On the Spectral Properties of Positive Irreducible Operators in an Arbitrary Banach Lattice and Problems of H. H. Schaefer

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1. Introduction[1), 2)]

After the work of M. A. Krein and R. A. Rutman [6] on positive compact operators, S. Karlin tried in [5] to generalize the theory to the case of positive operators, not necessarily compact, and has found among others the following interesting fact: The spectral properties of positive operators on the spectral circle are determined, in some respects, by the spectral properties at the single point, "the worst singularity at the spectral radius $r(T)$". A little later, in connection with these, H. H. Schaefer raised in [15] the following problems of a positive operator T:

a) If $r(T)$ is an isolated singularity of the resolvent $R(\lambda, T)$, is every singularity of $R(\lambda, T)$ on the spectral circle isolated?

b) If $r(T)$ is a pole of $R(\lambda, T)$, is every singularity of $R(\lambda, T)$ on the spectral circle a pole of $R(\lambda, T)$?

Since then many contributions have been done to this field: G.-C. Rota [12], H. H. Schaefer [16], [17], [18] and I. Sawashima [13].

The principal purpose of this note is to establish the following:

MAIN THEOREM. *Let E be an arbitrary Banach lattice and T be an bounded operator of $\mathfrak{L}(E)$ with the following properties:*

I) *T is positive.*

II) *T is irreducible.*

III) *$\lambda=r(T)$ is a pole of the resolvent $R(\lambda, T)$.*

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1) The principal results of this paper were announced in [11].

2) For the notations and terminologies, see section 2.

Then the spectrum of T on the spectral circle coincides with the set of k-th roots of unity multiplied by $r(T)$, each of which is a simple pole of $R(\lambda, T)$, where k is a fixed positive integer determined by T.

The authors have already proved the special case of this theorem where E is l_p $(1<p<\infty)$ in [9], L_p $(1<p<\infty)$ in [10] or $C(S)$ in [14][3].

In section 2 notations and terminologies are given.

In section 3 we apply Kakutani's representation of (AM) space, as H. H. Schaefer did in [18], to a certain subset of a Banach lattice E.

In section 4 we discuss positive irreducible operators and their direct consequences.

In sections 5), ⋯ , 8) we prove our main theorem. To prove the theorem we must show the following three assertions, namely,

A) *The point spectrum on the spectral circle satisfies the concluding condition of the main theorem.*

B) *The residual spectrum on the spectral circle is void.*

C) *The continous spectrum on the spectral circle is void.*

In section 5 we prove assertion A). The proof of this part was already established essentially by H. H. Schaefer in [18].

In section 6 we prove assertion B). The principal idea to prove this part can be found in the corresponding part of [14], i. e., the reduction theory of an operator which is not necessarily completely reducible.

In section 7 we extend the space E to an (AL) space L. This extension induces the extension of T to the space L.

In section 8, with the preparations established in section 7, we proceed to prove assertion C) as in the corresponding part of [10].

In the last section 9 problem b) of H. H. Schaefer is answered, as a consequence of the main theorem, affirmatively for positive irreducible operators in an arbitrary Banach lattice.

Finally we consider the problem to weaken the assumption of irreducibility in the main theorem. Theorem 9.2 and examples 9.1 and 9.2 solve this problem in some extents, which also solve partially the problem to weaken the assumption of irreducibility in H. H. Schaefer's problem b).

2. Notations and terminologies

We denote by E a *Banach lattice*, i. e., a Banach space and a vector lattice such that, for any $x, y \in E$,

$$|x| \leqq |y| \quad \text{implies} \quad \|x\| \leqq \|y\| .$$

3) We knew that H. H. Schaefer had proved in [19] and [20] the special case of this theorem where E is $C(S)$ or L_1 after this manuscript was prepared.

By a *normed lattice* we mean the space for which all the assumption of Banach lattice are satisfied with the only exception of the completeness assumption w. r. t. (with regard to) the norm. We make use as usual of the following notations and terminologies of a Banach lattice: $x\vee y$, $x\wedge y$, x_+, x_-, $|x|$, $\bigvee_\alpha x_\alpha$, $\bigwedge_\alpha x_\alpha$, ($\sigma$-)complete and so on. An *interval* $[a, b]$ is the set $\{x; a\leqq x\leqq b\}$. A subset $\{x_\alpha\}$ of E is called *bounded by* $e\in E$ if, for each α, holds the relation $|x_\alpha|\leqq e$. An element $x\in E$ is called *bounded w.r.t.* $e\in E$ if there exists a positive number c such that $|x|\leqq ce$. K is the positive cone of E, and $\mathfrak{L}(E)$ is the set of bounded linear operators in E. E^*, K^* and T^* are the duals of E, K and $T\in\mathfrak{L}(E)$ respectively. A subset F of E is called *solid* if

$$y\in F \quad \text{and} \quad |x|\leqq|y| \quad \text{imply} \quad x\in F.$$

Evidently a solid subspace is a sublattice of E which is not necessarily closed. An element $e\in K$ is called a *weak order unit* (Freudenthal unit) if $x\wedge e=0$ implies $x=0$, a *quasi-inner* element of K if the interval $[0, e]$ is total in E, i. e., the smallest solid subspace of E containing e is dense in E, and finally a *non-support* element of K if e is not a support point of the convex set K, i. e., for any non-zero functional $f\in K^*$ always holds $f(e)>0$. Dually a functional $f\in K^*$ is called *strictly positive* if, for any non-zero element $x\in K$, always holds $f(x)>0$. The w^*-limit and the w^*-topology are the ones w. r. t. the duality $\sigma(E^*, E)$, i. e., as functionals of E respectively. Let $P_\sigma(T)$, $R_\sigma(T)$, $C_\sigma(T)$, $\sigma(T)$, $\rho(T)$ and $R(\lambda, T)$ (or simply $R(\lambda)$) be assigned to the usual meanings. $r(T)$ is the *spectral radius* of T, i. e.,

$$r(T)=\max\{|\lambda|; \lambda\in\sigma(T)\}.$$

By Γ we denote the spectral circle of T, i. e.,

$$\Gamma=\{\lambda; |\lambda|=r(T)\}.$$

The *approximate spectrum* of T, denoted by $A_\sigma(T)$, is, by definition, the set of complex numbers λ for which there exists a sequence $x_n\in E$ with the properties:

$$\|Tx_n-\lambda x_n\|\to 0$$

and

$$\|x_n\|=1.$$

It is clear that

$$P_\sigma(T)\cup C_\sigma(T)\subset A_\sigma(T)\subset\sigma(T).$$

$T\in\mathfrak{L}(E)$ is called *positive* if T leaves K invariant, i. e.,

$$TK\subset K.$$

An operator $T\in\mathfrak{L}(E)$ is called *irreducible* if there exists no non-trivial closed solid subspace invariant under T. A positive operator $T\in\mathfrak{L}(E)$ is called *quasi-inner* if there exists a positive number $\lambda > r(T)$ such that $TR(\lambda, T)x$ is a quasi-inner point of K for every non-zero $x \in K$, and *semi-non-support* if, for any non-zero $x \in K$ and any non-zero $f \in K^*$, there exists a positive integer n such that $f(T^n x) > 0$. Further a positive operator $T \in \mathfrak{L}(E)$ is called *non-support* if, for any non-zero $x \in K$ and any non-zero $f \in K^*$, there exists a positive integer n_0 such that $f(T^n x) > 0$ whenever $n \geqq n_0$. Finally $C(S)$ stands for the set of all continuous functions defined on a compact Hausdorff space S.

3. Some properties of a Banach lattice

In this section we investigate some properties, fundamental for us or interesting in themselves, of an arbitrary Banach lattice. Its σ-completeness will be assumed only in the approximation lemma.

We denote by E_e the set of elements bounded w.r.t. e, then E_e is clearly the smallest solid subspace containing e. For each $x \in E_e$, define a new norm by

$$\|x\|_e = \inf \{c\,;\ |x| \leqq ce\}\,.$$

Under this norm E_e is easily seen to be a Banach space satisfying

$$\|x\| \leqq \|x\|_e \|e\|\,. \tag{3.1}$$

Inducing the order in E_e from E, E_e becomes a Banach lattice satisfying

$$\|x \vee y\|_e = \max(\|x\|_e, \|y\|_e) \qquad (x, y \geqq 0)\,.$$

Moreover e is the unit element in the sense of Kakutani [4]. Therefore, by his representation theory, E_e is isomorphic and isometric as a Banach lattice to the space $C(\mathfrak{M})$, the set of continuous functions defined on a compact Hausdorff space $\mathfrak{M}$. We write this representation by

$$x \longleftrightarrow \hat{x}\,.$$

Then clearly

$$\hat{e} = \mathbf{1}\,.$$

Hereafter we shall always assign to E_e the above defined new norm $\|\cdot\|_e$.

Since E_e is a sublattice of E, it can be seen that for any $x, y, z \in E_e$

$$x \vee y = z \text{ in } E \text{ if and only if } x \vee y = z \text{ in } E_e\,.$$

Moreover we can easily prove

LEMMA 3.1. *Let $x, x_\alpha \in E_e$. Then*

$$\bigvee_\alpha x_\alpha = x \textit{ in } E \textit{ if and only if } \bigvee_\alpha x_\alpha = x \textit{ in } E_e\,.$$

By this lemma we get

PROPOSITION 3.1. *For any $x, y \in E$ there exists in E the element*

$$\bigvee_\theta ((\cos\theta)x+(\sin\theta)y) \tag{3.2}$$

Proof. Put

$$e=|x|+|y|,$$

then clearly $x, y\in E_e$. Going over to Kakutani's representation space, we can see that the above supremum (3.2) exists in E_e. Lemma 3.1 then proves the proposition.

Remark 3.1. Let θ_n be a dense subset of the interval $[0, 2\pi]$, then the famous theorem of Dini shows that the above supremum (3.2) is, in the sense of $\|\cdot\|_e$, a limit point of the following sequence:

$$\bigvee_{1\leqq m\leqq n} ((\cos\theta_m)x+(\sin\theta_m)y).$$

By (3.1) this remains true in the sense of the initial norm $\|\cdot\|$.

The following proposition is also a direct consequence of lemma 3.1.

PROPOSITION 3.2. *E is (σ-)complete if and only if E_e is (σ-)complete for each $e\in K$.*

For $x, e\in K$, put

$$x_n=x\wedge ne.$$

Then x_n is a non-decreasing bounded sequence belonging to E. Without the assumption of σ-completeness for E, $\bigvee_n x_n$ may not exist. However we have

PROPOSITION 3.3. *An element $e\in K$ is a weak order unit if and only if, for each $x\in K$, holds the equality:*

$$\bigvee_n (x\wedge ne)=x.$$

Proof. Since $x\wedge e=0$ implies $x\wedge ne=0$, the 'if' part is evident. To prove the 'only if' part, we assume e to be a weak order unit and x to be an element of K. Let $y\in K$ satisfy

$$y\geqq x\wedge ne \qquad (n=1, 2, \cdots). \tag{3.3}$$

Put

$$e_0=e+x+y.$$

Then clearly e, x and y belong to E_{e_0}.

Since Kakutani's representation preserves order, we have

$$\hat{y}(\mathfrak{p})\geqq\min\{\hat{x}(\mathfrak{p}), n\hat{e}(\mathfrak{p})\} \qquad (\mathfrak{p}\in\mathfrak{M}, n=1, 2, \cdots). \tag{3.4}$$

Put

$$\mathfrak{N}=\{\mathfrak{p}; \hat{e}(\mathfrak{p})>0\}.$$

Then (3.4) implies

$$\hat{y}(\mathfrak{p})\geqq\hat{x}(\mathfrak{p}) \qquad (\mathfrak{p}\in\mathfrak{N}).$$

Since $\hat{x}$ and $\hat{y}$ are continuous on $\mathfrak{M}$, we get

$$\hat{y}(\mathfrak{p}) \geqq \hat{x}(\mathfrak{p}) \qquad (\mathfrak{p} \in \overline{\mathfrak{N}}) .$$

It is sufficient to prove

$$\overline{\mathfrak{N}} = \mathfrak{M} .$$

To prove $\overline{\mathfrak{N}} = \mathfrak{M}$, suppose the contrary, i.e.,

$$\overline{\mathfrak{N}} \neq \mathfrak{M} .$$

Then there exist

$$\mathfrak{p}_0 \in \overline{\mathfrak{N}}^c \quad \text{and} \quad \hat{z} \in C(\mathfrak{M})$$

which satisfy the relations:

$$\hat{z}(\mathfrak{p}_0) = 1 ,$$

$$\hat{z}(\mathfrak{p}) = 0 \qquad (\mathfrak{p} \in \overline{\mathfrak{N}})$$

and

$$\hat{z}(\mathfrak{p}) \geqq 0 \qquad (\mathfrak{p} \in \mathfrak{M}) .$$

Then clearly

$$\hat{e} \wedge \hat{z} = 0 ,$$

consequently

$$e \wedge z = 0 .$$

Since e is a weak order unit, we get from this

$$z = 0 .$$

This is a contradiction and the proposition is proved.

LEMMA 3.2. *Let e be a non-support element of K. Then, for each $x \in K$, $x \wedge ne$ converges to x strongly.*

Proof. For any $f \in K^*$, define

$$g(x) = \lim_n f(x \wedge ne) \qquad (x \in K) .$$

Then it is evident that g is a positively homogeneous and additive functional on K. The natural extension of this functional to E, denoted by the same letter g, satisfies

$$g \in E^* ,$$

$$0 \leqq g \leqq f$$

and

$$g(e) = f(e) .$$

Since e is a non-support element of K, these relations yield

$$f = g .$$

That is, for any $x \in K$ and for any $f \in K^*$ holds

$$\lim_n f(x\wedge ne)=f(x)\,. \tag{3.5}$$

Since $E^*=K^*-K^*$, above relation (3.5) holds for any $f\in E^*$. This implies that the non-decreasing sequence $x\wedge ne$ converges to x weakly. Then, e.g., by lemma 2 in S. Karlin [5], the sequence converges to x strongly for any $x\in K$.

From this lemma we get

THEOREM 3.1. *Let e be a positive element of a Banach lattice E. Then the following four conditions for e are equivalent to each other:*

(i) *e is a quasi-inner element of K.*

(ii) *The smallest closed solid subspace containing e coincides with E.*

(iii) *For any $x\in K$, the non-decreasing sequence $x\wedge ne$ converges to x strongly (weakly).*

(iv) *e is a non-support element of K.*

Proof. Since E_e, the smallest solid subspace containing e, is the smallest subspace containing $[0, e]$ conditions (i) and (ii) are equivalent. By lemma 3.2 (iv) implies (iii), and (iii) clearly implies (ii). Therefore we have only to prove that (ii) implies (iv). This can be proved from the following evident assertion:

If $f\in K^*$ and $f(e)=0$, then $f(x)=0$ for each $x\in E_e$.

Theorem 3.1, combined with proposition 3.3, proves the following well known

COROLLARY 3.1. *An element $e\in K$ is a weak order unit if it is a non-support element of K.*

Since the dual E^* of an arbitrary Banach lattice E is complete, there always exists, for any bounded non-decreasing sequence $f_n\in E^*$, $\bigvee_n f_n$. Concerning this we can prove without difficulty

PROPOSITION 3.4. *Let $f_n\in E^*$ be a bounded non-decreasing sequence. Then f_n converges to $\bigvee_n f_n$ in the w^*-topology.*

Remark 3.2. By this proposition we can see that for a monotone sequence in E^* the order convergence is equivalent to the w^*-convergence.

In the rest of this section we investigate the complexification of a Banach lattice. The *complexification*, denoted by $\tilde{E}$, of a Banach lattice E is a complex Banach space consisting of elements

$$z=x+iy \qquad (x, y\in E)\,.$$

We can define, according to proposition 3.1, the *extended absolute value* by

$$|z|=|x+iy|=\bigvee_{\theta}((\cos\theta)x+(\sin\theta)y)\,. \tag{3.6}$$

We also define as usual

$$\|z\|=\||z|\|\,.$$

Therefore, we have

THEOREM 3.2. *The complexification $\tilde{E}$ of a Banach lattice E (even if it is not σ-complete) has an extended absolute value* (3.6) *with properties:*

$$|z|\leqq|z'| \text{ implies } \|z\|\leqq\|z'\| \qquad (z, z'\in\tilde{E}),$$

$$|x|, |y|\leqq|x+iy|\leqq|x|+|y| \qquad (x, y\in E),$$

$$\|x\|, \|y\|\leqq\|x+iy\|\leqq\|x\|+\|y\| \qquad (x, y\in E),$$

$$|\lambda z+\mu z'|\leqq|\lambda||z|+|\mu||z'| \qquad (z, z'\in\tilde{E} \text{ and } \lambda, \mu \text{ be arbitrary complex numbers}).$$

In particular, if $z, z'\in\tilde{E}$ satisfy the relation

$$|z|\wedge|z'|=0\,,$$

then, for any complex numbers λ and μ, we have

$$|\lambda z+\mu z'|=|\lambda||z|+|\mu||z'|\,. \tag{3.7}$$

Remark 3.3. By this theorem the assumption concerning the extendability of absolute value of theorem 3.4 in H. H. Schaefer [18], p. 275 is not needed.

As we defined E_e, we can define $(\tilde{E})_e$ by the set of element $z\in\tilde{E}$ *bounded w.r.t. e*, i.e., for which there exists a positive number c such that

$$|z|\leqq ce\,.$$

By defining a similar norm as the real case, we can see that $(\tilde{E})_e$ is isometric to the complexification $(\tilde{E_e})$ of E_e, as complex Banach spaces with extended absolute value. Therefore we denote them simply by $\tilde{E}_e$. It is also seen easily that $\tilde{E}_e$ is isometric to $\tilde{C}(\mathfrak{M})$, the set of complex-valued continuous functions defined on $\mathfrak{M}$.

U. Krenger gave in [8]

APPROXIMATION LEMMA. *Let $\tilde{E}$ be the complexification of a σ-complete Banach lattice E. Then, for each $z\in\tilde{E}$ and each positive number ε, there exist $x_l\in E$ and complex numbers λ_l ($l=1,\cdots,n$) such that*

$$x_l\wedge x_m=0 \qquad (l\neq m)$$

and

$$|z-(\lambda_1x_1+\cdots+\lambda_nx_n)|\leqq\varepsilon|z|\,.$$

By the discussions above, we can give another proof of this lemma. Namely, put $|z|=e$ and let E_e be represented by $C(\mathfrak{M})$. By proposition 3.2, the compact space $\mathfrak{M}$ is basically disconnected, i.e., each F_σ open set in $\mathfrak{M}$ has an open closure[4]. From this property the lemma follows easily.

Concerning the complexification and the dualization of a Banach lattice, the following proposition was given in U. Krenger [8], p. 77 under the additional assumption that E is σ-complete.

PROPOSITION 3.5. *Let E be a Banach lattice and E^* be its dual, and let $\tilde{E}$ and $\tilde{E}^*$ be their complexifications respectively. Then $\tilde{E}^*$ can be considered as the complex dual of $\tilde{E}$. In other words there exists a bijective isometry between the complex Banach spaces $\tilde{E}^*$ and $\tilde{E}^{*}$, where $\tilde{E}^{*}$ is the complex dual of $\tilde{E}$.*

To prove the proposition we need the following two lemmas.

LEMMA 3.3. *Let $f_l \in K^*$ $(l=1, \cdots, n)$ satisfy*

$$f_l \wedge f_m = 0 \qquad (l \neq m).$$

Then, for every positive number ε and for every $x \in K$, there exists a decomposition $x_1, \cdots, x_n$ of x such that

$$x = x_1 + \cdots + x_n,$$

$$x_1, \cdots, x_n \in K$$

and

$$f_l(x_m) < \varepsilon \qquad (l \neq m). \tag{3.8}$$

Proof. We prove the lemma by mathematical induction.

(i) $n=2$. Since

$$0 = (f_1 \wedge f_2)(x) = \inf_{\substack{x_1 + x_2 = x \\ x_1,\, x_2 \in K}} (f_1(x_2) + f_2(x_1)),$$

there exist $x_1, x_2 \in K$, such that

$$x_1 + x_2 = x$$

and

$$f_1(x_2) + f_2(x_1) < \varepsilon.$$

These x_1 and x_2 clearly statisfy the desired condition.

(ii) $n-1 \Rightarrow n$. Put

$$f_1 + \cdots + f_{n-1} = g.$$

Then clearly

$$g \wedge f_n = 0.$$

By the discussion given above, there exist $y, x_n \in K$ such that

$$y + x_n = x,$$

$$g(x_n) < \varepsilon \quad \text{and} \quad f_n(y) < \varepsilon.$$

4) See, e.g., L. Gillman and M. Jerison [2].

Then, by the assumption of induction, there exists a decomposition $x_1, \cdots, x_{n-1}$ of y for the functionals $f_1, \cdots, f_{n-1}$. It is clear that $x_1, \cdots, x_{n-1}, x_n$ are the desired ones.

Let E and F be Banach lattices and $\tilde{E}$ and $\tilde{F}$ be their complexifications respectively, and let $T \in \mathfrak{L}(E, F)$. We define the operator $\tilde{T} \in \mathfrak{L}(\tilde{E}, \tilde{F})$ by

$$\tilde{T}(x+iy) = Tx + iTy \qquad (x, y \in E).$$

In particular, for $f \in E^*$, we define the functional $\tilde{f} \in \tilde{E}^*$ by

$$\tilde{f}(x+iy) = f(x) + if(y) \qquad (x, y \in E).$$

Under these notations we have

LEMMA 3.4. *Let $T \in \mathfrak{L}(E, F)$ be positive, then*

$$|\tilde{T}z| \leqq T|z| \qquad (z \in \tilde{E}).$$

In particular, for $f \in K^*$, we have

$$|\tilde{f}(z)| \leqq f(|z|) \qquad (z \in \tilde{E}).$$

Proof. By the definition of extended absolute value and the positivities of T and f, we can prove the lemma without difficulty.

Proof of proposition 3.5. For each

$$h = f + ig \in \tilde{E}^* \qquad (f, g \in E^*),$$

define as usual

$$\tilde{h} = \tilde{f} + i\tilde{g} \in \tilde{E}^*.$$

In other words, for each

$$z = x + iy \in \tilde{E} \qquad (x, y \in E),$$

$$\tilde{h}(z) = f(x) - g(y) + i(f(y) + g(x)).$$

Therefore we have

$$\|\tilde{h}\| \leqq 4\|h\|.$$

Then it is easy to see that this correspondence

$$h \longleftrightarrow \tilde{h}$$

is a bijective linear topological mapping between the complex Banach spaces $\tilde{E}^*$ and $\tilde{E}^*$. We have only to show that this mapping is an isometry, i. e.,

$$\|h\| = \|\tilde{h}\| \qquad (h \in \tilde{E}^*). \tag{3.9}$$

First we prove this for $f \in K^*$. Since $\tilde{f}$ is an extension of f, it is clear that

$$\|f\| \leqq \|\tilde{f}\|.$$

On the other hand, by lemma 3.4, we get

$$|\tilde{f}(z)| \leqq f(|z|) \leqq \|f\| \, \||z|\| = \|f\| \, \|z\| .$$

This holds for any $z \in \tilde{E}$, therefore we have

$$\|\tilde{f}\| \leqq \|f\| .$$

Next we show (3.9) for

$$h = \lambda_1 f_1 + \cdots + \lambda_n f_n ,$$

where $f_1, \cdots, f_n \in K^*$ satisfy

$$f_l \wedge f_m = 0 \qquad (l \neq m)$$

and $\lambda_1, \cdots, \lambda_n$ are complex numbers of absolute value 1[5]. By (3.7) in theorem 3.2, we get

$$|h| = f_1 + \cdots + f_n . \tag{3.10}$$

By lemma 3.4 and (3.10) we get, for each $z \in \tilde{E}$,

$$\begin{aligned} |\tilde{h}(z)| &= |\lambda_1 \tilde{f}_1(z) + \cdots + \lambda_n \tilde{f}_n(z)| \\ &\leqq |\lambda_1 \tilde{f}_1(z)| + \cdots + |\lambda_n \tilde{f}_n(z)| \\ &\leqq f_1(|z|) + \cdots + f_n(|z|) \\ &= (f_1 + \cdots + f_n)(|z|) \\ &= |h|(|z|) . \end{aligned}$$

From this we get

$$\|\tilde{h}\| \leqq \|h\| .$$

We then prove the converse inequality. For any positive number ε, there exists $x \in E$ such that

$$\|x\| = 1$$

and

$$\|h\| = \||h|\| < ||h|(x)| + \varepsilon .$$

Therefore we have

$$\|h\| < |h|(|x|) + \varepsilon . \tag{3.11}$$

For given ε and $f_1, \cdots, f_n$, establish the decomposition $x_1, \cdots, x_n$ of $|x|$ obtained in lemma 3.3, and define $z \in \tilde{E}$ by

$$z = \lambda_1^{-1} x_1 + \cdots + \lambda_n^{-1} x_n . \tag{3.12}$$

Then we have

$$|z| \leqq x_1 + \cdots + x_n = |x| .$$

Consequently

$$\|z\| \leqq \|x\| = 1 . \tag{3.13}$$

By (3.10), (3.12), (3.8) and a simple calculation, we get

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5) It can be seen easily that this restriction is not essential.

$$||h|(|x|)-\tilde{h}(z)|<2n(n-1)\varepsilon . \tag{3.14}$$

By (3.11) and (3.14) we get

$$\|h\|<|\tilde{h}(z)|+(2n^2-2n+1)\varepsilon . \tag{3.15}$$

Since ε is an arbitrary positive number, (3.15) and (3.13) show

$$\|h\|\leqq\|\tilde{h}\| .$$

Therefore (3.9) is proved in this case. Finally we prove the general case. Since E^* is complete, we can apply approximation lemma to E^*. For each $h\in\tilde{E}^*$ and for each positive number ε there exists $h'\in\tilde{E}^*$ of the preceding case such that

$$\|h-h'\|<\varepsilon$$

and

$$\|h'\|=\|\tilde{h}'\| .$$

Since the mapping $h\longleftrightarrow\tilde{h}$ is topological, we get conclusion (3.9).

Hereafter we shall omit the symbol $\sim$ from $\tilde{E}$, $\tilde{T}$ and $\tilde{f}$ if there arises no confusion.

4. Irreducible operaters

In the following part of this paper except for the last section we assume that the dimension of E is at least two[6]. First we establish the following

PROPOSITION 4.1. *For a positive operator $T\in\mathfrak{L}(E)$ the following three conditions are equivalent to each other:*

(i) *T is irreducible.*

(ii) *T is quasi-inner.*

(iii) *T is semi-non-support.*

Proof. The equivalence of (i) and (ii) is shown in H. H. Schaefer [18], p. 269, and that of (ii) and (iii) is clear by theorem 3.1.

Remark 4.1. It goes without saying that if E is the space L_p with σ-finite measure then T is irreducible if and only if it is indecomposable in the sense of [10].

Remark 4.2. If E is one-dimensional, then the proposition fails to hold. Indeed the zero operator is irreducible, but it is neither quasi-inner nor semi-non-support.

By theorems 1 and 2 in [13], we get

6) For this restriction, see remark 4.2 below.

PROPOSITION 4.2. *Let $T\in\mathfrak{L}(E)$ be positive and its resolvent $R(\lambda, T)$ has a pole at $\lambda=r(T)$. Then T is irreducible if and only if the following three assertions hold:*

1) *The eigenspace of T for $r(T)$ is one-dimensional.*
2) *The eigenspace of T for $r(T)$ contains a non-support element of K.*
3) *The eigenspace of T^* for $r(T)$ contains a strictly positive linear functional.*

PROPOSITION 4.3. *Let $T\in\mathfrak{L}(E)$ be positive and its resolvent has a pole at $\lambda=r(T)$. Further, if T is irreducible, then we have*

(i) $r(T)>0$.
(ii) *$\lambda=1$ is a simple pole of $R(\lambda, T)$.*
(iii) *The eigenspace of T^* for 1 is one-dimensional.*

We can extend these propositions partially to the following propositions 4.4 and 4.5, the proofs of which will be obtained by simply modifying the one of proposition 3.2 in H. H. Schaefer [18] p. 270.

PROPOSITION 4.4. *Let $T\in\mathfrak{L}(E)$ be a positive irreducible operator. Suppose that there exists a non-zero positive functional f of K^* satisfying*

$$T^*f\leqq r(T)f$$

and further that there exist a non-zero element x of E and a complex number λ_0 satisfying

$$Tx=\lambda_0 x \quad \text{and} \quad |\lambda_0|=r(T)\,.$$

Then the assertions 1), 2) and 3) of proposition 4.2 hold.

PROPOSITION 4.5. *Under the assumption of proposition 4.4, $r(T)$ must be positive.*

5. The point spectrum of T on the spectral circle

In this section we prove assertion A) of our main theorem, namely,

THEOREM 5.1. *Under the assumption of the main theorem, the point spectrum of T on Γ coincides with the set of k-th roots of unity multiplied by $r(T)$ each of which is a simple pole of $R(\lambda, T)$.*

The proof of this theorem is obtained from theorem 5.2 below.

Throughout this paper except in the last section we assume that T satisfies assumptions I), II) and III) in the main theorem. Then by proposition 4.3, we can assume $r(T)=1$. Proposition 4.2 also assures the unique existence of a non-support element $e\in K$ and a strictly positive linear functional $f_0\in K^*$ such that

$$\|e\|=1$$

and

$$f_0(e)=1.$$

Let P be the projection corresponding to the eigenvalue 1 and let

$$Q=I-P.$$

Then P is a positive projection and Q is a projection such that

$$Px=f_0(x)e \qquad (x\in E),$$

$$TP=PT=P$$

and

$$(I-T)Q=Q(I-T)=I-T.$$

It can be shown that the restriction of $\lambda I-T(\lambda>1)$ to QE has a bounded inverse and the operator norm of them is uniformly bounded in $\{\lambda;\ \lambda>1\}$. We denote this bound by b. All these assumptions and notations are used in sections 5, 6, 7 and 8.

We can show easily that T leaves E_e invariant. Since E_e is represented by $C(\mathfrak{M})$, we can treat, in stead of $E, C(\mathfrak{M})$ where theorem 3.3 in H. H. Schaefer [18] may be applied. However, the restriction of T on E_e is not known to satisfy the assumption of his theorem. Nevertheless we can reformulate his theorem into the following lemma 5.1(I) which is applicable for our purpose. The proof of lemma 5.1(I) will be obtained by checking the one of H. H. Schaefer's theorem 3.3, and that of lemma 5.1(II) is easy.

LEMMA 5.1. *Let, for a positive operator $U\in\mathfrak{L}(C(S))$ with $r(U)=1$, there exist a strictly positive linear functional f satisfying*

$$U^*f\leqq f.$$

(I) *If there exist $x_0\in C(S)$ and a complex number λ_0 such that*

$$Ux_0=\lambda_0 x_0, \quad |\lambda_0|=1 \quad \textit{and} \quad |x_0|=1.$$

then the operators S and S^{-1} defined by

$$Sx(t)=x_0(t)x(t) \qquad (t\in S,\ x\in C(S))$$

and

$$S^{-1}x(t)=x_0(t)^{-1}x(t) \qquad (t\in S,\ x\in C(S))$$

have the following properties:

$$SS^{-1}=S^{-1}S=I, \quad S|x_0|=x_0, \quad |Sx|=|S^{-1}x|=|x| \qquad (x\in C(S))$$

and

$$U=\lambda_0^{-1}S^{-1}US.$$

(II)[7] *If* λ_0 *is an eigenvalue of* U *and a n-th root of unity and also if the eigenspace of* U *for* 1 *is a one-dimensional subspace containing the element* $\mathbf{1}\in C(S)$, *then we have*

(i) *There exists an element* $x_0\in C(S)$ *satisfying*

$$Ux_0=\lambda_0 x_0\,, \quad |x_0|=x_0^k=\mathbf{1} \quad \textit{for} \quad k=\min\{m\,;\,\lambda_0^m=1,\, m\geqq 1\}\,.$$

(ii) *Let* S_j *be the subset* $\{t\,;\,x_0(t)=\lambda_0^{j-1}\}$ *of* S *and* y_j *be its characteristic function for each* $j=1,\cdots,k$. *Then* $y_1, y_2, \cdots, y_k$ *have the properties:*

$$y_j\in C(S) \qquad (j=1,2,\cdots,k)\,,$$

$$y_i\wedge y_j=0 \qquad (i\neq j)\,,$$

$$x_0=y_1+\lambda_0 y_2+\cdots+\lambda_0^{k-1}y_k\,,$$

$$|x_0|=y_1+y_2+\cdots+y_k$$

and

$$Uy_j=y_{j-1}{}^{8)} \qquad (j=1,2,\cdots,k)\,.$$

One of the direct consequences of this lemma is the following

THEOREM 5.2. *Let* $U\in\mathfrak{L}(E)$ *be a positive operator with* $r(U)=r$ *and satisfy conditions* 1), 2) *and* 3) *in proposition* 4.2 *(*T *being replaced by* U*). Then, for each eigenvalue* $\lambda_0 r$ *of* U *on* Γ *and its eigenvector* x_0, *there exist operators* $D, D^{-1}\in\mathfrak{L}(E)$ *satisfying*

$$DD^{-1}=D^{-1}D=I\,, \quad D|x_0|=x_0\,, \quad |Dx|=|D^{-1}x|=|x| \qquad (x\in E)$$

and

$$U=\lambda_0^{-1}D^{-1}UD\,.$$

Proof. Using conditions 2), 1) and the assumption that $\dim E\geqq 2$, we can show easily that $r(U)>0$. Therefore we assume $r(U)=1$. Let f be a strictly positive eigenfunctional of U^* for 1. It can be shown, as usual, that

$$U|x_0|=|x_0|\,. \tag{5.1}$$

By assumption this proves $|x_0|$ to be a non-support element of K. Put

$$F=E_{|x_0|}\,.$$

Then F is a Banach lattice with the norm $\|\cdot\|_{|x_0|}$ and hence it is represented by $C(\mathfrak{M})$ as in section 3. From (5.1) F is invariant under U. Moreover we can show that the operator $\hat{U}$ defined by $\hat{U}\hat{x}=\widehat{Ux}$[9] and the functional $\hat{f}$ defined by $\hat{f}(\hat{x})=f(x)$ satisfy the condition of lemma 5.1(I). Therefore, by this lemma, there exist operators $S, S^{-1}\in\mathfrak{L}(F)$ such that

7) This part of the lemma is needed in the next section.

8) For the case $j=1$, this must be understood as $Ty_1=y_k$.

9) $\hat{x}$ is the element of $C(\mathfrak{M})$ which corresponds to the element $x\in F$.

$$SS^{-1}=S^{-1}S=I, \quad S|x_0|=x_0, \quad |Sx|=|S^{-1}x|=|x| \qquad (x\in F)$$

and

$$Ux=\lambda_0^{-1}S^{-1}USx \qquad (x\in F).$$

It can be shown that S and S^{-1} have operator norm 1 w.r.t. the initial norm of E. Since F is dense in E, the unique extension of S and S^{-1}, denoted by D and D^{-1} respectively, satisfy the desired conditions.

From this theorem we can show easily that the assumption of this theorem imply, among others, assertions (ii), ⋯, (v) of theorem 3.3 in H. H. Schaefer [18]. For example the following corollary corresponds to (iv) there.

COROLLARY 5.1. *Suppose that U satisfies the assumption of theorem* 5.2 *and further that the point spectrum on the spectral circle contains an isolated point of this set. Then it coincides with $r(U)H$ where H is the set of k-th roots of unity for some $k\geqq 1$.*

Remark 5.1. If $U\in\mathfrak{L}(E)$ satisfies the assumption of proposition 4.4, then the conclusion of theorem 5.2 holds.

6. The voidness of the residual spectrum of T on the spectral circle

In this section we prove assertion B) of the main theorem, namely,

THEOREM 6.1. *Under the assumption of the main theorem, the residual spectrum is void on Γ.*

The proof of this theorem is, after a sequence of propositions, given at the end of this section.

We begin with the following

DEFINITION 6.1. *Let g be a non-zero positive linear functional and $P_g\in\mathfrak{L}(E^*)$ be the natural extension of the operator defined by*

$$P_g f=\bigvee_n (f\wedge ng) \qquad (\text{for each } f\in K^*).$$

By proposition 3.4 we can show that P_g is a lattice homomorphic projection satisfying

$$P_g f=w^*\text{-}\lim_n (f\wedge ng) \qquad (f\in K^*),$$

$$0\leqq P_g\leqq I, \quad \|P_g\|=1 \tag{6.1}$$

and

$$E^*_g\subset P_g E^*,$$

where E^*_g is the set of elements of E^* bounded w.r.t. g. Concerning this projection we have

LEMMA 6.1. (i) *If f_n is a bounded non-decreasing sequence of K^*, then*

$$P_g(w^*\text{-}\lim_n f_n)=P_g(\bigvee_n f_n)=\bigvee_n P_g f_n=w^*\text{-}\lim_n P_g f_n\,.$$

(ii) *If $T^*g=g$ then $T^*P_g=P_gT^*P_g$.*

Proof. (i) This is clear by proposition 3.4 and the following lattice equality:

$$\bigvee_m((\bigvee_n f_n)\wedge mg)=\bigvee_m\bigvee_n(f_n\wedge mg)=\bigvee_n\bigvee_m(f_n\wedge mg)=\bigvee_n P_g f_n\,.$$

(ii) From (6.1) and $T^*\geqq 0$ we have

$$T^*P_g\geqq P_gT^*P_g\,.$$

On the other hand, since T^* is w^*-continuous, it follows that

$$\begin{aligned}T^*P_gf=T^*(w^*\text{-}\lim_n(f\wedge ng))&=w^*\text{-}\lim_n T^*(f\wedge ng)\\ &\leqq w^*\text{-}\lim_n(T^*f\wedge ng)\\ &=P_gT^*f\qquad(f\in K^*)\end{aligned}$$

which implies

$$T^*P_g\leqq P_gT^*P_g\,.$$

Therefore

$$T^*P_g=P_gT^*P_g\,.$$

We write simply

$$P_{f_0}=P_0$$

and also

$$I-P_0=Q_0\,.$$

Then clearly

$$0\leqq Q_0\leqq I,\quad \|Q_0\|=1 \text{ or } 0,\quad \text{and}\quad P_0Q_0=0\,.$$

By lemma 6.1

$$T^*P_0=P_0T^*P_0 \tag{6.2}$$

and consequently

$$Q_0T^*=Q_0T^*Q_0\,. \tag{6.3}$$

By (6.2) P_0E^* is invariant under T^*. However it must be remarked that Q_0E^* is not necessarily invariant under T^*. We denote the restriction of T^* to P_0E^* by T^*_1 and the one of Q_0T^* to Q_0E^* by T^*_2.

Let us investigate the relation between the spectrum of T^* and T^*_1. In the case of $E=C(S)$, there has already been the reduction theory established recently by one of the authors (lemmas 1, 2, 3, 4 and propositions 1, 2 and 3 in [14]). This reduction theory may be extended without any significant modifications to our case where E is an arbitrary Banach lattice[10]. Therefore, we have

PROPOSITION 6.1. (i) *The spectral radius of* T^*_1 *is also* 1, *i.e.*,

$$r(T^*_1)=1\,.$$

(ii) *On the spectral circle* Γ, *the point spectrum of* T^*_1 *coincides with that of* T^*_1, *i.e.*,

$$P_\sigma(T^*)\cap\Gamma=P_\sigma(T^*_1)\cap\Gamma\,.$$

(iii) *On* Γ, *the spectrum of* T^* *coincides with that of* T^*_1, *i.e.*,

$$\sigma(T^*)\cap\Gamma=\sigma(T^*_1)\cap\Gamma\,.$$

(iv) *On* Γ, $\lambda=\lambda_0$ *is a pole of the resolvent* $R(\lambda, T)$ *if and only if it is a pole of* $R(\lambda, T^*_1)$.

In the case of $E=C(S)$, $E^*_{f_0}$ is dense in P_0E^* strongly. But in the present case this does not hold generally. Indeed, it is known that $E^*_{f_0}$ is only w^*-dense in P_0E^*. The following discussions are necessary to overcome this difficulty.

Let φ be the functional on E^* defined by $\varphi(f)=f(e)$, in other words, φ is the element of E^{**} which corresponds to the element e of E. Then φ satisfies

$$T^{**}\varphi=\varphi\,. \tag{6.4}$$

Since

$$T^*f_0=f_0\,, \tag{6.5}$$

T^* leaves the (AM) space $E^*_{f_0}$ invariant. We denote the restriction of T^* to $E^*_{f_0}$ by T^*_0. Under these notations we get

PROPOSITION 6.2. (i) T^*_0 *is a positive operator of* $\mathfrak{L}(E^*_{f_0})$ *with spectral radius* 1.

(ii) *The restriction of* φ *to* $E^*_{f_0}$, *denoted by* φ_0, *is a strictly positive functional of* $(E^*_{f_0})^*$ *and satisfies*

$$(T^*_0)^*\varphi_0=\varphi_0\,.$$

(iii) *The eigenspace of* T^*_0 *is one-dimensional containing* f_0 *which is a non-support element of the positive cone of* $E^*_{f_0}$.

(iv) $\lambda_0(\in\Gamma)$ *is an eigenvalue of* T^* *if and only if it is an eigenvalue of* T^*_0, *and then the corresponding eigenspaces are identical with the other.*

---

10) Indeed, lemmas 1, 2 and 3 in [14] hold even if we replace T^* and T^*_1 by T and T_1 respectively, under the assumption that E is an arbitrary Banach space not necessarily assigned to order relation and T is an arbitrary bounded operator of $\mathfrak{L}(E)$ and P is an arbitrary projection of $\mathfrak{L}(E)$ such that $TP=PTP$.

Proof. (i) is clear. Indeed, we can prove from (6.5) $\|T^*\|_{f_0}=1$.

Since e is a non-support element of K, φ is strictly positive on K^* satisfying (6.4). This proves (ii). Also this, combined with (iii) of proposition 4.3, proves (iv) as in the proof of proposition 1 in [14]. (iii) is a direct consequence of (iv).

By (ii) and (iii) in proposition 6.2 the operator T^*_0 satisfies conditions 1), 2) and 3) in proposition 4.2.

By (iv) in proposition 6.2 it is shown that 1 is an isolated point of $P_\sigma(T^*_0)\cap\Gamma$. Therefore, we can apply corollary 5.1 to the operator T^*_0. Thus using (iv) in proposition 6.2 again, we get

PROPOSITION 6.3. *Let λ_0 be an eigenvalue of T^* on Γ. Then λ_0 is a k-th root of unity for some positive integer k.*

LEMMA 6.2. *Let g_1 and g_2 be positive functionals of E^*. Then the following conditions are equivalent to each other.*

(i) $g_1\wedge g_2=0$.

(ii) $P_{g_1}(f_1)\wedge P_{g_2}(f_2)=0$ $(f_1,f_2\in K^*)$.

(iii) $P_{g_1}\cdot P_{g_2}=0$.

(iv) $P_{g_1}+P_{g_2}=P_{g_1+g_2}$.

From these we get

PROPOSITION 6.4. *Let $\lambda_0\neq 1$ be an eigenvalue of T^* on Γ. Then there exist operators $D, D^{-1}\in\mathfrak{L}(P_0E^*)$ satisfying*

$$DD^{-1}=D^{-1}D=I\,, \tag{6.6}$$

$$|Df|=|f| \qquad (f\in P_0E^*) \tag{6.7}$$

and

$$T^*_1=\lambda_0^{-1}D^{-1}T^*_1D\,. \tag{6.8}$$

Proof. By proposition 6.3, λ_0 is a k-th root of unity. We assume k to be the smallest positive integer satisfying this property. Then λ_0 is a primitive k-th root of unity and $k>1$. We represent $E^*_{f_0}$ by $C(\mathfrak{M})$ and consider the operator $(\widehat{T^*_0})$ belonging to $\mathfrak{L}(C(\mathfrak{M}))$. By propositions 6.2 and 6.3 and $\hat{f}_0=1$, the assumptions of lemma 5.1(II) are all satisfied by $(\widehat{T^*_0})$. Therefore, coming back to the original space, there exist elements $h, g_1, \cdots, g_k\in E^*_{f_0}$ such that

$$T^*h=\lambda_0 h\,,$$

$$|h|=f_0\,,$$

$$g_i\wedge g_j=0 \qquad (i\neq j)\,,$$

$$f_0=g_1+g_2+\cdots+g_k\,,$$

$$h=g_1+\lambda_0 g_2+\cdots \lambda_0^{k-1}g_k$$

and

$$T^*_0=\lambda_0^{-1}S^{-1}T^*_0 S \tag{6.9}$$

where S and S^{-1} are defined respectively by

$$\widehat{Sf}=\hat{h}\hat{f} \quad \text{and} \quad \widehat{S^{-1}f}=\frac{\hat{f}}{\hat{h}} \qquad (f\in E^*_{f_0}).$$

Making use of these elements $g_1, \cdots, g_k$, we extend the operators $S, S^{-1}\in L(E^*_{f_0})$ to operators $D, D^{-1}\in \mathfrak{L}(P_0E^*)$ as follows:

$$Df=(P_{g_1}+\lambda_0 P_{g_2}+\cdots+\lambda_0^{k-1}P_{g_k})f \qquad (f\in P_0E^*)$$

and

$$D^{-1}f=(P_{g_1}+\lambda_0^{-1}P_{g_2}+\cdots+\lambda_0^{-(k-1)}P_{g_k})f \qquad (f\in P_0E^*).$$

We must prove that these operators D, D^{-1} are the desired ones. By lemma 6.4, we have

$$P_0=P_{g_1}+P_{g_2}+\cdots+P_{g_k}$$

and

$$P_{g_i}\cdot P_{g_j}=0 \qquad (i\neq j).$$

By these equalities, we can prove

$$DD^{-1}=D^{-1}D=I.$$

Since $E^*_{f_0}$ is lattice isomorphic to $C(\mathfrak{M})$, we have

$$\hat{P}_{g_j}\hat{f}=\widehat{P_{g_j}f}=\bigvee_n(\hat{f}\wedge\widehat{ng_j})=\hat{g}_j\hat{f} \qquad (f\in E^*_{f_0}\cap K^*).$$

Consequently

$$\widehat{Sf}=\hat{h}\hat{f}=(\hat{g}_1+\lambda_0\hat{g}_2+\cdots+\lambda_0^{k-1}\hat{g}_k)\hat{f}$$

$$=(\hat{P}_{g_1}+\lambda_0\hat{P}_{g_2}+\cdots+\lambda_0^{k-1}\hat{P}_{g_k})\hat{f} \qquad (f\in E^*_{f_0}).$$

From this we get

$$Sf=(P_{g_1}+\lambda_0 P_{g_2}+\cdots+\lambda_0^{k-1}P_{g_k})f \qquad (f\in E^*_{f_0}).$$

This proves that D is an extension of S to the space P_0E^*. Let f be any positive element of P_0E^*. Then $f\wedge nf_0$ is a non-decreasing sequence belonging to $E^*_{f_0}$ which converges to f in the sense of $\sigma(E^*, E)$. Since D is an extension of S, we get from (6.9)

$$\lambda_0 DT^*_0(f\wedge nf_0)=T^*_0 D(f\wedge nf_0). \tag{6.10}$$

By lemma 6.1(i) it can be seen that, for every bounded non-decreasing sequence f_n of K^*, Df_n converges to $D(\bigvee f_n)$ in the w^*-topology. This property, combined with (6.10) and the fact that T^* is positive and w^*-continuous, shows

$$\lambda_0 DT^*f=T^*Df \qquad (f\in P_0E^*\cap K^*).$$

Since this equality, as can easily be shown, holds for any $f \in P_0E^*$, we get $\lambda_0 DT^*_1 = T^*_1 D$. Therefore, using (6.6), we get (6.8). Finally we have

$$\begin{aligned}|Df| &= |P_{g_1}f + \lambda_0 P_{g_2}f + \cdots + \lambda_0^{k-1} P_{g_k} f| \\ &\leqq |P_{g_1}f| + |P_{g_2}f| + \cdots + |P_{g_k}f| \\ &\leqq P_{g_1}|f| + P_{g_2}|f| + \cdots + P_{g_k}|f| \\ &= P_0|f| = |f| \qquad (f \in P_0E^*).\end{aligned}$$

Similarly D^{-1} is an extension of S^{-1} and satisfies

$$|D^{-1}f| \leqq |f| \qquad (f \in P_0E^*).$$

Therefore

$$|f| \leqq |D^{-1}Df| \leqq |Df| \leqq |f| .$$

This proves (6.7) and the proof is completed.

Proof of theorem 6.1. To prove the theorem we suppose the contrary, i. e., there exists a complex number λ_0 which belongs to $R_\sigma(T) \cap \Gamma$. Then clearly

$$\lambda_0 \in P_\sigma(T^*) \cap \Gamma \quad \text{and} \quad \lambda_0 \neq 1 .$$

Therefore, by proposition 6.4, there exist operators $D, D^{-1} \in \mathfrak{L}(P_0E^*)$ satisfying (6.6), (6.7) and (6.8). By proposition 6.1, $\lambda = 1$ is a pole of $R(\lambda, T^*_1)$. This fact, combined with formula (6.8), shows that $\lambda = \lambda_0$ is a pole of $R(\lambda, T^*_1)$. Again by proposition 6.1, $\lambda = \lambda_0$ is a pole of $R(\lambda, T^*)$. Therefore, by considering the facts $\sigma(T) = \sigma(T^*)$ and $R(\lambda, T)^* = R(\lambda, T^*)$, we can see that $\lambda = \lambda_0$ is a pole of $R(\lambda, T)$. Hence we have

$$\lambda_0 \in P_\sigma(T) \cap \Gamma .$$

This is a contradiction and the theorem is proved.

By theorem 5.1 and the above proof we get

COROLLARY 6.1. *Under the assumption of the theorem the point spectrum of T on Γ coincides with that of T^* on Γ, i. e.,*

$$P_\sigma(T) \cap \Gamma = P_\sigma(T^*) \cap \Gamma .$$

7. The space L and the extension of T to L

For every $x \in E$, define a new norm

$$\|x\|_L = f_0(|x|) .$$

Then this norm $\|\cdot\|_L$ makes E a normed lattice. Since

$$\|x\|_L \leqq \|f_0\| \, \|x\| , \tag{7.1}$$

the topology defined by this norm $\|\cdot\|_L$ is weaker than that defined by $\|\cdot\|$. The completion of E under this new norm is denoted by L, an element x of which consists of mutually equivalent fundamental sequences $\{x_n\}$ in the norm $\|\cdot\|_L$. Let us write conveniently

$$x=\{x_n\} .$$

If there exist fundamental sequences $\{x_n\}$ and $\{y_n\}$ which represent x and y respectively such that

$$x_n \geqq y_n \qquad (n=1, 2, \cdots)$$

then we define

$$x \geqq y .$$

Under these definitions we get

PROPOSITION 7.1. *L is a Banach lattice under the norm and order defined above, and the new order is an extension of the old one defined in E, in other words, for $x, y \in E$,*

$x \geqq y$ in E if and only if $x \geqq y$ in L.

Proof. We only check an essential point of the proof which seems to be obvious. First, for $x_n, y_n \in E$, the following relations are well known:

$$|x_n \vee y_n - x_m \vee y_m| \leqq |x_n - x_m| + |y_n - y_m| \tag{7.2}$$

and

$$|x_n \vee y_n - x_n| \leqq |x_n - y_n| . \tag{7.3}$$

Then, remembering $f_0 \in K^*$, we get from them

$$\|x_n \vee y_n - x_m \vee y_m\|_L \leqq \|x_n - x_m\|_L + \|y_n - y_m\|_L . \tag{7.2'}$$

and

$$\|x_n \vee y_n - x_n\|_L \leqq \|x_n - y_n\|_L . \tag{7.3'}$$

From (7.3′) we conclude the following assertion:

(*) If $\{x_n\}$ and $\{y_n\}$ are mutually equivalent fundamental sequences, then $\{x_n \vee y_n\}$ and $\{x_n \wedge y_n\}$ are both fundamental sequences which are equivalent to the given ones.

Using (*) and (7.2′), we can show that L is a lattice, indeed, for $x=\{x_n\}$ and $y=\{y_n\}$, $\{x_n \vee y_n\}$ is a fundamental sequence representing $x \vee y$.

It is clear that f_0 can be extended uniquely to L, denoted by the same letter f_0. Then f_0 is a strictly positive linear functional belonging to L^* with norm 1 and satisfies the relation

$$f_0(x) = \|x_+\|_L - \|x_-\|_L .$$

and

$$f_0(|x|) = \|x\|_L .$$

By (7.1), we have

$$L^* \subset E^*, \tag{7.4}$$

that is, for every element $f \in L^*$, the restriction of f to E is an element of E^*. More precisely, it can be shown that L^* is isomorphic as a Banach lattice to $E^*_{f_0}$. Denote the positive cone of L by K_L. Then, by (7.4), we get

PROPOSITION 7.2. *e is a non-support element of K_L.*

Since a non-support element is a weak order unit, L is, by Kakutani [3], isomorphic as a Banach lattice to a concrete L_1 space on a compact Hausdorff space with finite measure. It can be seen that the operator norms of T and $R(\lambda, T)$ $(\lambda > 1)$ w.r.t. the norm $\|\cdot\|_L$ are 1 and $\frac{1}{\lambda - 1}$ respectively. Therefore they can be extended uniquely to operators of $\mathfrak{L}(L)$ without changing their norms. Let us denote them by T_L and $R_L(\lambda, T)$ respectively. Then clearly

$$R_L(\lambda, T) = R(\lambda, T_L) \qquad (\lambda > 1).$$

Further we denote by P_L and Q_L the extensions to L of the projections P and Q defined in section 5. Then these extensions satisfy the relations similar to the old ones, e. g.,

$$P_L(x) = f_0(x)e \qquad (x \in L).$$

From now on we investigate the properties of T_L. In the first place we have

PROPOSITION 7.3. *T_L is a positive operator of $\mathfrak{L}(L)$ and the eigenspace of T_L (resp. T_L^*) for 1 is one-dimensional and contains the non-support element e (resp. the strictly positive functional f_0).*[♯)]

Proof. It is sufficient to prove that the eigenspaces are both one-dimensional. The eigenspace of T_L^* is one-dimensional by (7.4). To show this for T_L, assume the contrary, i. e., the eigenspace of T_L be not one-dimensional. Then there exists $x \in L$ such that

$$T_L x = x \quad \text{and} \quad Q_L x = y \neq 0.$$

From these relations we get

$$T_L y = y. \tag{7.5}$$

We can assume here

$$\|y\|_L = 1. \tag{7.6}$$

Put

$$T_{L,n} = \frac{I + T_L + \cdots + T_L^{n-1}}{n}$$

and

$$T_n = \frac{I + T + \cdots + T^{n-1}}{n}.$$

Then $T_{L,n}$ coincides in E with T_n, and

♯) Added in proof: Indded, it can be shown that T_L is irreducible.

$$\|T_{L,n}\|_L=1 .$$

Since E is dense in L in the sense of norm $\|\cdot\|_L$, there exists $z\in E$ such that

$$\|y-z\|_L<\frac{1}{2\|Q_L\|_L} .$$

Therefore we have

$$\|T_{L,n}Q_Ly-T_{L,n}Q_Lz\|_L<\frac{1}{2} ,$$

consequently

$$\|T_{L,n}y-T_nQz\|_L<\frac{1}{2} .$$

Then, by (7.5),

$$\|y-T_nQz\|_L<\frac{1}{2} . \tag{7.7}$$

By theorem 5 in S. Karlin [5],

$$\|T_nQz\|\to 0 \qquad (n\to\infty) .$$

Therefore

$$\|T_nQz\|_L\to 0 \qquad (n\to\infty) .$$

This contradicts (7.6) and (7.7), and the proof is completed.

LEMMA 7.1. *For $x\in L$, $|x|\leqq e$ and $\lambda>1$, holds the relation*

$$\|R(\lambda, T_L)Q_Lx\|_L\leqq\|R(\lambda, T)\|_{QE}\|Q\|\ \|f_0\|$$

where $\|\cdot\|_{QE}$ is the operator norm of $R(\lambda, T)$ restricted on QE in the sense of the old norm $\|\cdot\|$.

Proof. Since E is dense in L, there exist $x_n\in E$ $(n=1, 2, \cdots)$ satisfying

$$\|x-x_n\|_L<\frac{1}{n} . \tag{7.8}$$

Here we can assume

$$|x_n|\leqq e . \tag{7.9}$$

For, if this is not the case, then, for x_n satisfying (7.8), the sequence

$$(x_n\wedge e)\vee(-e)$$

satisfies (7.8) and (7.9). From (7.8) we get

$$\|R_L(\lambda)(Q_Lx-Qx_n)\|_L\leqq\frac{\|Q_L\|_L}{n(\lambda-1)} .$$

Consequently

$$\|R_L(\lambda)Q_Lx\|_L\leqq\|R(\lambda)Qx_n\|_L+\frac{\|Q_L\|_L}{n(\lambda-1)} .$$

Let $n\to\infty$. Then, using (7.9), we get

$$\|R_L(\lambda)Q_Lx\|_L \leqq \varliminf_{n\to\infty} \|R(\lambda)Qx_n\|_L \leqq \varliminf_{n\to\infty} \|R(\lambda)Qx_n\| \, \|f_0\|$$

$$\leqq \varliminf_{n\to\infty} \|R(\lambda)\|_{QE}\|Q\| \, \|x_n\| \, \|f_0\|$$

$$\leqq \|R(\lambda)\|_{QE}\|Q\| \, \|f_0\| \, .$$

We do not know if the value $\lambda=1$ is a pole of $R(\lambda, T_L)$, however the preceding lemma shows a weaker result, namely,

PROPOSITION 7.4. *For every positive number ε, there exists a positive number η such that*

$$x, u \in L\,, \quad |x| \leqq e\,, \quad \|u\|_L < \eta$$

and

$$T_Lx - x \geqq u$$

imply

$$\|Q_Lx\|_L < \varepsilon\,.$$

Proof. If the proposition is false, then there exist x_n and u_n $(n=1, 2, \cdots)$ in L and a positive number ε such that

$$|x_n| \leqq e\,, \qquad \|u_n\|_L < \frac{1}{n} \tag{7.10}$$

$$\|Qx_n\|_L \geqq \varepsilon \tag{7.11}$$

and

$$T_Lx_n - x_n \geqq u_n\,. \tag{7.12}$$

Put

$$Q_Lx_n = z_n\,,$$

then

$$T_Lz_n - z_n \geqq u_n \tag{7.13}$$

and

$$z_n \in Q_LL\,.$$

For any λ $(\lambda>1)$, we have from (7.13)

$$(\lambda I - T_L)z_n \leqq (\lambda-1)z_n + |u_n|\,.$$

Since $R(\lambda, T_L)$ is positive for $\lambda>1$, we have

$$z_n \leqq (\lambda-1)R(\lambda, T_L)z_n + R(\lambda, T_L)|u_n|\,,$$

$$z_{n+} \leqq (\lambda-1)(R(\lambda, T_L)z_n)_+ + R(\lambda, T_L)|u_n|\,,$$

$$\|z_{n+}\|_L \leqq (\lambda-1)\|(R(\lambda, T_L)z_n)_+\|_L + \|R(\lambda, T_L)|u_n|\,\|_L\,,$$

therefore

$$\|z_{n+}\|_L \leqq (\lambda-1)\|R(\lambda, T_L)z_n\|_L + \frac{\|u_n\|_L}{\lambda-1}\,.$$

For a fixed λ, let $n\to\infty$. Then, making use of lemma 7.1 and (7.10), we get

$$\overline{\lim_{n\to\infty}}\, \|z_{n+}\|_L \leqq (\lambda-1)\, \overline{\lim_{n\to\infty}}\, \|R(\lambda, T_L)Q_L x_n\|_L$$
$$\leqq (\lambda-1)\|R(\lambda, T)\|_{QE}\|Q\|\,\|f_0\|$$
$$\leqq (\lambda-1)b\|Q\|\,\|f_0\|,$$

where b is the number defined in p. 158. This holds for any λ ($\lambda>1$), therefore

$$\lim_{n\to\infty} \|z_{n+}\|_L = 0.$$

Since $z_n \in Q_L L$, we have

$$\|z_{n+}\|_L = \|z_{n-}\|_L .$$

Consequently

$$\|z_n\|_L \to 0 \qquad (n\to\infty).$$

This contradicts (7.11) and the proposition is proved.

Since L is a Banach lattice the complexification $\tilde{L}$ of L can be defined. By consulting remark 3.1, it can be shown that $\tilde{L}$ is nothing other than the (norm-)completion of the complex normed space $\tilde{E}$ the norm of which is defined by $f_0(|x|)$. To denote the extension of operators in L to $\tilde{L}$ we use the same letter, e.g., $\tilde{T}_L$ is the extension of T_L to $\tilde{L}$. Then, by proposition 7.4, we get

PROPOSITION 7.5. *For every positive number ε, there exists a positive number η such that*

$$x, u \in \tilde{L}, \qquad |x| \leqq e, \qquad \tilde{T}_L x - x = u$$

and

$$\|u\|_L < \eta$$

imply

$$\|\tilde{Q}_L x\|_L < \varepsilon .$$

Proof. It can be shown easily that both the real and imaginary parts of x and corresponding parts of u satisfy the conditions of this proposition. Then proposition 7.4 proves proposition 7.5.

Hereafter we denote by $\eta(\varepsilon)$ the positive number η determined by ε in proposition 7.5.

PROPOSITION 7.6. *For every positive number ε, the positive number $\eta\left(\frac{\varepsilon}{2}\right)$ defined above satisfies the following condition:*

Let

$$x, y \in L, \qquad x\wedge y = 0, \qquad x+y = e, \qquad T_L x - x = u$$

and

$$\|u\|_L < \eta\left(\frac{\varepsilon}{2}\right).$$

Then

$$\|x\|_L<\varepsilon \quad or \quad \|y\|_L<\varepsilon .$$

Proof. By proposition 7.5 we get

$$\|Q_L x\|_L<\frac{\varepsilon}{2} \quad \text{and} \quad \|Q_L y\|_L<\frac{\varepsilon}{2} .$$

Without loss of generality we can assume

$$f_0(x)\leqq f_0(y) .$$

Then, as in the proof of lemma 4 in [10], we get

$$0\leqq|x|\leqq f_0(x)e+|Q_L x|\leqq 2|Q_L x| .$$

Consequently

$$\|x\|_L\leqq 2\|Q_L x\|_L<\varepsilon .$$

PROPOSITION 7.7. *For every positive integer k and every positive number ε which is smaller than $\frac{1}{k}$, the positive number $\eta\left(\frac{\varepsilon}{2}\right)$ satisfies the following condition:*

Let $x_l\in L$ $(l=1, 2, \cdots, k)$ be such that

$$x_l\wedge x_m=0 \qquad (l\neq m)$$

$$e=x_1+x_2+\cdots+x_k$$

$$\|T_L x_l-x_l\|<\eta\left(\frac{\varepsilon}{2}\right) \qquad (l=1, 2, \cdots, k).$$

Then at least one l $(1\leqq l\leqq k)$ satisfies

$$\|e-x_l\|_L<\varepsilon .$$

Proof. By proposition 7.6 we can prove this proposition as one of the authors has proved lemma 5 in [10].

8. The voidness of the continuous spectrum on the spectral circle

In this section we prove assertion C) of the main theorem, namely,

THEOREM 8.1. *Under the assumption of the main theorem, the continuous spectrum is void on Γ.*

To prove this theorem, it is sufficient to show

$$A_\sigma(T)\cap\Gamma\subset P_\sigma(T)\cap\Gamma .$$

Throughout this section, as we mentioned in section 3 for E, the symbol $\sim$ is omitted in the complexification of L and also in the extensions of operators to this complexificated space.

We begin with the following

DEFINITION 8.1. *The normalized approximate spectrum of T_L, denoted by $NA_\sigma(T_L)$, is the set of complex number λ_0 for which there exist $x_n, u_n \in L$*

$$T_L x_n - \lambda_0 x_n = u_n , \tag{8.1}$$

$$|x_n| = e \tag{8.2}$$

and

$$\|u_n\|_L \to 0 \qquad (n \to \infty) . \tag{8.3}$$

With this definition we get

PROPOSITION 8.1. *The approximate spectrum of T on Γ is contained in the normalized approximate spectrum of T_L, i.e.,*

$$A_\sigma(T) \cap \Gamma \subset NA_\sigma(T_L) \cap \Gamma .$$

Proof. Assume

$$\lambda_0 \in A_\sigma(T) \cap \Gamma .$$

By definition there exist $x_n, u_n \in E$ which satisfy, besides (8.1),

$$\|x_n\| = 1$$

and

$$\|u_n\| \to 0 \qquad (n \to \infty) . \tag{8.4}$$

From (8.1) we get

$$T|x_n| \geqq |x_n| - |u_n| . \tag{8.5}$$

Put

$$z_n = Q|x_n| .$$

Then, as in the proof of proposition 7.4, we get from (8.5)

$$f_0(z_{n+}) = f_0(z_{n-}) \to 0 .$$

Consequently

$$\|z_n\|_L \to 0 . \tag{8.6}$$

(8.4) shows

$$\|u_n\|_L \to 0 . \tag{8.7}$$

Since lemma 6 in [14] remains true in the case where E is an arbitrary Banach lattice, we get from the assumptions

$$\underline{\lim}_{n \to \infty} \|x_n\|_L \geqq 1 . \tag{8.8}$$

If we consider x_n and u_n to be elements of L, then relations (8.1), (8.6), (8.7) and (8.8) assure, as in the proof of proposition 4 in [10], the existence of new sequences which satisfy relations (8.1), (8.2) and (8.3). This proves that

$$\lambda_0 \in NA_\sigma(T_L) .$$

LEMMA 8.1. *Let* $\lambda_0 \in NA_\sigma(T_L) \cap \Gamma$ *and* $\lambda_0 \neq 1$. *Then for the sequence* x_n *described in definition* 8.1 *holds the relation:*

$$\|P_L x_n\|_L \to 0. \qquad (n \to \infty)$$

Proof. Since

$$f_0(u_n) = f_0(T_L x_n - \lambda_0 x_n) = (T_L^* f_0 - \lambda_0 f_0)(x_n) = (1-\lambda_0) f_0(x_n)$$

and

$$P_L x_n = f_0(x_n) e,$$

the lemma is evident.

The properties of $T_L \in \mathfrak{L}(L)$, shown in the previous section, allow us to develop the discussion for $NA_\sigma(T_L) \cap \Gamma$ along the same line as one of the authors did in [10]. Therefore we get the following proposition the corresponding one of which is found in the proof of proposition 6 in [10].

PROPOSITION 8.2. *If* λ_0 *belongs to* $NA_\sigma(T_L) \cap \Gamma$, *then, for every integer* l, λ_0^l *belongs to* $NA_\sigma(T_L) \cap \Gamma$.

In the previous paper [10], it was a direct consequence of this proposition that λ_0 is a k-th root of unity. However in the present case, since $R(\lambda, T_L)$ is not known to have a pole at $\lambda = 1$, we must prove

PROPOSITION 8.3. *If* λ_0 *belongs to the normalized approximate spectrum of* T_L *on* Γ, *then it is a* k-*th root of unity for a positive integer* k.

Proof. To prove the proposition, assume the contrary, i. e., let the complex number λ_0 belonging to $NA_\sigma(T_L) \cap \Gamma$ be not a root of unity. Then, by proposition 8.2, there exist complex numbers λ_m $(m=1, 2, \cdots)$ such that

$$\lambda_m \in NA_\sigma(T_L) \cap \Gamma$$

and

$$|\lambda_m - 1| < \frac{1}{m}.$$

By definition and lemma 8.1 there exist $x_m \in L$ such that

$$|x_m| = e,$$

$$\|T_L x_m - \lambda_m x_m\|_L < \frac{1}{m}$$

and

$$\|P_L x_m\|_L < \frac{1}{m}.$$

Then

$$\|T_L x_m - x_m\|_L \leqq \|T_L x_m - \lambda_m x_m\|_L + \|(\lambda_m - 1) x_m\|_L < \frac{2}{m}.$$

Therefore, by proposition 7.5, we get

$$\|Q_L x_m\|_L \to 0 .$$

Hence

$$\|x_m\|_L \leqq \|P_L x_m\|_L + \|Q_L x_m\|_L \to 0 .$$

This is a contradiction and the proof is completed.

We assume hereafter that $\lambda_0 \in NA_\sigma(T_L) \cap \Gamma$ and $\lambda_0 \neq 1$, then by proposition 8.3, λ_0 is a primitive k-th root of unity for a positive integer $k>1$. Also we put as in [10]

$$c = \frac{1}{1-\cos\frac{2\pi}{k}} .$$

Proposition 8.3 corresponds to proposition 6 in section 7 of [10]. The discussion of the remaining parts of section 7 in [10] can be applied to our present case with only slight modifications. In the first place, since we are dealing with normalized approximate spectrum, proposition 7.5 stands for the assumption that $\lambda=1$ is a pole of $R(\lambda, T_L)$. Indeed, propositions 7.6 and 7.7 in the present paper correspond to lemmas 4 and 5 in [10] respectively. In the proof of proposition 7 in [10] we have made use of the property that from every bounded sequence we can select a weakly convergent subsequence. This property cannot be used in the present case. However, instead, we can use here the property that from every bounded sequence x_n we can select a subsequence $x_{n(m)}$ such that $f_0(x_{n(m)})$ is convergent. By this modification the proof of proposition 7 can be applied to the present case. Therefore we can get for T_L and $\lambda_0 \in NA_\sigma(T_L)$ propositions which correspond to propositions 8, 9, 10 and 11 in [10]. Consequently we get the following proposition which corresponds to theorem 7 in [10].

PROPOSITION 8.4. *If λ_0 belongs to the normalized approximate spectrum of T_L on Γ, then it belongs to the point spectrum of T_L on Γ, i.e.,*

$$NA_\sigma(T_L) \cap \Gamma \subset P_\sigma(T_L) \cap \Gamma .$$

Proof. In the proof of theorem 7 in [10] if we replace the definition of δ_0 and δ by the following ones (i) and (ii) respectively, then the discussions there remain true in the present case.

(i) Determine δ_0 smaller than

$$\frac{\eta\left(\frac{1}{2k}\right)}{2k(2k-1)c} \quad \text{and} \quad \frac{\eta\left(\frac{1}{8c}\right)}{2k(2k-1)} .$$ [11]

(ii) For every positive number $\varepsilon\left(<\frac{1}{2c}\right)$, determine a positive number δ to

11) For the notation $\eta(\cdot)$ see p. 170.

satisfy

$$\delta \leqq \min\left\{\delta_0, \frac{\eta\left(\frac{\varepsilon}{4}\right)}{k(2k-1)}\right\}.$$

PROPOSITION 8.5. *If λ_0 belongs to the point spectrum of T_L on Γ, then it belongs to the point spectrum of T on Γ, i. e.,*

$$P_\sigma(T_L) \cap \Gamma \subset P_\sigma(T) \cap \Gamma .$$

Proof. To prove the proposition, assume the contrary, i. e.,

$$\lambda_0 \overline{\in} P_\sigma(T) . \tag{8.9}$$

Put

$$\varphi_n(\lambda) = \frac{\lambda_0^{nk-1} + \lambda_0^{nk-2}\lambda + \cdots + \lambda^{nk-1}}{nk\lambda_0^{nk-1}}$$

and

$$p(\lambda) = \lambda_0 - \lambda .$$

Then we have

(i) $\varphi_n(\lambda_0) = 1$

and

(ii) $$\|p(T)\varphi_n(T)\| = \left\|\frac{\lambda_0^{nk}I - T^{nk}}{nk\lambda_0^{nk-1}}\right\| = \left\|\frac{I - T^{nk}}{nk}\right\| = \left\|(I-T)\left(\frac{I+T+\cdots+T^{nk-1}}{nk} - P\right)\right\| \to 0 ,$$

where the last part of the discussion is justified by theorem 5 in S. Karlin [5] which asserts

$$\left\|\frac{I+T+\cdots+T^{n-1}}{n} - P\right\| \to 0 . \tag{8.10}$$

By theorem 6.1 and (8.10) we also have

(iii) The range of $p(T)$ is dense in E and $\|\varphi_n(T)\|$ is bounded uniformly w. r. t. n.

The above conditions (i), (ii) and (iii) correspond to conditions (1), (2) and (5) of theorem 3.9 in N. Dunford [1]. From the theorem we have, for each $x \in E$,

$$\|\varphi_n(T)x\| \to 0 . \tag{8.11}$$

By assumption, there exists $x_0 \in L$ such that

$$\|x_0\|_L = 1 \tag{8.12}$$

and

$$T_L x_0 = \lambda_0 x_0 . \tag{8.13}$$

Since E is dense in L, there exists $x_0' \in E$ such that

$$\|x_0 - x_0'\|_L < \frac{1}{3}.$$

Then the equality

$$\|\varphi_n(T_L)\|_L = 1$$

yield

$$\|\varphi_n(T_L)x_0 - \varphi_n(T)x_0'\|_L < \frac{1}{3}. \tag{8.14}$$

By (8.13) we get

$$\varphi_n(T_L)x_0 = x_0.$$

Therefore (8.14) is reduced to the formula

$$\|x_0 - \varphi_n(T)x_0'\|_L < \frac{1}{3}. \tag{8.15}$$

This holds for any positive integer n.

By (8.11) there exists a positive integer n such that

$$\|\varphi_n(T)x_0'\| < \frac{1}{3\|f_0\|}.$$

Therefore

$$\|\varphi_n(T)x_0'\|_L < \frac{1}{3}. \tag{8.16}$$

(8.15) and (8.16) imply

$$\|x_0\|_L < \frac{2}{3}.$$

This contradicts (8.12) and the proof is completed.

Proof of theorem 8.1. By propositions 8.1, 8.4 and 8.5 we get

$$A_\sigma(T) \cap \Gamma \subset P_\sigma(T) \cap \Gamma. \tag{8.17}$$

This proves the theorem.

Remark 8.1. Since the converse inclusion in (8.17) is evident, it can be easily proved that $\sigma(T)$, $P_\sigma(T)$, $P_\sigma(T_L)$ and $NA_\sigma(T_L)$ are identical on Γ with each other.

9. Concluding section

Theorems 5.1, 6.1 and 8.1 together show that our main theorem holds true.[#]

As a consequence of the main theorem, H. H. Schaefer's problem b) mentioned in the introduction is answered affirmatively for positive irreducible operators in an arbitrary Banach lattice.

#) Added in proof: Though in sections 4, ..., 8 it has been always assumed that the dimension of E is at least two, the main theorem holds trivially in the case where E is one-dimensional.

As another consequence of the main theorem, we can see that the additional condition C) is not necessary in theorem 5 of [13], namely,

THEOREM 9.1. *Let* $T\in\mathfrak{L}(E)$ *be positive and* $\lambda=r(T)$ *be a pole of* $R(\lambda, T)$. *Then* T *is a non-support operator if and only if* T *is irreducible and the spectrum of* T *on* Γ *consists only of one point* $r(T)$.

In the rest of this section we try to generalize the foregoing theory. Condition II) in the main theorem is, under conditions I) and III), equivalent to three conditions 1), 2) and 3) in proposition 4.2. By weakening 1) in the assumption of the main theorem, we get

THEOREM 9.2. *Let* $T\in\mathfrak{L}(E)$ *satisfy conditions* I), III), 2) *and* 3) *mentioned above and, in place of* 1), *satisfy the following condition:*

1′) *The eigenspace of* T *for* $r(T)$ *is finite-dimensional.*

Then the spectrum of T *on* Γ *is a finite union of sets, each of which is the set of* k_j*-th roots of unity multiplied by* $r(T)$, *and each point of the sets is a simple pole of* $R(\lambda, T)$, *where* k_j $(j=1, 2, \cdots, h)$ *are positive integers and* h *is the dimension of the eigenspace of* T *for* $r(T)$.

Remark 9.1. By this theorem the residual and continuous spectrum are both void on Γ.

To prove the theorem we prepare five propositions below, in each of which the assumptions of theorem 9.2 are also assumed. Since the theorem is trivial in case $r(T)=0$, we assume further $r(T)=1$ for the sake of simplicity.

PROPOSITION 9.1. *Let* F *be the eigenspace of* T *for* 1, *then* F *is a vector lattice w.r.t. the lattice operation defined already in* E, *i.e.,*

$$\text{for } x, y\in F, \quad x\vee y \text{ and } x\wedge y \text{ belong to } F.$$

Consequently

$$\text{for } x, y\in F, \quad x\wedge y=0 \text{ in } F \text{ if and only if } x\wedge y=0 \text{ in } E. \tag{9.1}$$

Proof. Since T is positive, we have for $x, y\in F$

$$T(x\vee y)\geqq Tx\vee Ty=x\vee y. \tag{9.2}$$

The existence of a strictly positive eigenfunctional of T^* for 1 assures the equality in (9.2), and therefore the proposition is valid.

We denote the positive cone of F by K_F, i.e.,

$$K_F=K\cap F.$$

Then by Sz. Nagy [21] we get

PROPOSITION 9.2. *There exists a positive base* $e_1, e_2, \cdots, e_h$ *of* K_F, *namely,*

$$e_1, e_2, \cdots, e_h\in K_F$$

is a base of F such that, for

$$x=\alpha_1 e_1+\alpha_2 e_2+\cdots+\alpha_h e_h \in F,$$

$$x\in K_F \quad \textit{if and only if} \quad \alpha_1, \alpha_2, \cdots, \alpha_h \geqq 0.$$

By (9.1) we get

COROLLARY 9.1. *Under the notation of proposition 9.2 we have*

$$e_l \wedge e_m = 0 \qquad (l\neq m).$$

Put

$$e=e_1+e_2+\cdots+e_h.$$

Since, F contains a non-support element of K, it can be seen without difficulty that e is also a non-support element of K. Then we have

PROPOSITION 9.3. *For each $x\in K$ and for each l ($1\leqq l\leqq h$), the sequence $x\wedge ne_l$ converges strongly to an element $x_l\in K$ (as $n\to\infty$) such that*

$$x=x_1+x_2+\cdots+x_h$$

and

$$x_l\wedge x_m=0 \qquad (l\neq m).$$

Proof. Put

$$x\wedge ne_l = x_{n,l}$$

and

$$x\wedge ne = x_{n,0}.$$

Then it is easy to see that

$$0\leqq x_{n,l}\leqq x_{n+p,l} \qquad (0\leqq l\leqq h,\ 0<p),$$

$$x_{n,l}\wedge x_{n,m}=0 \qquad (l\neq m,\ 1\leqq l, m\leqq h) \tag{9.3}$$

and

$$x_{n,0}=x_{n,1}+x_{n,2}+\cdots+x_{n,h}. \tag{9.4}$$

Consequently

$$x_{n+p,0}-x_{n,0}=(x_{n+p,1}-x_{n,1})+(x_{n+p,2}-x_{n,2})+\cdots+(x_{n+p,h}-x_{n,h}).$$

Therefore, for each l ($1\leqq l\leqq h$) and for each positive integers n and p,

$$0\leqq x_{n+p,l}-x_{n,l}\leqq x_{n+p,0}-x_{n,0}.$$

Then, making use of theorem 3.1, we can see that the sequence $x_{n,l}$ converges strongly. If we denote the limit of this sequence by x_l, then (9.3) and (9.4) prove the proposition.

Under the above notations, let $P_{e_l}\in\mathfrak{L}(E)$ be the natural extension of the operator defined by

$$P_{e_l}x=x_l \qquad (x\in K).$$

Then P_{e_l} is a positive projection of $\mathfrak{L}(E)$ such that

$$|P_{e_l}x| \wedge |P_{e_m}y| = 0 \qquad (x, y \in E \text{ and } l \neq m)$$

and

$$P_{e_1} + P_{e_2} + \cdots + P_{e_h} = I.$$

Put

$$P_{e_l}E = E_l \quad \text{and} \quad P_{e_l}K = K_l \qquad (1 \leqq l \leqq h).$$

Then E_l is invariant under T and E is the direct sum of $E_1, E_2, \cdots, E_h$. We denote the restriction of T on E_l by T_l. We also have $K_l = E_l \cap K$. Under these notations we get by theorem 3.1

PROPOSITION 9.4. *For each* l $(1 \leqq l \leqq h)$, e_l *is a non-support element of* K_l *and satisfies*

$$T_l e_l = e_l.$$

Let f be a strictly positive eigenfunctional of T^* for 1, and denote the restriction of f to E_l by f_l $(l = 1, 2, \cdots, h)$. Under these notations we get easily

PROPOSITION 9.5. *For each* l $(1 \leqq l \leqq h)$, f_l *is a strictly positive eigenfunctional of* T_l^* *for* 1.

Proof of theorem 9.2. Since each E_l reduces T, by the well known (complete) reduction theory each T_l satisfies III). It is clear that T_l satisfies I). By propositions 9.2 and 9.4 T_l satisfies 1) and 2). Finally by proposition 9.5 T_l satisfies 3). Therefore for each T_l the main theorem is applicable. Then, again by the (complete) reduction theory, theorem 9.2 is valid.

As a consequence of theorem 9.2, problem b) of H. H. Schaefer is answered affirmatively for a positive operator $T \in \mathfrak{L}(E)$ which satisfies 1′), 2) and 3) mentioned above.

It is natural to ask if condition 1) or 1′) is not necessary. However, if neither 1) nor 1′) is assumed, then, all the other conditions being satisfied, the essential parts of the main theorem are shown to fail. Indeed, example 9.1 shows that $\sigma(T) \cap \Gamma$ contains a point which is not contained in $P_\sigma(T) \cap \Gamma$. Moreover, example 9.2 shows that there exists an eigenvalue on Γ which is not a pole of $R(\lambda, T)$. As a preparation for them we consider the four-dimensional l_p $(1 \leqq p \leqq \infty)$ space E_0, elements of which will be denoted by

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}.$$

In this space we define the operator $A_{a,b}$ (or simply A) by [12)]

12) Hereafter we do not distinguish an operator from its matrix representation.

$$A_{a,b}=\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & a \\ b & a & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix}.$$

Let

$$a, b \geqq 0, \qquad a+b=1, \quad \text{and} \quad |a-b| \geqq \frac{1}{2}. \tag{9.5}$$

Then the operator A is positive and irreducible. Moreover, we can show

$$r(A)=1, \qquad \|A\|=1$$

and

$$\sigma(A)=P_\sigma(A)=\{\pm 1, \ \pm i(a-b)\}.$$

The eigenspace of A for 1, which is one-dimensional, contains the element

$$\xi_0=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Similar situation holds for A^*. By solving a linear equation we get for $\lambda \in \rho(A)$

$$R(\lambda, A)=(\lambda I-A)^{-1}=\frac{1}{2(\lambda^2-1)}\begin{pmatrix} \lambda & \lambda & 1 & 1 \\ \lambda & \lambda & 1 & 1 \\ 1 & 1 & \lambda & \lambda \\ 1 & 1 & \lambda & \lambda \end{pmatrix}$$

$$+\frac{1}{2(\lambda^2+(a-b)^2)}\begin{pmatrix} \lambda & -\lambda & a-b & b-a \\ -\lambda & \lambda & b-a & a-b \\ b-a & a-b & \lambda & -\lambda \\ a-b & b-a & -\lambda & \lambda \end{pmatrix}.$$

Therefore, putting

$$c=a-b, \tag{9.6}$$

we get

$$R(\lambda, A)=\sum_{m=-1}^{\infty} A_m(\lambda-1)^m \qquad (|\lambda-1|<1), \tag{9.7}$$

where, for $m \geqq 0$, A_m is determined by the following relation:

$$A_m=\frac{1}{4}\left(-\frac{1}{2}\right)^{m+1}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}+\frac{1}{4}\left(-\frac{1}{2}\right)^{m}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$$

$$+\frac{(-1)^m}{2ci}\left(\frac{1}{(1-ci)^{m+1}}-\frac{1}{(1+ci)^{m+1}}\right)\begin{pmatrix}1&-1&c&-c\\-1&1&-c&c\\-c&c&1&-1\\c&-c&-1&1\end{pmatrix}$$

$$+\frac{(-1)^{m-1}}{2ci}\left(\frac{1}{(1-ci)^{m}}-\frac{1}{(1+ci)^{m}}\right)\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix}.$$

Therefore, for all m, we have

$$\|A_m\|\leqq d\,, \tag{9.8}$$

where d is a positive constant independent of m and a, b under the condition (9.5).

Let E be the l_p space whose range is the space E_0 mentioned above, i. e., E is the set of elements $x=\{x_n\}$ for which

$$x_n\in E_0$$

and

$$\|x\| \text{ is finite.}$$

Here as usual we mean

$$\|x\|=\begin{cases}(\sum_{n=1}^{\infty}\|x_n\|^p)^{1/p} & (1\leqq p<\infty)\\ \sup_n\{\|x_n\|\} & (p=\infty)\,.\end{cases}$$

Again we define

$$x\geqq 0 \text{ by } x_n\geqq 0 \qquad (n=1, 2, \cdots)\,.$$

Then E is clearly a Banach lattice. Indeed E is nothing other than the usual l_p $(1\leqq p\leqq\infty)$. With these preparations established we give

Example 9.1. Let E be the space mentioned above, and for each $x=\{x_n\}$, define the operator $T\in\mathfrak{L}(E)$ by

$$Tx=T\{x_n\}=\{T_nx_n\}\,,$$

where, by definition,

$$T_n = A_{1-\frac{1}{4n},\ \frac{1}{4n}}\ .$$

Let us show that this example has the desired property. Since the norm of the operator T is the supremum of the norms of the operators T_n, T belongs to $\mathfrak{L}(E)$ with $\|T\|=1$. It is clear that T is positive. It is also clear that the eigenspace of T for 1 contains a non-support element, indeed the element

$$e=\left\{\frac{1}{2^n}\xi_0\right\} \qquad (\text{for } 1 \leqq p < \infty)$$

and

$$e=\{\xi_0\} \qquad (\text{for } p=\infty)\ .$$

This proves also that

$$r(T)=1\ .$$

We can see similarly that the eigenspace of T^* for 1 contains a strictly positive functional. For any $x=\{x_n\}$ define the operator $S(\lambda, T)$ by

$$S(\lambda, T)x=\{R(\lambda, T_n)x_n\}\ .$$

Then relations (9.7) and (9.8) show that

$$\|S(\lambda, T)\|=\sup_n \|R(\lambda, T_n)\|<\infty \qquad (|\lambda-1|<1)\ .$$

Therefore $S(\lambda, T)$ belongs to $\mathfrak{L}(E)$. Then it can be seen without difficulty that

$$S(\lambda, T)=R(\lambda, T) \qquad (|\lambda-1|<1)\ .$$

Relations (9.7) and (9.8) also imply that $\lambda=1$ is a pole of $R(\lambda, T)$. Thus for T all conditions of theorem 9.2 are satisfied except for 1′). However, it can be easily seen that $\pm i$ belong to $C_\sigma(T) \cup R_\sigma(T)$.

Example 9.2. Let E be the space mentioned in example 9.1. For any $x=\{x_n\}\in E$, defines the operator T' by

$$T'x=T'\{x_n\}=\{T_{n-1}x_n\}\ ,$$

where we define

$$T_0=A_{1,0}$$

and, for $n\geqq 1$, T_n is the operator defined in example 9.1.

As regards conditions I), III), 2) and 3) the situation is the same as the former one. However, either of the eigenvalues $\pm i$ is not a pole of $R(\lambda, T)$. Indeed, it is not an isolated point of $P_\sigma(T)$.

By these examples problem b) of H. H. Schaefer is solved negatively without the assumption of irreducibility in the space l_p even if T satisfies conditions 2) and 3). As for his problem a) there are two ways of interpretation, namely, under the assumption described in section 1:

a′) Is every element of $\sigma(T)\cap\Gamma$ an isolated point of $\sigma(T)\cap\Gamma$?

a″) Is every element of $\sigma(T)\cap\Gamma$ an isolated point of $\sigma(T)$?

Under these interpretations example 9.2 provides a negative answer to problem a″) but does not to problem a′).

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