma\left(\frac{s+i\rho}{2}\right)\Gamma\left(\frac{s-i\rho}{2}\right)}{\Gamma(s)}\right|^2 d\rho\,.$$

References

[1] Orihara, A., On some integral formulae containing Bessel functions, *Publ. of Research Institute for Mathematical Science, Kyoto University*, 1, 1 (1965).

[2] Takahashi, R., Sur les représentations unitaires des groupes de Lorentz généralisés, *Bull. Sci. Math. France*, 91 (1963).

[3] Биленкин, Н. Я., Специальные функции и теория представлений групп. Москва (1965).

[4] Ромм, Б. Д., Разложение на неприводимые представления сужения представлений основной серии собственной группы Лоренца на вещественную группу Лоренца. *Д. А. Н. СССР*, 152 (1) (1963).